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NZI rings

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Abstract: A ring R is called NZI if for any $a \in R$, $l(a)$ is an N-ideal of R. In this paper, we first study some basic properties and basic extensions of NZI rings. Next, we study the strong regularity of NZI rings and obtain the following results: (1) Let R be a left SF-ring. Then R is a strongly regular ring if and only if R is an NZI ring; (2) If R is an NZI left MC2 ring and every simple singular left $R-$module is nil-injective, then R is reduced; (3) Let R be an NZI ring. Then R is a strongly regular ring if and only if R is a von Neumann regular ring; (4) Let R be an NZI ring. Then R is a clean ring if and only if R is an exchange ring.

Key words: Reduced ring, SF ring, strongly regular ring, N-ideal, NZI ring, nil-injective

1. Introduction

All rings considered in this paper are associative with identity, and all modules are unital. The symbols $J(R)$, $P(R)$, $N(R)$, $E(R)$, $Z_l(R)$ (resp., $Z_r(R)$) and $\mathrm{Soc}({}_RR)$ (resp., $\mathrm{Soc}(R_R)$) stand respectively for the Jacobson radical, the prime radical, the set of all nilpotent elements, the set of all idempotent elements, the left (resp., right) singular ideal and the left(resp., right) socle of R. For any nonempty subset X of R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of X and the set of left annihilators of X in R, respectively. Especially, if $X = a$, we write $r(X) = r(a)$ and $l(X) = l(a)$.

A ring R is called (von Neumann) regular [9] if for every $a \in R$, there exists $b \in R$ such that $a = aba$. A ring R is called strongly regular [18] if for every $a \in R$, there exists $b \in R$ such that $a = a^2b$. A ring R is called reduced [19] if $N(R) = 0$. Rege [18] proved that R is strongly regular if and only if R is reduced regular. According to Cohn [6], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is said to be semicommutative if $ab = 0$ implies $aRb = 0$. Obviously, R is a semicommutative ring if and only if $l(a)$ is an ideal of R for any $a \in R$. A ring R is called Abelian if every idempotent of R is contained in the centry $C(R)$. Evidently, semicommutative rings are Abelian. For several years, the applications of semicommutative rings have been studied by many authors. Kim, Nam and Kim [12, Theorem 4] proved that if R is a semicommutative ring and every simple singular left R-module is YJ-injective, then R is a reduced weakly regular ring. A ring R is called a left (right) SF [19] if all simple left (right) R-modules are flat. It is well known that if R is left

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SF and semicommutative, then R is strongly regular. Clearly, R is a semicommutative ring, and then R is a von Neumann regular ring if and only if R is a strongly regular ring. The present paper is such an attempt in this direction; in other words, we shall give some weaker conditions such as NZI rings.

2. NZI rings

Let R be a ring. A left ideal L of R is called an N-ideal, for every $b \in N(R) \cap L$, $bR \subseteq L$. A ring R is called NZI if for any $a \in R$, $l(a)$ is an N-ideal of R. Clearly, semicommutative rings are NZI, but the converse is not true, in general.

Example 2.1 *Let F be a field, $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} | x, y, z \in F \right\}$ and let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$, $B = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in l_R(A)$. Then $0 = BA = \begin{pmatrix} xa & xb + yc \\ 0 & zc \end{pmatrix}$, that is, $xa = zc = 0$ and $xb + yc = 0$.*

Case 1. *If $a \neq 0$, $c \neq 0$, then $x = y = z = 0$, so $l_R(A) = 0$, this implies $l_R(A)$ is an N-ideal.*

Case 2. *If $a \neq 0$, $c = 0$, then $x = 0$, so $l_R(A) = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Since $N(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is an ideal of R, $N(R) \bigcap l_R(A) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is an ideal. Hence $l_R(A)$ is an N-ideal.*

Case 3. *If $a = 0$, $c \neq 0$, then $z = 0$, so $l_R(A) = \left\{ \begin{pmatrix} x & -xbc^{-1} \\ 0 & 0 \end{pmatrix} | x \in F \right\}$, $N(R) \bigcap l_R(A) = 0$, this implies $l_R(A)$ is an N-ideal.*

Case 4. *If $a = 0$, $c = 0$, $b \neq 0$, then $x = 0$, the process is similar to **Case 2**.*

Case 5. *If $a = 0$, $b = 0$, $c = 0$, then $l_R(A) = R$. Clearly, $l_R(A)$ is an N-ideal.*

So we can attain R is an NZI ring.

Set $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$, then $BA = 0$, $BRA = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \neq 0$, this implies R is not a semicommutative ring.

According to Anderson and Camillo [1], a ring R is called 2-primal if $N(R) = P(R)$. According to Zhao and Yang [24], a ring R is called NCI if $N(R) = 0$ or $N(R)$ contains a nonzero ideal of R, and R is called NI if $N(R)$ is an ideal of R. Clearly, 2-primal rings are NI and NI rings are NCI, and no reversal holds.

