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Geodesics of the Cheeger-Gromoll Metric

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Abstract

The main purpose of the paper is to investigate geodesics on the tangent bundle with respect to the Cheeger-Gromoll metric.

Key Words: Geodesics, Cheeger-Gromoll metric, Horizontal and vertical lift.

1. Introduction

In [1] Cheeger and Gromoll study complete manifolds of nonnegative curvature and suggest a construction of Riemannian metrics useful in that contex. Inspired by a paper of Cheeger and Gromoll, in [4] Musso and Tricerri defined a new Riemannian metric ^{CG}g on tangent bundle of Riemannian manifold which they called the Cheeger-Gromoll metric. The Levi-Civita connection of ^{CG}g and its Riemannian curvature tensor are calculated by Sekizawa in [5] (for more details see [2],[3]). The main purpose of this paper is to investigate geodesics of the Cheeger-Gromoll metrics on tangent bundle.

Let M_n be a Riemannian manifold with metric g. We denote by $\Im^p_q(M_n)$ the set of all tensor fields of type (p,q) on M_n. Manifolds, tensor field and connections are always assumed to be differentiable and of class C^∞.

Let $T(M_n)$ be a tangent bundle of M_n, and π the projection $\pi : T(M_n) \to M_n$. Let the manifold M_n be covered by system of coordinate neighbourhoods (U, x^i), where $(x^i), i = 1, ..., n$ is a local coordinate system defined in the neighbourhood U. Let (y^i) be the Cartesian coordinates in each tangent spaces $T_p(M_n)$ at $P \in M_n$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}\right\}$, P being an arbitrary point in U whose coordinates are x^i. Then we can introduce local coordinates (x^i, y^i) in open set $\pi^{-1}(U) \subset T(M_n)$. We call them coordinates induced in $\pi^{-1}(U)$ from (U, x^i). The projection π is represented by $(x^i, y^i) \to (x^i)$. We use the notations $x^I = (x^i, x^{\bar\imath})$ and $x^{\bar\imath} = y^i$. The indices $I, J, ...$ run from 1 to 2n, the indices $\bar\imath, \bar\jmath, ...$ run from n+1 to 2n .

Let $X \in \Im^1_0(M_n)$, which locally are represented by $X = X^i\partial_i$, $\left(\partial_i = \dfrac{\partial}{\partial x^i}\right)$. Then the vertical and horizontal lifts ^VX and ^HX of X (see [6]) are given, respectively by

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$$^{V}X = X^{i}\partial_{\bar{i}}, \left(\partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}\right) \tag{1}$$

and

$$^{H}X = X^{i}\partial_{i} - \Gamma^{i}_{jk}x^{\bar{j}}X^{k}\partial_{\bar{i}} \tag{2}$$

where Γ^{i}_{jk} are the coefficents of the Levi-Civita connection ∇.

Suppose that we are given on M_n a tensor field $S \in \Im^{p}_{q}(M_n), q > 1$. We then define a tensor field $\gamma S \in \Im^{p}_{q-1}(T(M_n))$ in $\pi^{-1}(U)$ by [6, p. 12]

$$\gamma S = (x^{\bar{e}}S^{j_1...j_p}_{ei_2...i_q})\partial_{\bar{j}_1} \otimes ... \otimes \partial_{\bar{j}_p} \otimes dx^{i_2} \otimes .. \otimes dx^{i_q}$$

with respect to the induced coordinates $(x^i, x^{\bar{i}})$. The tensor field γS defined in each $\pi^{-1}(U)$ determine global tensor field on $T(M_n)$. We easily see that for any $\varphi \in \Im^{1}_{1}(M_n)$, $\gamma\varphi$ has components $(\gamma\varphi) = \begin{pmatrix} 0 \\ x^{\bar{i}}\varphi^{j}_{i} \end{pmatrix}$ with respect to the induced coordinates $(x^i, x^{\bar{i}})$ and $(\gamma\varphi)(^{V}f) = 0$, $f \in \Im^{0}_{0}(M_n)$ i.e. $\gamma\varphi$ is a vertical vector field on $T(M_n)$.

