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A fixed point theorem for a compact and connected set in Hilbert space

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Abstract

Let $(H, <>)$ be a real Hilbert space and let K be a compact and connected subset of H. We show that every continuous mapping $T : K \to K$ satisfying a mild condition has a fixed point.

Key Words: Fixed point, nonexpansive mapping, Hilbert space a fixed point theorem for a compact and connected set in Hilbert space

1. Introduction

Let K be a nonempty,close,convex and bounded subset of a real Hilbert space H. Let $T : K \to K$ be a continuous mapping.

If K is compact, then by The Schauder Fixed Point Theorem [**8**] (a generalization of [1]), T has a fixed point. If $K = B$ is the closed unit ball of H and the dimension of H is finite, then by The Brouwer Fixed Point Theorem [**1**], T has a fixed point. In the case where the dimension of H is infinite this is no longer the case [**4, p.198** and **207**].In this case, it is necessary to impose some extra conditions to assure the existence of a fixed point of T. The conditions imposed are usually compactness or monotonicity or nonexpansiveness of T [**6**]. For instance, Browder [**2**], Browder [**3**] – Göhde [**5**] and Kirk [**7**] discovered in 1965 independently that the nonexpansiveness of T is a guarantee the existence of a fixed point of T .

In the present paper, we impose an extra condition on T to obtain the same result. The extra condition imposed is this: For a certain number $r > 0$, the inclusion $T(\partial B_r) \subseteq B_r$ holds. Here, B_r denotes the closed ball, $B_r = \{x \in H : ||x|| \leq r\}$ and ∂B_r is the boundary of the ball B_r.

Moreover we impose some new conditions for The Schauder Fixed Point Theorem [**8**], and for Theorem 1 in [**2**]. In addition, we obtain some results related to the imposed conditions. To explain these conditions let us define a subset $A^T(x_0)$ of K, for $x_0 \in K$, as

$$A^T(x_0) = \{x \in K : ||x - x_0|| \leq | \, ||T(x)|| - ||x_0|| \, |\}.$$

The main theorem of this paper obtains The Schauder Fixed Point Theorem, which states that every continuous mapping on a compact and convex subset of a Banach space has a fixed point, for compact and

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connected subset K of H. To get this we replace the convexity of K in this theorem with the condition $K = A^T(x_0)$ for some $x_0 \neq 0$. To obtain Theorem 1 in [2] for continuous mapping T, we replace the nonexpansiveness of T with the following conditions:

(a) The existence of a point a in K satisfying $||a|| = ||T(a)||$;

and

$(b)((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset$,

where f is the mapping defined by, $f(x) = (x + T(x)) \ / \ 2$, $r = \sup\{||x|| : x \in K\}$ and $P : B_r \to K$ is the nearest point projection, that is, for $x \in B_r$,

$$||x - P(x)|| = \inf\{||x - u|| : u \in K\}.$$

We remark that the mapping f also applies K into itself.

Proposition 1 *Let K be a nonempty and convex subset of H. Let $T : K \to K$ be a continuous mapping. Suppose that there exists a point $a \in K$ such that*

$$||a|| = ||T(a)|| = ||f(a)||.$$

Then a is a fixed point of T.

Proof. Let $||a|| = ||T(a)|| = ||f(a)||$. The equality $||a|| = ||f(a)||$ is equivalent to

$$< a, a >= \frac{1}{4} < a + T(a), a + T(a) > .$$

Developing this product, and taking into account the condition $||T(a)|| = ||a||$, we obtain the equality

$$||a||^2 =< a, T(a) > .$$

This equality in turn implies that $||a - T(a)||^2 =< a - T(a), a - T(a) >= 0$ so that $T(a) = a$. □

The following corollary is now obvious. Since the inequality $||a|| \leq ||f(a)||$ in turn implies that $||a|| \leq ||T(a)||$.