Theorem 2.2 *NZI rings are 2-primal.*

Proof Let R be an NZI ring. It is clear that $P(R) \subseteq N(R)$. For any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n = 0$. By hypothesis, $a^{n-1}Ra = 0$. If $n - 1 = 1$, $aRa = 0$, then $(Ra)^2 = 0$, $Ra \subseteq P(R)$, $a \in P(R)$. If $n - 1 > 1$, $a^{n-2}RaRa = 0$, if $n - 2 = 1$, then $aRaRa = 0$, $(Ra)^3 = 0$, $Ra \subseteq P(R)$, $a \in P(R)$. Similar to the above, we can attain $N(R) \subseteq P(R)$. This shows that NZI rings are 2-primal. $\square$

Corollary 2.3 *NZI rings are NI.*

Example 2.4 *Let Z_4 be the ring of integers modulo 4. Consider the ring* $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in Z_4 \right\}$ *with the usual matrix operations.*

Since $N(Z_4) = P(Z_4) = \{\bar{0}, \bar{2}\}$ *is an ideal of* Z_4, Z_4 *is 2-primal, by Bass [4, Proposition 2.5],* R *is 2-primal.*

Set $A = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$, *then* $B \in N(R)$ *and* $BA = 0$. *But* $B \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$, *this shows that* R *is not an NZI ring. So the converse of Theorem 2.2 is not true.*

According to Chen [5], a ring R is called nil-semicommutative if $a, b \in R$ satisfy $ab \in N(R)$, then $arb \in N(R)$ for any $r \in R$. It is known that NI rings are nil-semicommutative by Chen [5].

Corollary 2.5 *NZI rings are nil-semicommutative.*

Example 2.6 *Let F be a field,* $R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} | a_1, a_2, ..., a_6 \in F \right\}$.

Since $N(R) = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$ *is an ideal of* R, R *is an NI ring.*

Set $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *then* $A^2 = 0$, $AB = 0$, *but* $ARB = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$; *this shows that* R *is not an NZI ring.*

By Example 2.6, the converse of Corollary 2.5 is not true, in general.

Corollary 2.7 *NZI rings are directly finite.*

Proof Let R be an NZI ring and $ab = 1$ for any $a, b \in R$. Write $e = ba$, then $e^2 = e$ and $ae = aba = a$. Set $h = a - ea$, so $he = ae - eae = ae - ea = a - ea = h$, $eh = ea - e^2a = 0$, $h^2 = heh = 0$, then $h \in N(R) \bigcap l(1 - e)$. Since R is an NZI ring, $hR(1 - e) = 0$, $hb(1 - e) = 0$, $hb = ab - eab = 1 - e$. Hence $1 - e = (1 - e)^2 = hb(1 - e) = 0$, $e = ba = 1$. This shows that R is directly finite.

An element k of R is called left minimal if Rk is a minimal left ideal of R, and an idempotent e of R is said to be left minimal idempotent if e is a left minimal element of R. We use $ME_l(R)$ to denote the set of all left minimal idempotents of R. A ring R is called left min-abel [22] if every element of $ME_l(R)$ is left semicentral in R, and R is said to be strongly left min-abel [22] if every element of $ME_l(R)$ is central in R. Clearly, Abelian rings and semicommutative rings are left min-abel. □

Theorem 2.8 *NZI rings are left min-abel.*

Proof Let $0 \neq e \in ME_l(R)$ and $a \in R$. Write $h = ae - eae$. If $h \neq 0$, then $eh = 0$, $he = h$ and $h^2 = 0$, hence $h \in N(R) \bigcap l(h)$. By the hypothesis, $hR \subseteq l(h)$, $hRh = 0$. Since $Rh = R(ae - eae) \subseteq Re$, Re is a minimal left ideal of R and $Rh \neq 0$, $Re = Rh$, thus $0 = R(hRh) = ReRe = Re$, $e = 0$, but Re is minimal left ideal of R, which is a contradiction. Hence $h = 0$ and $ae = eae$ for all $a \in R$, which implies that R is left min-abel. □

Lemma 2.9 *Let R be a ring. Then $R[x]$ is left min-abel.*

Proof If I is a minimal left ideal of $R[x]$, then $I \neq 0$, so there exists $f(x) \in I$ and $f(x) \neq 0$. Since I is a minimal left ideal of $R[x]$, $I = R[x]f(x)$. Since $xf(x) \neq 0$, $0 \neq R[x]xf(x) \subseteq R[x]f(x) = I$. Since I is a minimal left ideal of $R[x]$, $R[x]xf(x) = R[x]f(x) = I$, there exists $g(x) \in R[x]$ such that $f(x) = g(x)xf(x)$, but the degree of x appeared on both sides cannot be equal. So there is no minimal left ideal in $R[x]$. Hence $R[x]$ is left min-abel.