Let there be given in $U \subset M_n$ a vector field $X = X^{i}\partial_{i}$ and a covector field $g_X = g_{ij}X^{i}dx^{j}$. Then we define a function $\gamma g_X \in \Im^{0}_{0}(M_n)$ in $\pi^{-1}(U) \subset T(M_n)$ by $\gamma g_X = x^{\bar{j}}g_{ij}X^{i}$ with respect to the induced coordinates $(x^i, x^{\bar{i}})$. Now, let r be the norm a vector $y = (y^i) = (x^{\bar{i}})$, i.e. $r^2 = g_{ij}x^{i}x^{\bar{j}}$. The Cheeger-Gromoll metric ^{CG}g on tangent bundle $T(M_n)$ is given by

$$^{CG}g(^{H}X,^{H}Y) =^{V} (g(X,Y)), \tag{3}$$

$$^{CG}g(^{H}X,^{V}Y) = 0, \tag{4}$$

$$^{CG}g(^{V}X,^{V}Y) = \frac{1}{1+r^2}\left[^{V}(g(X,Y)) + (\gamma g_X) + (\gamma g_Y)\right] \tag{5}$$

for all vector field $X, Y \in \Im^{1}_{0}(M_n)$, where $^{V}(g(X,Y)) = (g(X,Y)) \circ \pi$.

It is obvious that the Cheeger-Gromoll metric ^{CG}g is contained in the class of natural metrics (Recall that by a natural metric on tangent bundles we shall mean a metric which satisfies conditions (3) and (4)).

2. Expressions in Adapted Frames

In each local chart $U \subset M_n$, we put $X_{(j)} = \frac{\partial}{\partial x^{j}}, j = 1, ..., n.$ Then from (1) and (2), we see that these vector fields have, respectively, local expressions

$$^{H}X_{(j)}=\delta_j^h\partial_h+(-\Gamma_{sj}^h x^s)\partial_{\bar{h}} \tag{6}$$

$$^{V}X_{(j)}=\delta_j^h\partial_{\bar{h}} \tag{7}$$

with respect to the natural frame $\{\partial_h,\partial_{\bar{h}}\}$, where δ_j^h-Kronecker delta. These $2n$ vector fields are linear independent and generate, respectively, the horizontal distribution of ∇ and the vertical distribution of $T(M_n)$ We have call the set $\left\{{}^{H}X_{(j)},{}^{V}X_{(j)}\right\}$ the frame adapted to the affine connection ∇ in $\pi^{-1}(U)\subset T(M_n)$. On putting

$$\begin{aligned} e_{(j)} &= {}^{H}X_{(j)}, \\ e_{(\bar{j})} &= {}^{V}X_{(j)}, \end{aligned}$$

we write the adapted frame as $\{e_\beta\}=\left\{e_{(j)},e_{(\bar{j})}\right\}$. The indices $\alpha,\beta,...$ run over the range $\{1,...,2n\}$ and indicate the indices with respect to the adapted frame.
Using (1), (2), (6) and (7) we have

$$\begin{aligned} {}^{H}X &= \begin{pmatrix} X^j\delta_j^h \\ -X^j\Gamma_{sj}^h x^{\bar{s}} \end{pmatrix} = X^j\begin{pmatrix} \delta_j^h \\ -\Gamma_{sj}^h x^{\bar{s}} \end{pmatrix} = X^j e_{(j)} \\ {}^{V}X &= \begin{pmatrix} 0 \\ X^h \end{pmatrix} = \begin{pmatrix} 0 \\ X^j\delta_j^h \end{pmatrix} = X^j\begin{pmatrix} 0 \\ \delta_j^h \end{pmatrix} = X^j e_{(\bar{j})}, \end{aligned}$$

i.e. the lifts ${}^{H}X$ and ${}^{V}X$ have respectively components

$$\begin{aligned} {}^{H}X &= ({}^{H}X^\beta) = \begin{pmatrix} {}^{H}X^j \\ {}^{H}X^{\bar{j}} \end{pmatrix} = \begin{pmatrix} X^j \\ 0 \end{pmatrix} \\ {}^{V}X &= ({}^{V}X^\beta) = \begin{pmatrix} {}^{V}X^j \\ {}^{V}X^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ X^j \end{pmatrix} \end{aligned}$$