Corollary 2 *Let K be a nonempty and convex subset of H and let $T : K \to K$ be a continuous mapping. If there exists a point $a \in K$ such that $||T(a)|| \leq ||a|| \leq ||f(a)||$, then a is a fixed point of T.*

Proposition 3 *Let K be a nonempty and convex subset of H and let $T : K \to K$ be a continuous mapping. If there exists a point $x_0 \in K$ such that both sets $A^T(x_0)$ and $A^f(x_0)$ are at most countable then x_0 is a fixed point of the mapping T.*

Proof. Remark that $A^f(x_0) = \{x \in K : \| x - x_0 \| \leq \ |\| f(x) \| - \| x_0 \||\}$. Let both $A^T(x_0)$ and $A^f(x_0)$ be at most countable. We shall show that $\| x_0 \| = \| T(x_0) \|$ and $\| x_0 \| = \| f(x_0) \|$. Let show that $\| x_0 \| = \| T(x_0) \|$. On the contrary, assume that $\| x_0 \| \neq \| T(x_0) \|$. Let $\varepsilon = (|\| x_0 \| - \| T(x_0) \||) \ / \ 2$. Since T is continuous, there exists $\delta > 0$ such that if $x \in K$ and $\| x - x_0 \| < \delta$ then $| \ \| T(x) \| - \| T(x_0) \| | < \varepsilon$. Let $\delta_0 = \min\{\varepsilon, \delta\}$. We claim that $B_{\delta_0}(x_0) \cap K \subset A^T(x_0)$. Indeed, let $x \in K$ and $\| x - x_0 \| < \delta_0$. Then ,

$$\| x - x_0 \| < 2\varepsilon - \varepsilon \leq | \| x_0 \| - \| T(x_0) \| | - | \| T(x) \| - \| T(x_0) \| | .$$

From here we conclude that

$$\| x - x_0 \| \leq | \| T(x) \| - \| x_0 \| | .$$

That means $x \in A^T(x_0)$. This implies that $B_{\delta_0}(x_0) \cap K \subset A^T(x_0)$. But this is impossible, since K is convex and $A^T(x_0)$ is at most countable. To prove that $\| x_0 \| = \| f(x_0) \|$, it is enough to replace T with f and repeat the proof. By Proposition 1, x_0 is a fixed point of T. □

Next is the main theorem of this paper.

Theorem 4 *Let K be a nonempty, compact and connected subset of H and let $T : K \to K$ be a continuous mapping. Assume that there exists a point $x_0 \in K$, $x_0 \neq 0$, such that $A^T(x_0) = K$. Then T has a fixed point.*

Proof. Let $a \in K$ be fixed. Now we define the sequences $\alpha_n = \| T^n(a) - x_0 \|$ and $\beta_n = | \| T^n(a) \| - \| x_0 \| |$. Since $A^T(x_0) = K$ for some $x_0 \in K$, $x_0 \neq 0$, the sequence $(T^n(a))_{n \in N}$ is in the set $A^T(x_0)$. Hence we have

$$\| a - x_0 \| \leq | \| T(a) \| - \| x_0 \| | \leq \| T(a) - x_0 \|$$

$$\| T(a) - x_0 \| \leq | \| T^2(a) \| - \| x_0 \| | \leq \| T^2(a) - x_0 \| ...,$$

and so on. In this way we get,

$$\| T^n(a) - x_0 \| \leq | \| T^{n+1}(a) \| - \| x_0 \| | \leq \| T^{n+1}(a) - x_0 \|,$$

for all $n = 0, 1, 2 \ldots$ (Here $T^n = T \circ T \circ T \circ ...n - times$ and $a = T^0(a)$)

Hence, for all n, we get

$$\alpha_n \leq \beta_{n+1} \leq \alpha_{n+1}.$$

The last inequalities show that $(\alpha_n)_{n \in N}$ and $(\beta_n)_{n \in N}$ are increasing sequences. Moreover, since K is bounded, they converge and approach the same limit. On the other hand, since K is compact, the sequence $(T^n(a))_{n \in N}$ has a convergent subsequence. We show this subsequence $(T^{n_k}(a))_{k \in N}$. Let $\lim_{n \to \infty} T^{n_k}(a) = b \in K$. Since T is continuous, we get

$$\lim_{n \to \infty} \| T^{n_k + p}(a) \| = \| T^p(b) \|,$$

for all $p = 0, 1, 2...$ Now $\alpha_{n_k + p} = \| T^{n_k + p}(a) - x_0 \|$ and $\beta_{n_k + p} = | \| T^{n_k + p}(a) \| - \| x_0 \| |$ are all subsequences of $(\alpha_n)_{n \in N}$ and $(\beta_n)_{n \in N}$. Since they have the same limit, we get

$$\| b - x_0 \| = | \| b \| - \| x_0 \| | = \| T(b) - x_0 \| = | \| T(b) \| - \| x_0 \| | = \cdots . \tag{1}$$