By Lemma 2.9, there exists a left min-abel ring which is not NZI. Hence the converse of Theorem 2.8 is not true.

According to Wei and Chen [20], a ring R is called nil-injective if for any $a \in N(R)$, $lr(a) = Ra$. Obviously, YJ-injective modules are nil-injective, but the converse is not true, in general, by Wei and Chen [20]. According to Nicholson and Yousif [17], a ring R is called left MC2 if every left minimal idempotent is right minimal. It is known that R is strongly left min-abel if and only if R is left min-abel and left MC2 by Wei [22]. □

Theorem 2.10 *Let R be an NZI and left MC2 ring. If every simple singular left R-module is YJ-injective, then R is reduced, and $RbR + l(b) = R$ for any $b \in R$.*

Proof Let $a^2 = 0$. If $a \neq 0$, then $l(a) \neq R$, there exists a maximal left ideal M of R such that $l(a) \subseteq M$. If M is not essential in ${}_RR$, then $M = Re = l(1-e)$ for some $e \in E(R)$. Since $R(1-e) \cong R/l(1-e) = R/M$ is a simple left R-module, $R(1-e)$ is a minimal left ideal of R, $1-e$ is a left minimal idempotent element. By hypothesis, R is an NZI ring, R is a left min-abel ring. R is a left MC2 ring, hence R is a strong left min-abel ring, and $1-e$ is central. Since $a \in l(a) \subseteq M = l(1-e)$, $a(1-e) = 0 = (1-e)a$, $1-e \in l(a) \subseteq l(1-e)$, which is a contradiction. Hence M is an essential left ideal of R, thus R/M is YJ-injective. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r + M$. Note that f is a well-defined R-homomorphism. Then there exists a left R-homomorphism $g : R \to R/M$ such that $g(a) = f(a)$. Hence $1 + M = f(a) = g(a) = ag(1) = ac + M$, $g(1) = c + M$, $1 - ac \in M$. Since R is an NZI ring, $ac \in l(a)$, then $1 \in M$, which is a contradiction. Therefore $a = 0$, and R is reduced.

Suppose that there exists $0 \neq c \in R$ such that $RcR + l(c) \neq R$, then there exists a maximal left ideal M of R containing $RcR + l(c)$. If M is not essential in ${}_RR$, then $M = Re = l(1-e)$ for some $e \in E(R)$. Since $R(1-e) \cong R/l(1-e) = R/M$ is a simple left R-module, $R(1-e)$ is a minimal left ideal of R, $1-e$ is a left minimal idempotent element. By the hypothesis, R is an NZI ring, R is a left min-abel ring. R is a left MC2 ring, hence R is a strong left min-abel ring, $1-e$ is central. Since $c \in RcR + l(c) \subseteq M = l(1-e)$, $c(1-e) = 0 = (1-e)c$, $1-e \in l(c) \subseteq l(1-e)$, which is a contradiction. Hence M is an essential left ideal of R, thus R/M is YJ-injective. Let $f : Rc \longrightarrow R/M$ be defined by $f(rc) = r + M$. Note that f is a well-defined R-homomorphism. Then $1 + M = f(c) = cd + M$, $d \in R$, hence $1 - cd \in M$. Since $cd \in RcR \subseteq M$, so $1 \in M$, which is a contradiction. Therefore $RcR + l(c) = R$ for any $c \in R$.

It is clear that Let R be an NZI and left MC2 ring. If every simple singular left R-module is nil-injective, then R is reduced. □

Now we consider whether the result holds if we omit the condition R is a left MC2 ring.

Example 2.11 *Let F be a field,* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} | x, y, z \in F \right\}$.

Clearly, $E(R)=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&u\\0&0\end{pmatrix},\begin{pmatrix}0&u\\0&1\end{pmatrix},u\in F\right\}$

$I_1=\begin{pmatrix}0&F\\0&F\end{pmatrix}=R\begin{pmatrix}0&0\\0&1\end{pmatrix}=R\begin{pmatrix}0&u\\0&1\end{pmatrix}$ $(u\neq 0)$ *is a maximal left ideal of* R *and a summand of* R, *so* I_1 *is not an essential left ideal of* R.

$I_2=\begin{pmatrix}F&F\\0&0\end{pmatrix}$ *is an essential maximal left ideal of* R.

$I_3=\begin{pmatrix}F&0\\0&0\end{pmatrix}=R\begin{pmatrix}1&0\\0&0\end{pmatrix}$ *is a minimal left ideal of* R.

$I_4=\left\{\begin{pmatrix}x&xu\\0&0\end{pmatrix}|x\in F,u\neq 0\right\}$ *is a minimal left ideal of* R.