with respect to the adapted frame $\{e_\beta\}$. From (3)–(5) we see that the Cheeger-Gromoll metric ${}^{CG}g$ has components

$$({}^{CG}\tilde{g}_{\beta\gamma}) = \begin{pmatrix} {}^{CG}g_{jl} & {}^{CG}g_{j\bar{l}} \\ {}^{CG}g_{\bar{j}l} & {}^{CG}g_{\bar{j}\bar{l}} \end{pmatrix} = \begin{pmatrix} g_{jl} & 0 \\ 0 & \frac{1}{1+r^2}(g_{jl}+g_{js}g_{lt}x^{\bar{s}}x^{\bar{t}}) \end{pmatrix}$$

with respect to the adapted frame $\{e_\beta\}$.

For the Levi-Civita connection of the Cheeger-Gromoll metric we have the following.

Theorem 1 *[5] Let (M_n, g) be a Riemannian manifold and equip its tangent bundle $T(M_n)$ with the Cheeger-Gromoll metric ${}^{CG}g$. Then the corresponding Levi-Civita connection ${}^{CG}\nabla$ satisfies the following:*

$$\begin{cases} {}^{CG}\nabla_{{}^HX}{}^HY = {}^H(\nabla_X Y) - \frac{1}{2}{}^V(R(X,Y)y), \\ {}^{CG}\nabla_{{}^HX}{}^VY = \frac{1}{2\alpha}{}^H(R(y,Y)X) + {}^V(\nabla_X Y), \\ {}^{CG}\nabla_{{}^VX}{}^HY = \frac{1}{2\alpha}{}^H(R(y,X)Y), \\ {}^{CG}\nabla_{{}^VX}{}^VY = -\frac{1}{\alpha}({}^{CG}g({}^VX,\gamma\delta){}^VY + {}^{CG}g({}^VY,\gamma\delta){}^VX) \\ \qquad + \frac{1+\alpha}{\alpha}{}^{CG}g({}^VX,{}^VY)\gamma\delta - \frac{1}{\alpha}{}^{CG}g({}^VX,\gamma\delta){}^{CG}g({}^VY,\gamma\delta)\gamma\delta. \end{cases} \tag{8}$$

for any $X, Y \in \Im_0^1(M_n)$, where R and $\gamma\delta$ denotes respectively the curvature tensor of ∇ and the canonical vertical vector field on $T(M_n)$ with components

$$\gamma\delta = \begin{pmatrix} 0 \\ x^{\bar{i}}\delta_i^j \end{pmatrix} = \begin{pmatrix} 0 \\ x^{\bar{j}} \end{pmatrix} = x^{\bar{j}}\partial_{\bar{j}} = x^{\bar{j}}e_{(\bar{j})}.$$

With respect to the adapted frame $\{e_\alpha\}$ of $T(M_n)$, we write ${}^{CG}\nabla_{e_\alpha}e_\beta = {}^{CG}\Gamma_{\alpha\beta}^{\gamma}e_\gamma$ where ${}^{CG}\Gamma_{\alpha\beta}^{\gamma}$ denote the Christoffel symbols constructed by ${}^{CG}g$. The particular values of ${}^{CG}\Gamma_{\alpha\beta}^{\gamma}$ for different indices, on taking account of (8) are then found to be