That is,

$$\| T^n(b) - x_0 \| = | \| T^n(b) \| - \| x_0 \| | = \| T^{n+1}(b) - x_0 \| .$$

From here we conclude that, for all $n \in N$,

$$< x_0, T^n(b) > = \| x_0 \| \| T^n(b) \| .$$

The last equality implies that there exist positive real numbers t_n such that all equalities below,

$$x_0 = t_0 b = t_1 T(b) = t_2 T^2(b) = \cdots = t_n T^n(b) = \cdots \tag{2}$$

hold. Now there exist two cases:

Case 1. Assume that the equality $\| T^n(b) \| = \| T^{n+1}(b) \|$ holds for some n. Then the point $T^n(b)$ is a fixed point of T. Indeed by (2),

$$\| x_0 \| = t_n \| T^n(b) \| = t_{n+1} \| T^{n+1}(b) \| .$$

Hence $t_n = t_{n+1}$. Again by (2), we get $T^n(b) = T^{n+1}(b) = T(T^n(b))$.

Case 2. Suppose that $\| T^n(b) \| \neq \| T^{n+1}(b) \|$ for all n. In this case x_0 is a fixed point of T. Indeed, by taking square of the equalities in (1), we obtain,

$$\| x_0 \| = (\| b \| + \| T(b) \|)/2 = (\| T(b) \| + \| T^2(b) \|)/2 = \cdots . \tag{3}$$

From here, we get for all n

$$\| b \| = \| T^2(b) \| = \| T^{2n}(b) \| \text{ and } \| T(b) \| = \| T^3(b) \| = \| T^{2n+1}(b) \| . \tag{4}$$

Together with (2), the equalities in (4) imply that both

$$\| x_0 \| = t_0 \| b \| = t_2 \| T^2(b) \| = t_{2n} \| T^{2n}(b) \| \tag{5}$$

and

$$\| x_0 \| = t_1 \| T(b) \| = t_3 \| T^3(b) \| = t_{2n+1} \| T^{2n+1}(b) \| . \tag{6}$$

Hence relations (4), (5) and (6) give us for all n

$$t_0 = t_2 = t_{2n} \text{ and } t_1 = t_3 = t_{2n+1}.$$

By (2), $b = T^2(b) = T^{2n}(b)$. Let $g : K \to R$ be a function defined by

$$g(x) = |\| b \| - \| T^2(x) \|| - |\| T(b) \| - \| T^2(x)) \|| .$$

Clearly g is continuous and $g(b) < 0$ and $0 < g(T(b))$ hold. Since K is connected, by The Intermediate Value Theorem, there exists a point c in K such that

$$g(c) = |\| b \| - \| T^2(c) \|| - |\| T(b) \| - \| T^2(c)) \|| = 0.$$

That is, $\| T^2(c) \| = (\| b \| + \| T(b) \|) / 2$. By (3), $\| x_0 \| = \| T^2(c) \|$. On the other hand since $T(c) \in A^T(x_0) = K$, we get $\| x_0 - T(c) \| \leq |\| x_0 \| - \| T^2(c) \|| = 0$. Hence $x = T(c)$. Similarly since $c \in A^T(x_0)$, we get $\| x_0 - c \| \leq |\| x_0 \| - \| T(c) \|| = 0$. Hence $x_0 = c$. From here we conclude that x_0 is a fixed point of T. □

We use the symbol $\overline{A}$ to denote the closure of a set A.

Corollary 5 *Let K and T be as in Theorem 4. If T has no fixed point then*

$$A^T(x) \cap \overline{K/A^T(x)} \neq \emptyset$$

for all $x \in K$, $x \neq 0$. That is, for all $x \in K$, there exists a point y in K such that

$$||x - y|| = |\ ||x|| - ||T(y)||\ | \, .$$

Proof. On the contrary, suppose that $A^T(x) \cap \overline{K/A^T(x)} = \emptyset$ for some $x \in K$. Then the set $A^T(x)$ is both open and closed in K. Since K is connected and $x \in A^T(x) \neq \emptyset$ we must have $A^T(x) = K$. By Theorem 4, T has a fixed point, which is not the case. □

Lemma 6 *Let K be a nonempty, closed, convex and bounded subset of H and $0 \in K$. Let $\dot{T} : K \to K$ be a continuous mapping. Assume that there exists at least one point $a \in K$ such that the equality $||T(a)|| = ||a||$ holds. Then,*