$I_5=N(R)=\begin{pmatrix}0&F\\0&0\end{pmatrix}=R\begin{pmatrix}0&1\\0&0\end{pmatrix}$ *is a minimal left ideal of* R.

Clearly, I_3, I_4, $I_5\subseteq I_2$ *and* I_3, I_4, I_5 *are not maximal left ideals of* R *and* $I_2=I_3\bigoplus I_5$. *By Example 2.1,* R *is an NZI ring, Set* W *is a simple singular left* R*-module,* $W\cong R/L$, *where* L *is an essential maximal left ideal of* R. *Since* I_2 *is the only essential maximal left ideal of* R, $W\cong R/I_2$. *Now we prove* $W\cong R/I_2$ *is injective.*

Let $f: I_2\longrightarrow R/I_2$ *be any left* R*-homomorphism and* $f\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)=\begin{pmatrix}u&v\\0&w\end{pmatrix}+I_2$, *then*
$\begin{pmatrix}u&v\\0&w\end{pmatrix}+I_2=\begin{pmatrix}1&0\\0&0\end{pmatrix}f\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\left(\begin{pmatrix}u&v\\0&w\end{pmatrix}+I_2\right)=\begin{pmatrix}u&v\\0&0\end{pmatrix}+I_2=\bar{0}$.

Similar to the above, $f\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)=\bar{0}$.

For any $\begin{pmatrix}a&b\\0&0\end{pmatrix}\in I_2$, $f\left(\begin{pmatrix}a&b\\0&0\end{pmatrix}\right)=f\left(\begin{pmatrix}a&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}b&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)$
$=\begin{pmatrix}a&0\\0&0\end{pmatrix}f\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)+\begin{pmatrix}b&0\\0&0\end{pmatrix}f(\begin{pmatrix}0&1\\0&0\end{pmatrix})=\bar{0}$.

Hence $f=0$ *and* f *can be extended to R. Then* R *is an NZI ring and every simple singular left* R*-module is injective, so every simple singular left* R*-module is nil-injective.*

Since $\begin{pmatrix}1&0\\0&0\end{pmatrix}\in ME_l(R)$ *and* $\begin{pmatrix}0&1\\0&0\end{pmatrix}R\begin{pmatrix}1&0\\0&0\end{pmatrix}=0$, *but* $\begin{pmatrix}1&0\\0&0\end{pmatrix}R\begin{pmatrix}0&1\\0&0\end{pmatrix}\neq 0$, *because* $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\neq 0$, *so* R *is not a left MC2 ring.*

Since $N(R)=\begin{pmatrix}0&F\\0&0\end{pmatrix}\neq 0$, R *is not reduced.*

Therefore the result of Theorem 2.10 is not true if we omit the condition R is a left MC2 ring.

Theorem 2.12 *Let* R *be an NZI ring. If every simple left* R*-module is nil-injective, then* R *is reduced.*

Proof Let $a\in R$ and $a^2=0$. If $a\neq 0$, then $l(a)\neq R$, so there exists a maximal left ideal of R such that $l(a)\subseteq M$. Since R/M is simple left R-module, R/M is nil-injective. Let $f: Ra\longrightarrow R/M$ be defined by $f(ra)=r+M$. Note that f is a well-defined R-homomorphism. Then $1+M=f(a)=ac+M$ where $c\in R$, hence $1-ac \in M$. Since R is an NZI ring, $ac \in l(a)$, then $1 \in M$, which is a contradiction. Therefore $a = 0$, and R is reduced. $\square$

Remark 2.13 *Let F is a field. Since $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ is an NZI ring but not reduced ring, there exists a simple left R-module W is not nil-injective.*

Let $W = R\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$ be a minimal left ideal of R, then W is a simple left R-module. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

Let $f : R\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow R\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then f is a well defined left R-homomorphism. If W is nil-injective, then there exists $g : R \longrightarrow W$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = g\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = 0$, which is a contradiction.

This shows that W is the simple left R-module but not nil-injective.

Kim, Nam and Kim [12, Theorem 4] proved that if R is a semicommutative ring and every simple singular left module is YJ-injective, then R is a reduced weakly regular ring. By Theorem 2.10, we have the following corollary.

Corollary 2.14 *Let R be an NZI and left MC2 ring. If every simple singular left R-module is YJ-injective, then R is a reduced weakly regular ring.*

A ring R is called MELT if every maximal essential left ideal is an ideal of R. Wei [21, Theorem 16] proved that R is a strongly regular ring if and only if R is a semicommutative MELT ring and every simple singular left modules are YJ-injective. By Theorem 2.10, we have the following corollary.