$$\begin{cases} {}^{CG}\Gamma_{ji}^{h} = \Gamma_{ji}^{h}, \quad {}^{CG}\Gamma_{ji}^{\bar{h}} = -\frac{1}{2}R_{jik}^{h}x^{\bar{k}} \\ {}^{CG}\Gamma_{j\bar{i}}^{h} = -\frac{1}{2\alpha}R_{\bullet jki}^{h\bullet}x^{\bar{k}}, \quad {}^{CG}\Gamma_{j\bar{i}}^{\bar{h}} = \Gamma_{ji}^{h} \\ {}^{CG}\Gamma_{\bar{j}i}^{h} = -\frac{1}{2\alpha}R_{\bullet ikj}^{h\bullet}x^{\bar{k}}, \quad {}^{CG}\Gamma_{\bar{j}i}^{\bar{h}} = 0 \\ {}^{CG}\Gamma_{\bar{j}\bar{i}}^{h} = 0 \\ {}^{CG}\Gamma_{\bar{j}\bar{i}}^{\bar{h}} = -\frac{1}{\alpha}(x_{\bar{j}}\delta_i^h + x_{\bar{i}}\delta_j^h) + \frac{1+\alpha}{\alpha}g_{ji}x^{\bar{h}} - \frac{1}{\alpha}x_{\bar{j}}x_{\bar{i}}x^{\bar{h}} \end{cases} \tag{9}$$

with respect to the adapted frame, where $x_{\bar{j}} = g_{ji}x^{\bar{i}}, R_{\bullet ikj}^{h\bullet} = g^{ht}g_{js}R_{tik}^{h}$.

3. Results

Let $\tilde{C} : [0,1] \to T(M_n)$ be a curve on $T(M_n)$ and suppose that $\tilde{C}$ is expressed locally by $x^A = x^A(t)$, i.e., $x^h = x^h(t)$, $x^{\bar{h}} = x^{\bar{h}}(t) = y^h(t)$ with respect to induced coordinates $(x^h, x^{\bar{h}})$ in $\pi^{-1}(U) \subset T(M_n)$, t being a parameter. Then the curve $C = \pi \circ \tilde{C}$ on M_n is called the projection of the curve $\tilde{C}$ and denoted by $\pi\tilde{C}$ which is expressed locally by $x^h = x^h(t)$. Let $X^h(t)$ be a vector field along C. Then , on $T(M_n)$ we define a curve $\tilde{C}$ by

$$\begin{cases} x^h = x^h(t) \\ x^{\bar{h}} = X^h(t). \end{cases} \qquad (10)$$

If the curve (10) satisfies at all points the relation

$$\frac{\delta X^h}{dt} = \frac{dX^h}{dt} + \Gamma^h_{ji}\frac{dx^j}{dt}X^i = 0,$$

then the curve $\tilde{C}$ is said to be a horizontal lift of the curve C and denoted by ${}^HC$ [6,p.172]. If X^h is the tangent vector field $\frac{dx^h}{dt}$ to C, then the curve $\tilde{C}$ defined by (10) is called the natural lift of the curve C and denoted by C^*.

The geodesics of the connection ${}^{CG}\nabla$ is given by the differential equations

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + {}^{CG}\Gamma^A_{CB}\frac{dx^C}{dt}\frac{dx^B}{dt} = 0, \qquad (11)$$

with respect to induced coordinates $(x^h, x^{\bar{h}})$, where t is the arc length of a curve on $T(M_n)$.

We find it more convenient to refer equations (11) to the adapted frame $\{e_\alpha\}$. From (6) and (7) we see that the matrix of change of frames $e_\beta = A_\beta{}^H \partial_H$ has components of the form

$$A = (A_\beta{}^B) = \begin{pmatrix} \delta^k_j & 0 \\ -\Gamma^h_{sj}x^{\bar{s}} & \delta^k_j \end{pmatrix}$$

The inverse of the matrix A is given by

$$\tilde{A} = (\tilde{A}^\alpha{}_A) = \begin{pmatrix} \delta^h_i & 0 \\ \Gamma^h_{si}x^{\bar{s}} & \delta^h_i \end{pmatrix}.$$

Using $\tilde{A}$, we now write

$$\theta^\alpha = \tilde{A}^\alpha{}_A dx^A$$

or

$$\theta^h = \tilde{A}^h{}_A dx^A = \delta^h_i dx^i = dx^h,$$

for $\alpha = h$

$$\theta^{\bar{h}} = \tilde{A}^{\bar{h}}{}_A dx^A = \Gamma^h_{si}x^{\bar{s}}dx^i + \delta^h_i dx^{\bar{i}} = dy^h + \Gamma^h_{si}y^s dx^i = \delta y^h,$$

for $\alpha = \bar{h}$ and put

$$\frac{\theta^h}{dt} = A^h{}_A \frac{dx^A}{dt} = \frac{dx^h}{dt},$$
$$\frac{\theta^{\bar{h}}}{dt} = A^{\bar{h}}{}_A \frac{dx^A}{dt} = \frac{\delta y^h}{dt}$$

along a curve $x^A = x^A(t)$ on $T(M_n)$.