(*a*) *The set $F = \{x \in K : ||x|| = ||f(x)||\,\} = A^f(0) \cap \overline{K/A^f(0)}$ is nonempty.*

(*b*) *The quantity $\delta(f) = \inf\{||x|| : x \in F\}$ is zero iff $T(0) = 0$.*

Proof. (*a*) If $f(0) = 0$, then there is nothing to prove. If

$$a \in A^f(0) = F \cup \{x \in K : ||x|| < ||f(x)||\}$$

then, $||a|| \leq ||f(a)||$. By Corollary 2, a is a fixed point of both T and f. Hence $a \in F \neq \emptyset$. Hence we suppose that $0 < ||f(0)||$ and $a \notin A^f(0)$. Let $g : K \to R$ be a continuous function defined by, $g(x) = ||x|| - ||f(x)||$. Since $g(0) < 0$ and $g(a) > 0$ and since K is connected, by the intermediate value theorem, there is a point $b \in K$ such that $g(b) = 0$. Hence $b \in F \neq \emptyset$.

b) Since $F \neq \emptyset$, the quantity $\delta(f) = \inf\{||x|| : x \in F\}$ exists. This quantity is zero iff $T(0) = 0$. Indeed, if $T(0) = 0$ then $0 \in F$ so that $\delta(f) = 0$. Conversely, if $\delta(f) = 0$ then there is a sequence $(x_n)_{n \in N}$ in F such that $||x_n|| \to 0$, as $n \to \infty$. Since f is continuous on K and since $||x_n|| = ||f(x_n)||$, we see that $f(0) = 0$. This implies that $T(0) = 0$, too □

Let K be a nonempty, closed , convex and bounded subset of H. In this case the quantity $\sup\{||x|| : x \in K\} = r$ exists and $K \subset B_r$. Let $P : B_r \to K$ be the nearest point projection, that is, for $x \in B_r$, $||x - P(x)|| = \inf\{||x - u|| : u \in K\}$. Now we give the next corollary below.

Corollary 7 *Let K , T and a be as in lemma 6. Suppose that*

$$((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset.$$

Then T has a fixed point.

Proof. Let $((f \circ P)(\partial B_r)) \cap \partial B_r \neq \emptyset$. In this case, there is a $x \in \partial B_r$ such that $||((f \circ P)(x)|| = ||x|| = r$. The equality $||((f \circ P)(x)|| = r$ implies that both

$$||P(x)|| \leq ||((f \circ P)(x)|| \text{ and } ||T(P(x))|| \leq ||((f \circ P)(x)||.$$

As $(f \circ P)(x) = (P(x) + T(P(x)) \ / \ 2$, we have $||P(x)|| = ||T(P(x))|| = ||((f \circ P)(x)||$. By Proposition 1, $P(x)$ is a fixed point of $\dot{T}$. □

In the previous corollary the nearest point projection can be replaced with the radial retraction, which uniquely defined for a ball in any strictly convex normed space.

Theorem 8 *Let $K = B$ be the closed unit ball of H and let $T : K \to K$ be a continuous mapping. Suppose that for each $r \geq \delta(f)$, the inclusion $T(\partial B_r) \subseteq B_r$ holds. Then T has a fixed point in B.*

Proof. If we take $a \in K$ with $||a|| = 1$ and repeat the proof of Lemma 6(a), we see that the set $F \neq \emptyset$. By Lemma 6(b), If $\delta(f) = 0$ then zero is a fixed point of T so that there is nothing to prove in this case. Hence we suppose that $\delta(f) > 0$.

Case 1. $f(\partial B) \cap \partial B \neq \varnothing$. In this case, there is a point $x \in \partial B$ such that $||f(x)|| = 1 = ||x||$. As $f(x) = (x + T(x)) \ / \ 2$, the equality $||f(x)|| = ||x||$ implies that $||T(x)|| \geq ||x||$. Since $||x|| = 1$, this is possible only if $||T(x)|| = ||x||$. By Proposition 1, $T(x) = x$.