Corollary 2.15 *Let R be an MELT NZI and left MC2 ring. If every simple singular left module is YJ-injective, then R is a strongly regular ring.*

Let R be a ring and $a \in R$. a is called clean element if a is a sum of an unit and an idempotent of R, and x is called exchange element if there exists $e \in E(R)$ such that $e \in Rx$ and $1-e \in R(1-x)$. A ring R is called clean ring [15] if every element of R is clean, a ring R is called exchange [15] if every element of R is exchange. It is well known that clean rings are always exchange, but the converse is not true unless R is Abelian.

Lemma 2.16 *If R is an NZI ring and idempotents can be lifted modulo $J(R)$, then $R/J(R)$ is Abelian.*

Proof Let $\bar{R} = R/J(R)$ and $\bar{a} \in E(\bar{R})$. Since idempotents can be lifted modulo $J(R)$, there exists $e \in E(R)$ such that $e + J(R) = \bar{e} = \bar{a}$. For any $\bar{x} \in \bar{R}$, write $h = xe - exe$, then $he = h$, $eh = 0$, $h^2 = 0$. so $h \in N(R) \bigcap l(h)$. Since R is an NZI ring, $hRh = 0$, $(RhR)^2 = 0$. So RhR is a nilpotent ideal of R, this gives

$RhR \subseteq P(R) \subseteq J(R)$, hence $h \in J(R)$, $xe - exe \in J(R)$, so $\bar{x}\bar{e} = \bar{e}\bar{x}\bar{e}$, that is, $\bar{x}\bar{a} = \bar{a}\bar{x}\bar{a}$. Hence $\bar{a}$ is left semicentral. This implies that $\bar{R}$ is Abelian. $\square$

Theorem 2.17 *Let R be an NZI ring. Then R is clean if and only if R is exchange.*

Proof One direction is trivial. Now we show the other direction. Let R be an exchange ring. $R/J(R)$ is exchange and idempotents can be lifted modulo $J(R)$. Hence $R/J(R)$ is Abelian by Lemma 2.16. Therefore $R/J(R)$ is clean, by Camillo and Yu [7, Proposition7], R is clean. $\square$

Corollary 2.18 *Let R be an NZI ring, $x \in R$ and $n \geq 1$. If x^n is clean, then x is clean.*

Proof Since x^n is clean, $x^n = u + f$, where $u \in U(R)$, $f \in E(R)$. Write $e = u^{-1}(1-f)u$. Then $e \in E(R)$, $u(x^n - e) = ux^n - ue = u(u+f) - (1-f)u = (x^n - 1)x^n \in Rx$, $e = x^n + u^{-1}(x^n - x^{2n}) \in Rx$, $1-e = 1-x^n-u^{-1}(x^n-x^{2n}) = (1-u^{-1}x^n)(1-x^n) = (1-u^{-1}x^n)(1+x+x^2+...+x^{n-1})(1-x) \in R(1-x)$, so x is exchange. Hence there exists $e \in E(R)$ such that $e \in Rx$, $1-e \in R(1-x)$. Let $e = yx$, $1-e = z(1-x)$, where $y = ey \in R$, $z = (1-e)z \in R$. By computing, we have $(y-z)(x-(1-e)) = 1 - y(1-e) - ze$. Since $(y(1-e))^2 = (ze)^2 = 0$ and R is an NZI ring, by Theorem 2.2, $y(1-e) + ze \in J(R)$. Hence $1 - y(1-e) - ze$ is invertible in R, by Corollary 2.7, $x - (1-e)$ is invertible, so x is clean. $\square$

Corollary 2.19 *Let R be an NZI ring and $x \in R$. If x^2 is clean, then x and $-x$ are clean.*

Theorem 2.20 *If R is an NZI left idempotent reflexive ring, then R is Abelian.*

Proof For any $e \in E(R)$, write $h = ea - eae$. Then $eh = h$, $he = 0$, $h^2 = 0$. R is an NZI ring and $h \in N(R) \bigcap l(e)$, hence $hRe = 0$. Since R is a left idempotent reflexive ring, $eRh = 0$, $h = eh \in eRh = 0$. This shows that e is central. $\square$

Lemma 2.21 *If R is a left PP Abelian ring, then R is a semicommutative ring.*

Proof Since R is a left PP ring, ${}_RRa$ is a projective module for any $a \in R$. Then there exists $e \in E(R)$ such that $a = ae$, $l(a) = l(e) = R(1-e)$. Since $1-e$ is central, $R(1-e)$ is an ideal of R. Hence $l(a)$ is an ideal of R, R is a semicommutative ring. $\square$

Corollary 2.22 *If R is a left PP Abelian ring, then R is an NZI ring.*

It is known that R is a left quasi-duo ring if and only if $Rx + R(xy-1) = R$ for any $x, y \in R$.

