A note on the lévy constant for continued fractions

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Abstract

In this note, we study the lévy constant of continued fraction expansions. We show that for all $x \in [0, 1)$, the upper lévy constant of $x$ is finite except a set with Hausdorff dimension one-half.

Key Words: Continued fractions, lévy constant, Hausdorff dimension.

1. Introduction

It is well known that every irrational number $x \in [0, 1)$ has a unique standard continued fraction expansion of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}} =: [a_1, a_2, a_3, \ldots],$$

where each partial quotient $a_n(x) \in \mathbb{N}$ is uniquely defined by the number $x$.

For any $n \geq 1$ and $a_1, \ldots, a_n \in \mathbb{N}$, define a CF-interval of rank $n$ as

$$I(a_1, a_2, \ldots, a_n) = \{x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n\}.$$

Therefore, (see [5], section 12), $I(a_1, \ldots, a_n)$ is the interval with endpoints $\frac{p_n}{q_n}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$, where $p_n$ and $q_n$ are defined by following recurrence relations

$$p_{-1} = 1; \quad p_0 = 0; \quad p_n = a_n p_{n-1} + p_{n-2}, \quad n \geq 1.$$

$$q_{-1} = 0; \quad q_0 = 1; \quad q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 1. \quad (1)$$

Thus, the length of $I(a_1, a_2, \ldots, a_n)$ is

$$|I(a_1, a_2, \ldots, a_n)| = \left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{1}{q_n(q_n + q_{n-1})}. \quad (2)$$

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For an irrational number \( x \in [0, 1) \), we call
\[
\beta^*(x) = \limsup_{n \to \infty} \frac{\log q_n(x)}{n} \quad \text{and} \quad \beta_*(x) = \liminf_{n \to \infty} \frac{\log q_n(x)}{n},
\]
the upper lévy constant and lower lévy constant of \( x \), respectively. If \( \beta^*(x) = \beta_*(x) \), we say the lévy constant of \( x \) exists and denote the common value by \( \beta(x) \). A famous result of P. Lévy [6] says that for almost all \( x \), the lévy constant exists and
\[
\beta(x) = \frac{\pi^2}{12 \log 2} \approx 1.18657.
\]

\( \beta^*(x) \) and \( \beta_*(x) \) describe the exponential growth rates of \( q_n(x) \) in \( n \). Faiver [2] showed that every quadratic number has a lévy constant. It is easy to see that for any irrational number \( x \in [0, 1) \), one has
\[
\beta^*(x) \geq \log \frac{\sqrt{5} + 1}{2},
\]
then Faiver [3] also established that for all \( \lambda \geq \log \frac{\sqrt{5} + 1}{2} \), there exists an \( x \in I \) such that \( \beta(x) = \lambda \) by employing an ergodic theorem. Later, Baxa [1] showed the following more general result by elementary means.

**Theorem 1.1** For any \( \log \frac{\sqrt{5} + 1}{2} \leq \lambda_* \leq \lambda^* < \infty \), there exist uncountably many \( x \in [0, 1) \) such that \( \beta_*(x) = \lambda_* \) and \( \beta^*(x) = \lambda^* \).

In 2006, Wu [7] improved Baxa’s result by showing the following theorem.

**Theorem 1.2** For any \( \log \frac{\sqrt{5} + 1}{2} \leq \lambda_* \leq \lambda^* < \infty \), let
\[
E(\lambda_*, \lambda^*) = \{ x \in [0, 1) : \beta_*(x) = \lambda_*, \ \beta^*(x) = \lambda^* \}.
\]

Then
\[
\dim H E(\lambda_*, \lambda^*) \geq \frac{\lambda_* - \log \frac{\sqrt{5} + 1}{2}}{\lambda^*}.
\]

In this note, we consider the set of \( x \in [0, 1) \) whose upper lévy constant is infinite and obtain

**Theorem 1.3** Let
\[
E^\infty = \left\{ x \in [0, 1) : \limsup_{n \to \infty} \frac{\log q_n(x)}{n} = \infty \right\}.
\]

Then
\[
\dim H E^\infty = \frac{1}{2}.
\]

Here and in what follows, \( \dim H \) denotes the Hausdorff dimension of a subset of \([0, 1)\), and \( |\cdot| \) denotes the diameter. We sketch, very briefly, the definition and some basic properties of Hasdorff dimension. If \( E \subset R \) and \( \delta > 0 \), define for each \( s \geq 0 \),
\[
H^s(E) = \liminf_{\delta \to 0} \left\{ \sum_{n=1}^{\infty} |I_n|^s : E \subset \bigcup_{n=1}^{\infty} I_n, |I_n| \leq \delta, n = 1, 2, \ldots \right\},
\]
\[
\dim H E = \inf \{ s \geq 0 : H^s(E) = 0 \} = \sup \{ s \geq 0 : H^s(E) = \infty \}.
\]
The following two facts are basic in calculating Hasdorff dimension of various sets.