If we therefore write down the form equivalent to (11), namely,

$$\frac{d}{dt}(\frac{\theta^\alpha}{dt}) + {}^{CG}\Gamma^\alpha_{\gamma\beta}\frac{\theta^\gamma}{dt}\frac{\theta^\beta}{dt} = 0$$

with respect to adapted frame and taking account of (9), then we have

$$\begin{cases} (a) & \dfrac{\delta^2 x^h}{dt^2} + \dfrac{1}{\alpha}R^h_{kji}y^k\dfrac{\delta y^j}{dt}\dfrac{dx^i}{dt} = 0, \\ (b) & \dfrac{\delta^2 y^h}{dt^2} + [-\dfrac{1}{\alpha}(y_j\delta^h_i + y_i\delta^h_j) + \dfrac{1+\alpha}{\alpha}g_{ji}y^h - \dfrac{1}{\alpha}y_jy_iy^h]\dfrac{\delta y^j}{dt}\dfrac{\delta y^i}{dt} = 0, \end{cases} \tag{12}$$

where $y^i = x^{\bar{i}}$. Thus we have the following theorem.

Theorem 2 *Let $\tilde{C}$ be a curve on $T(M_n)$ and locally expressed by $x^h = x^h(t)$, $x^{\bar{h}} = y^h(t)$ with respect to induced coordinates $(x^h, x^{\bar{h}})$ in $\pi^{-1}(U) \subset T(M_n)$. The curve $\tilde{C}$ is a geodesic of ${}^{CG}g$, if it satisfies the equations (12).*

If a curve $\tilde{C}$ satisfying (12) lies on a fibre given by $x^h = const$, then by virtue of $\dfrac{dx^h}{dt} = 0$ and $\dfrac{\delta y^h}{dt} = \dfrac{dy^h}{dt} + \Gamma^h_{ij}\dfrac{dx^i}{dt}y^j = \dfrac{dy^h}{dt}$, the equations (12) reduces to

$$\frac{d^2y^h}{dt^2} + [-\frac{1}{\alpha}(y_j\delta^h_i + y_i\delta^h_j) + \frac{1+\alpha}{\alpha}g_{ji}y^h - \frac{1}{\alpha}y_jy_iy^h]\frac{dy^j}{dt}\frac{dy^i}{dt} = 0. \tag{13}$$

Hence we have this final theorem

Theorem 3 *If a geodesic lies on a fibre of $T(M_n)$ with metric ${}^{CG}g$, the geodesic is expressed by equation (13).*

Let $C = \pi \circ C^H$ be a geodesic of ∇ on M_n. Then $\dfrac{\delta^2 x^h}{dt^2} = 0$. Using this condition and condition $\dfrac{\delta y^j}{dt} = \dfrac{\delta X^h}{dt} = 0$, we have

Theorem 4 *The horizontal lift of a geodesic on M_n is always geodesic on $T(M_n)$ with the metric ${}^{CG}g$.*

Let now $C = \pi \circ C^*$ be a geodesic of ∇ on M_n, i.e. $\dfrac{\delta^2 x^h}{dt^2} = \dfrac{\delta}{dt}\left(\dfrac{dx^h}{dt}\right) = 0$. On the other hand, from definition of the natural lift of the curve, we obtain

$$\frac{\delta y^h}{dt} = \frac{\delta}{dt}\left(\frac{dx^h}{dt}\right) = 0. \tag{14}$$

Then from (12) and (14) we easily see that the natural lift of a curve on M_n defined $x^h = x^h(t)$ is geodesic on $T(M_n)$ with the metric ${}^{CG}g$. Thus we have

Theorem 5 *The natural lift C^* of a any geodesic on M_n is a geodesic on $T(M_n)$ with the metric ${}^{CG}g$.*

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