Theorem 2.23 *If R is an NZI ring, then $Rx + R(xa-1) = R$ for any $x \in R$ and $a \in N(R)$.*

Proof Since R is an NZI ring, R is 2-primal by Theorem 2.2, $a \in N(R) = P(R)$, $xa \in P(R) = N(R)$, then $1 - xa$ is invertible. Hence $R(1-xa) = R$. $\square$

Example 2.24 *Let F be a field, $R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$. By Yu [23, Proposition 2.1], R is a left quasi-duo ring. By Lam and Dugas [14, Theorem 3.2], for any $x \in R$, $a \in N(R)$, $Rx + R(xa - 1) = R$.*

Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N(R)$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, then $AB = 0$. But $A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. So R is not an NZI ring.

By Example 2.24, we know that the converse of Theorem 2.23 is not true, in general. In fact, Example 2.24 implies: (1) left quasi-duo ring is not NZI; (2) the upper triangular matrix ring of NZI ring is not NZI.

A ring R is called left GPP ring if for every $a \in R$, there exists a positive integer n such that $a^n \in P_r(R)$, where $P_r(R) = \{x \in R | Rx \text{ is projective as a left } R\text{-module}\}$.

Theorem 2.25 *If R is an NZI and left GPP ring, then $x = u + a$ for any $x \in R$, where $a \in N(R)$ and $u \in P_r(R)$.*

Proof Let R be a left GPP ring. For any $x \in R$, there exists a positive integer n such that $x^n \in P_r(R)$, then there exists $e \in E(R)$ such that $x^n e = x^n$ and $r(x^n) = r(e)$. For any $y \in r(xe)$, $xey = 0$, $x^n ey = 0$, $ey \in r(x^n) = r(e)$, $ey = eey = 0$, hence $y \in r(e)$. This shows that $r(xe) \subseteq r(e)$, then $r(xe) = r(e)$, $xe \in P_r(R)$. For any $z \in R$, $(1-e)ze \in N(R)$ and $(1-e)ze(1-e) = 0$, by hypothesis, $(1-e)zeR(1-e) = 0$. $(1-e)z^2(1-e) = (1-e)z(1-e)z(1-e) = [(1-e)z(1-e)]^2$, hence $(1-e)z^n(1-e) = [(1-e)z(1-e)]^n$. So $(x - xe)^{n+1} = x[(1-e)x(1-e)]^n(1-e) = x[(1-e)x^n(1-e)](1-e)$. Since $x^n e = x^n$, $x^n(1-e) = 0$, $(x - xe)^{n+1} = 0$. Hence $x - xe \in N(R)$. Write $u = xe \in P_r(R)$, $a = x - xe \in N(R)$. Then $x = u + a$, we are done.

Let R be a ring and $\sigma : R \longrightarrow R$ be an automorphism of R, $R[x;\sigma]$ is $R[x]$ as a set. Give $R[x;\sigma]$ a ring structure with multiplication: $xa = \sigma(a)x$, where $a \in R$. We shall call this extension the skew polynomial ring of R. Specially, $\sigma = Id_R : R \longrightarrow R$, $r \longmapsto r$. Then $R[x;\sigma] = R[x]$. □

Example 2.26 *Let D be a division ring and $R = D \bigoplus D$ with componentwise multiplication. Clearly, R is a reduced ring, so R is an NZI ring.*

Let $\sigma : R \longrightarrow R$ be defined by $\sigma(s,t) = (t,s)$. Then σ is an automorphism of R. Write $S = R[x;\sigma]$. Then S is not an NCI ring by Hwang, Joen and Park [10, Example 2.7]. So $R[x;\sigma]$ is not an NZI ring.

So the skew polynomial ring of NZI ring is not NZI, in general.

We may suspect that a ring R is NZI if and only if $R[x]$ is NZI. However, the following example erases the possibility.

Example 2.27 *Let Z_2 be the field of integers modulo 2 and $A = Z_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant terms in noncommuting indeterminates a_0, a_1, a_2, b_0, b_1, b_2, c over Z_2. Note that A is a ring without identity and consider an ideal of the ring $Z_2 + A$, say I, generated by*

a_0b_0, $a_0b_1+a_1b_0$, $a_0b_2+a_1b_1+a_2b_0$, $a_1b_2+a_2b_1$, a_2b_2, a_0rb_0, a_2rb_2,

b_0a_0, $b_0a_1+b_1a_0$, $b_0a_2+b_1a_1+b_2a_0$, $b_1a_2+b_2a_1$, b_2a_2, b_0ra_0, b_2ra_2,

$(a_0+a_1+a_2)r(b_0+b_1+b_2)$, $(b_0+b_1+b_2)r(a_0+a_1+a_2)$, and $r_1r_2r_3r_4$, where $r, r_1, r_2, r_3, r_4 \in A$. Then clearly $A^4 \in I$.