**Lemma 1.4** Let $E \subset R$ and let $s \geq 0$ be given. Suppose for each $\delta > 0$ there is a sequence of intervals $\{I_n\}$ such as $E \subset \bigcup I_n$, $|I_n| \leq \delta$ for all $n$, and $\sum_{n=1}^{\infty} |I_n|^s \leq 1$. Then $\dim_H E \leq s$.

**Lemma 1.5** Let $E \subset R$ be a Borel set and $\mu$ be a measure with $\mu(E) > 0$. If for any $x \in E$

$$\lim inf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s.$$  

where $B(x,r)$ denotes the open ball with center at $x$ and radius $r$. Then $\dim_H E \geq s$.

Lemma 1 is obvious; for Lemma 2, see ([4], Proposition 2.3).

2. **Proof of Theorem 1.3**

In this section, we show Theorem 1.3 in detail and divide the proof into two parts: upper bound and lower bound.

I. **Upper bound.** $\dim_H E \leq \frac{1}{2}$.

**Proof.** By (1), we have $$a_nq_{n-1} \leq q_n \leq 2a_nq_{n-1}.$$ 

Successive application of this inequality gives $$a_1a_2\cdots a_n \leq q_n \leq 2^na_1a_2\cdots a_n. \quad (3)$$

Thus we get the following alternative description of $E^\infty$:

$$E^\infty = \left\{ x \in [0,1) : \limsup_{n \to \infty} \frac{\log a_1(x) + \log a_2(x) + \cdots + \log a_n(x)}{n} = \infty \right\}.$$  

Let $$E^{(m)} = \left\{ x \in [0,1) : \limsup_{n \to \infty} \frac{\log a_1(x) + \log a_2(x) + \cdots + \log a_n(x)}{n} > m \right\}.$$  

Then $E^\infty$ can be written

$$E^\infty = \bigcap_{m=1}^{\infty} E^{(m)} = \lim_{n \to \infty} E^{(m)},$$

and for every $x = [a_1, a_2, a_3, \cdots] \in E^{(m)}$, there exist infinitely many positive integers $n_i$ such that

$$\frac{\log a_1(x) + \cdots + \log a_{n_i}(x)}{n_i} > m, \quad i = 1, 2, 3, \cdots$$

So that, for any $\delta > 0$, the family of the CF-intervals

$$A(m, \delta) = \left\{ I(a_1, a_2, \cdots, a_{n_i}) : \frac{\log a_1(x) + \cdots + \log a_{n_i}(x)}{n_i} > m, |I(a_1, a_2, \cdots, a_{n_i})| \leq \delta, n_i \in \mathbb{N} \right\}$$

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is a $\delta$–cover of $E^{(m)}$.

Note that, for any two $CF$-intervals, say $I$ and $I'$, the following relation holds:

$$I \cap I' \neq \phi \implies I \subseteq I' \text{ or } I' \subseteq I.$$  

In fact, if $I = I(a_1, \cdots, a_n)$ and $I' = I(a'_1, \cdots, a'_{n+k})$ $(k \geq 0)$ have a common point $x = [x_1, x_2, \cdots]$, then $a_1 = x_1, a_2 = x_2, \cdots, a_n = x_n = a'_n$. It follows that $I' \subseteq I$.

We remove from $A(m, \delta)$ all those the $CF$-intervals which are contained in other $CF$-interval in $A(m, \delta)$, and denote the complement by $A(m, \delta)$. Then, $A(m, \delta)$ is a non-overlapping $\delta$–cover of $E^{(m)}$.

Now we define a family of measures \{\mu_t : t > 1\} as

$$\mu_t(I(a_1, a_2, \cdots, a_n)) = e^{-np(t) - t \log \sum_{i=1}^{n} \log a_i},$$  

where \(p(t) = \log \zeta(t) = \sum_{n \geq 1} \frac{\lambda(t)}{n^t}, \text{ (}t > 1\).$$

By (2) and (3), we have for any $\epsilon > 0$,

$$\log |I(a_1, a_2, \cdots, a_n)| \leq -(\epsilon + t) \log(a_1 a_2 \cdots a_n) = -t \sum_{i=1}^{n} \log a_i - t \sum_{i=1}^{n} \log a_i.$$  