Case 2. $f(\partial B) \cap \partial B = \varnothing$. In this case, for all $x \in \partial B$, $||f(x)|| < 1$ so that $||f(x)|| < ||x||$. We fix a $y \in \partial B$. Since $||f(y)|| < ||y||$ and $||f(0)|| > 0$, as in the proof of Lemma 6(a), the function $g(x) = ||x|| - ||f(x)||$ vanishes at some point $a \in B$. That is, $||a|| = ||f(a)||$. This point a belongs to F so that $||a|| \geq \delta(f)$. As $a \in \partial B_r$, where $r = ||a||$ and since $T(\partial B_r) \subseteq B_r$, we have $||T(a)|| \leq ||a||$. On the other hand, since $||a|| = ||f(a)||$, we also have $||T(a)|| \geq ||a||$. Hence $||T(a)|| = ||a||$. By Proposition 1, $T(a) = a$. Hence T has a fixed point in B. □

The next corollaries are now obvious.

Corollary 9 *Suppose that for each $x \in B$ with $||x|| \geq \delta(f)$, we have $||T(x)|| \leq ||x||$. Then T has a fixed point in B.* □

Corollary 10 *Let K be a closed, convex and bounded subset of H with $0 \in K$, and let $T : K \to K$ be a continuous mapping. Suppose that the inclusion $T(\partial C) \subseteq C$ holds for all convex and closed subsets of K. Then T has a fixed point in K.*

Theorem 11 *Let H be infinite dimensional and let $K = B$ be the closed unit ball of H. Let $T : K \to K$ be a continuous mapping. Set*

$$M_i = \{x \in B : x = x_i e_i\},$$

where $e_i = (0, 0, ..., y_i, 0, ..)$, $y_i = 1$. Then $F \cap M_i \neq \varnothing$, for all i.

Proof. Let $H = \ell_2$. We give the proofs without loss of generality for $M = M_1$.

(a) If $f(0) = 0$ then $0 \in F \cap M$. If $a = (x_1, 0, 0, ...) \in F$ where $|\ x_1\ | = 1$ then $a \in F \cap M$. So we suppose that $\{0, e_1, -e_1\} \cap F = \varnothing$. Now define the function $g : M \to R$, $g(x) = \parallel x \parallel - \parallel f(x) \parallel$. Then since $g(0) < 0$ and $g(e_1) > 0$ and since M is convex, by the intermediate value theorem, there is a point $b \in M$ such that

$g(b) = 0$. That is $b \in F \cap M \neq \emptyset$. □

For the next corollary we put $\sup\{\| x \| : x \in F \cap M_i\} = \| x_i \| = r_i$ for some $x_i \in F \cap M_i$. If $r_i = 1$ for some i, then $\| x_i \| = \| f(x_i) \| = 1 \geq \| T(x_i) \|$. By Proposition 1, x_i is a fixed point of T.

Corollary 12 *Let B be the closed unit ball of H and $T : B \to B$ be a continuous mapping. If the inclusion $T(\partial B_{r_i}) \subseteq B_{r_i}$ for some i, then T has a fixed point .*

Proof. We remark that $r_i = \| x_i \|$ for some $x_i \in F \cap M_i$, for all i. If $r_i = 1$ for some i, then

$$\| x_i \| = \| f(x_i) \| = 1 \geq \| T(x_i) \| .$$

By Proposition 1, x_i is a fixed point of T. Let $0 < r_i < 1$ for all i. Then since $T(\partial B_{r_i}) \subseteq B_{r_i}$ and $x_i \in \partial B_{r_i}$, $\| T(x_i) \| \leq r_i = \| x_i \| = \| f(x_i) \|$. By Proposition 1, x_i is a fixed point of T. □

Example 13 *Let $H = \ell_2$ and B its closed unit ball.*

1- *For $x = (x_1, x_2, ..) \in B$, let $T(x) = (1 - \| x \|, \| x \|, x_3, x_4, ...)$. Then clearly T takes B into itself. By a simple calculation, we have $r_1 = 1 \ \sqrt{3}$, $r_2 = 1$ and $r_i = \sqrt{2} - 1$, for all $i = 3, 4, ...$ It is clear that $T(\partial B_{r_2}) \subseteq B_{r_2}$. By Corollary 12, T has a fixed point. Moreover, T is not a nonexpansive mapping.*

2- *For $x = (x_1, x_2, ..) \in B$, let $T(x) = (1 - \| x \|, x_2, x_3, x_4, ...)$. Then we have, $r_1 = 1 \ / \ 2$ and $r_i = 1$ for all $i \geq 2$. It is clear that $T(\partial B_{r_1}) \subseteq B_{r_1}$. By Corollary 12, T has a fixed point.*

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