Next let $R = (Z_2 + A)/I$ and consider $R[x] \cong (Z_2 + A)[x]/I[x]$. Notice that $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) \in I[x]$ and $(a_0 + a_1x + a_2x^2)^4 \in I[x]$, but $(a_0 + a_1x + a_2x^2)c(b_0 + b_1x + b_2x^2) \notin I[x]$ because $a_0cb_1 + a_1cb_0 \notin I$, hence $R[x]$ is not an NZI ring. By Kim and Lee [13, Example 2.1], we know that R is an NZI ring.

Proposition 2.28 *(1) Let R be a ring and $\triangle$ a multiplicatively closed subset of R consisting of central regular elements. Then R is NZI if and only if $\triangle^{-1}R$ is NZI.*

(2) Let R be a ring, then $R[x]$ is NZI if and only if $R[x, x^{-1}]$ is NZI.

Proof (1) Let $u^{-1}x \in \triangle^{-1}R$. For any $0 \neq v^{-1}y \in N(\triangle^{-1}R) \bigcap l_{\triangle^{-1}R}(u^{-1}x)$, then $yx = 0$ and $y \in N(R)$, $y \in N(R) \bigcap l_R(x)$. Since R is an NZI ring, $(v^{-1}y) \triangle^{-1} Ru^{-1}x = \triangle^{-1}yRx = 0$. So $\triangle^{-1}R$ is NZI.

(2) One direction is trivial. Now we show the other direction. Let $\triangle = \{1, x, x^2, ..., \}$. Clearly, $\triangle$ is a multiplicatively closed subset of $R[x]$. Since $R[x, x^{-1}] = \triangle^{-1}R[x]$, it follows that $R[x, x^{-1}]$ is NZI by (1).

A ring R is called Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \ + a_mx^m$, $g(x) = b_0 + b_1x + \ + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. □

In general, Armendariz rings are not NZI.

Example 2.29 *Let F be a field, $F < X, Y >$ the free algebra on X, Y over F and I denotes the ideal $(X^2)^2$ of $F < X, Y >$, where (X^2) is the ideal of $F < X, Y >$ generated by (X^2). Let $R = F < X, Y > /I$ and $x = X + I$. By Antoine [2, Example 4.8], R is an Armendariz ring. According to Chen [5, Example 2.2], R is not a nil-semicommutative ring. So R is not an NZI ring.*

In general, NZI rings are not Armendariz.

Example 2.30 *Let F be a field, $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} | x, y, z \in F \right\}$, by Example 2.1, R is an NZI ring, but not an Abelian ring, so R is not an Armendariz ring.*

Proposition 2.31 *Let R be an Armendariz ring. Then the following statements are equivalent:*

(1) R is NZI;

(2) $R[x]$ is NZI;

(3) $R[x; x^{-1}]$ is NZI.

Proof (1) $\Longrightarrow$ (2): Let $f(x) = \sum_{i=0}^{n} a_ix^i \in N(R)$, $g(x) = \sum_{j=0}^{m} b_jx^j \in R[x]$ and $f(x)g(x) = 0$. Since R is an NZI ring, by Theorem 2.2, R is a 2-primal ring. Hence $R[x]$ is a 2-primal ring by Dirkenmeier, Heatherly and Lee [8]; then by Hiranv, Huynh and Park [11, Theorem1], $N(R[x]) = P(R[x]) \subseteq J(R[x]) = I[x] \subseteq N(R)[x]$, where I is a nil-ideal of R. Hence $f(x) \in N(R[x]) \subseteq N(R)[x]$, $a_i \in N(R)$. Since R is Armendariz , $f(x)g(x) = 0$ and $a_ib_j = 0$, $a_iRb_j = 0$. Let $h(x) = \sum_{t=0}^{l} c_tx^t \in R[x]$. Then $a_ic_tb_j = 0$, $f(x)h(x)g(x) = 0$. This shows that $R[x]$ is an NZI ring.

(2) $\Longrightarrow$ (3) is similar to (1) $\Longrightarrow$ (2).

(3) $\Longrightarrow$ (1) is trivial. □

3. Strong regularity of NZI rings

Lemma 3.1 *The following conditions are equivalent for a ring R.*

(1) R is a reduced ring;

(2) R is an n-regular ring and semicommutative ring;

(3) R is an n-regular ring and NZI ring.