By the definition of $A(m, \delta)$, for every $I = I(a_1, a_2, \cdots, a_n) \in A(m, \delta)$ with $m \geq \frac{p(t)}{t}$, we have

$$-t \sum_{i=1}^{n} \log a_i \leq -\epsilon \cdot mn \leq -np(t).$$  

Combining (4), (5) and (6), we get for any $\epsilon > 0$, and $I = I(a_1, a_2, \cdots, a_n) \in A(m, \delta)$ with $m \geq \frac{p(t)}{t}$,

$$\log |I(a_1, a_2, \cdots, a_n)| \leq e^{-np(t) - t \log \sum_{i=1}^{n} \log a_i} = \mu_t(I(a_1, a_2, \cdots, a_n)).$$

Since $A(m, \delta)$ is a non-overlapping $\delta$–cover of $E^{(m)}$, we sum the above inequality to have

$$\sum_{I \in A(m, \delta)} |I| \leq \sum_{I \in A(m, \delta)} \mu_t(I) = \mu_t \left( \bigcup_{I \in A(m, \delta)} I \right) \leq 1.$$  

By Lemma 1.4, we get for any $t > 1, \epsilon > 0$ and $m \geq \frac{p(t)}{t}$,

$$\dim H E^{(m)} \leq \frac{t + \epsilon}{2}.$$  

Letting $\epsilon \to 0$ and since $t > 1$ is arbitrary, we obtain

$$\dim H E_{\infty} \leq \frac{1}{2}.$$  

\[\square\]
II. Lower bound. \( \dim_H E^\infty \geq \frac{1}{2} \).

Proof. Put

\[
F = \left\{ x \in [0, 1) : 2^n \leq a_n(x) < 2^{n+1}, \text{ for all } n \geq 1 \right\}.
\]  

(7)

It is easy to check that \( F \subset E^\infty \). So it is enough to prove \( \dim_H F \geq \frac{1}{2} \). To give a precise view on the structure of \( F \), we shall make use of a kind of symbolic space defined as follows.

\[
\mathcal{D}_n = \left\{ (a_1, \cdots, a_n) \in \mathbb{N}^n : 2^k \leq a_k < 2^{k+1}, \text{ for all } 1 \leq k \leq n \right\}.
\]

For any \((a_1, \cdots, a_n) \in \mathcal{D}_n\), call

\[
J(a_1, \cdots, a_n) = \overline{\{ x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n \}}
\]

an admissible CF-intervals of rank \( n \), where “\( \overline{\cdot} \)” denotes the closure of a set in \([0, 1)\). It is observable that

\[
F = \bigcap_{n=1}^{\infty} \bigcup_{(a_1, \cdots, a_n) \in \mathcal{D}_n} J(a_1, \cdots, a_n).
\]

Let \( \mu \) be a probability measure supported on \( F \) such that for every admissible intervals \( J(a_1, \cdots, a_n) \),

\[
\mu(J(a_1, \cdots, a_n)) = \frac{1}{\sharp \mathcal{D}_n} = \frac{1}{2^{1+2+\cdots+n}},
\]

(8)

where \( \sharp \) denotes the cardinality.

Now we estimate the \( \mu \)-measure of arbitrary ball \( B(x, r) \) with center \( x \in F \) and radius \( r \) small enough. Choose \( n \geq 1 \) such that

\[
|J(a_1, \cdots, a_{n+1})| \leq r < |J(a_1, \cdots, a_n)|.
\]

Calculations show

\[
|J(a_1, \cdots, a_n)| \leq \sum_{1 \leq i \leq 4} |J(a_1, \cdots, a_{n-1}, a_n + i)|.
\]

So that, from \( a_n \geq 2 \) and \( r < |J(a_1, \cdots, a_n)| \) we have

\[
B(x, r) \subset J(a_1, \cdots, a_{n-1}).
\]

(9)

On the other hand, from (2), (3) and (7), We have

\[
r \geq |J(a_1, \cdots, a_{n+1})| > \frac{1}{2q_{n+1}} \geq \frac{1}{2^{2n+3}a_1^2a_2^2 \cdots a_{n+1}^2} > \frac{1}{2^{2n+3+(n+1)(n+4)}}.
\]

(10)

Combining (8), (9) and (10), we get

\[
\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq \liminf_{r \to 0} \frac{(n-1)n}{2n + 3 + (n+1)(n+4)} = \frac{1}{2}.
\]

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By using lemma 1.5, we obtain $\dim_H F \geq \frac{1}{2}$, which shows $\dim_H E^\infty \geq \frac{1}{2}$ since $F \subset E^\infty$. This completes the proof.

References