Proof (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1): Suppose $a^2 = 0$, by hypothesis, and $a = aba$ for some $b \in R$. Since $a \in N(R) \bigcap l(a)$ and R is an NZI ring, $aR \subseteq l(a)$. So $aRa = 0$ and $a = aba \in aRa = 0$. Thus R is a reduced ring. □

Lemma 3.2 *Let R be an NZI ring. If $a \in R$ is a von Neumann regular element, then a is a strongly regular element.*

Proof Since $a \in R$ is a von Neumann regular element, there exists $b \in R$ such that $a = aba$. Write $e = ba$. Then $a = ae$, $e^2 = e$. Write $h = a - ea$. Then $he = h$, $eh = 0$, $h^2 = 0$, $h \in N(R) \bigcap l(h)$. By hypothesis, $hRh = 0$. Especially, $hbh \in hRh = 0$, $(ab - eab)(a - ea) = 0$, $a = aba = (ab + 1 - eab)ea = (ab + 1 - eab)ba^2 \in Ra^2$. Write $g = ab$. Then $a = ga$, $g = g^2$. Write $t = a - ag$. Then $t^2 = 0$, $tg = 0$, $t \in N(R) \bigcap l(g)$. By hypothesis, $tRg = 0$. Especially, $tbg \in tRg = 0$, so $(ab - agb)g = 0$, $abg = abab = agbab$, then $g = g^2 = abab = agbab = a^2b^2ab \in a^2R$. Hence $a = ga \in a^2R$. □

Theorem 3.3 *The following conditions are equivalent for a ring R.*

(1) R is a strongly regular ring;

(2) R is a semicommutative ring and von Neumann regular ring;

(3) R is an NZI ring and von Neumann regular ring.

Let R be a ring and $a \in R$. a is called Π -regular element [3] if there exists $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$, and a is said to be strongly Π-regular element [3] if $a^n = ba^{n+1}$. A ring R is called Π-regular if every element of R is Π-regular element, and R is called strongly Π-regular if every element of R is strongly Π-regular element. A ring R is called strongly clean [16] if $a = e + u$ where $e \in E(R)$, $u \in U(R)$ and $eu = ue$ for every $a \in R$.

Corollary 3.4 *If R is an NZI and Π-regular ring, then R is a strongly Π-regular ring; consequently, R is a strongly clean ring.*

Proof Since R is Π-regular, for any $x \in R$, there exists a positive integer n such that x^n is von Neumann regular. Since R is an NZI ring, x^n is a strongly regular element by Lemma 3.2, $x^n \in R(x^n)^2 \subseteq Rx^{n+1}$, thus R is strongly Π-regular. By Nicholson [16], strongly Π-regular rings are always strongly clean. □

Theorem 3.5 *If R is a left SF and NZI ring, then R is a reduced ring.*

Proof Let $a^2 = 0$ and $a \neq 0$. If $Ra + r(aR) \neq R$, there exists a maximal left ideal M of R such that $Ra + r(aR) \subseteq M$. Since R is a left SF ring and $a \in M$, there exists $b \in M$ such that $a = ab$, then $a(1-b) = 0$, $a \in N(R) \bigcap l(1-b)$. By the hypothesis, $aR \subseteq l(1-b)$, $aR(1-b) = 0$. Hence $1-b \in r(aR) \subseteq M$, $1 = (1-b)+b \in M$, which is a contraction. So $Ra+r(aR) = R$. Let $1 = x+y$, $x \in Ra$ and $y \in r(aR) \subseteq r(a)$. Write $x = ca$, $c \in R$. $a = a \cdot 1 = a(x+y) = ax = aca$. Write $e = ac$. We have $e^2 = e$, $a = ea$. Write $h = a - ae$. We have $he = 0$, $eh = h$ and $h^2 = 0$, $h \in N(R) \bigcap l(e)$. Since R is an NZI ring, $hRe = 0$, especially, $hce \in hRe = 0$. $hc = (a-ae)c = ac - aec$, $hce = (ac-aec)e = ace - aece = e - aece = 0$, then $e = aece = a(ac)ce = a^2c^2e = 0$. Hence $a = ea = 0$. □

Corollary 3.6 *If R is a left SF and NZI ring, then R is a strongly regular ring.*

Theorem 3.7 *If R is a left SF ring and $R/Z_l(R)$ is an NZI ring, then R is a strongly regular ring.*

Proof Let R be a left SF ring and $R/Z_l(R)$ a left SF ring. By hypothesis, $R/Z_l(R)$ is a strongly regular ring. If $Z_l(R) \neq 0$, there exists $0 \neq a \in Z_l(R)$. Write $T = Z_l(R) + r(a)$. If $T \neq R$, there exists a maximal right ideal M and $M \supseteq T$. Since $M/Z_l(R)$ is a right ideal of $R/Z_l(R)$ and $R/Z_l(R)$ is a strongly regular ring, $M/Z_l(R)$ is an ideal of $R/Z_l(R)$, then M is an ideal of R. Hence there exists a maximal left ideal L and $M \subseteq L$. There exists $b \in L$ such that $a = ab$. So $1-b \in r(a) \subseteq L$, $1 = (1-b)+b \in L$, which is a contraction. Let $1 = x+y$, where $x \in Z_l(R)$ and $y \in r(a)$. Then $a(1-x) = ay = 0$, but $x \in Z_l(R)$, $l(1-x) = 0$. It implies that $a = 0$. This shows that R is a strongly regular ring. □

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