Cover for Modules and Injective Modules

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Abstract

Let \( R \) be a commutative ring with identity and \( M \) be an \( R \)-module with \( \text{Spec}(M) \neq \emptyset \). A cover of the \( R \)-submodule \( K \) of \( M \) is a subset \( C \) of \( \text{Spec}(M) \) satisfying that for any \( x \in K, x \neq 0 \), there is \( N \in C \) such that \( \text{ann}(x) \subset (N : M) \). If we denote by \( J = \bigcap_{N \in C} (N : M) \) and assume that \( M \) is finitely generated, then \( JM = M \) implies that \( M = 0 \). \( M \) is called \( C \)-injective provided each \( R \)-homomorphism \( \phi : (N : M) \to M \) with \( N \in C \) can be lifted to an \( R \)-homomorphism \( \lambda : R \to M \). If \( R \) is a commutative Noetherian ring and \( C' = \text{Spec}(R) \), where \( C' = \{(N : M) | N \in C\} \), then every \( C \)-injective \( R \)-module is injective.

Key Words: Commutative ring, \( D \)-prime module cover, prime submodule, injective module, quasi-injective and injective hull.

Definition. Let \( M \) be an \( R \)-module. A proper submodule \( P \) of \( M \) is a prime submodule, if \( rm \in P \), for \( r \in R \) and \( m \in M \) implies that either \( m \in P \) or \( rM \subset P \). The set of all prime submodules of \( M \) is called the spectrum of \( M \) and denoted by \( \text{Spec}(M) \).

Definition. Let \( M \) be an \( R \)-module. A subset \( C \) of \( \text{Spec}(M) \) is a cover of \( M \), if for every \( 0 \neq x \in M \) there exists \( P \in C \) such that \( \text{ann}(x) \subset (P : M) \). If \( C \) is a finite set, then \( C \) is called a finite cover.

Definition. An \( R \)-module \( M \) is called \( D \)-prime provided that \( M \neq 0 \) and \( \text{ann}(N) = \text{ann}(M) \), for all non-zero submodule \( N \) of \( M \).

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1. Cover for Modules and Localization

**Lemma 1.** Let $M$ be a non-zero $R$-module and $C$ a cover of $M$ and $J = \bigcap_{P \in C} (P : M)$ if $JM = M$, then $M = 0$.

**Proof.** Suppose that $M \neq 0$ and $JM = M$, then there exists $r \in R$ such that $r - 1 \in J$ and $rM = 0$, so $rm = 0$ for all $m \in M$ and $r \in \text{ann}(m)$. Hence $r \in J$, that is a contradiction. $\Box$

**Lemma 2.** Let $R$ be a Noetherian ring, $M$ is a finitely generated $R$-module, $C$ a cover of $M, I \subset \bigcap_{P \in C} (P : M)$. Then $\bigcap_{n=1}^{\infty} I^n M = 0$.

**Proof.** Let $\bigcap_{n=1}^{\infty} I^n M = K$. Then by Krull’s Theorem $IK = K$ and by Lemma 1, $K = 0$. $\Box$

**Lemma 3.** Let $C$ be a finite subset of $\text{Spec}(M)$ such that $(P : M)$ is maximal for every $P \in C$, and $J = \bigcap_{P \in C} (P : M)$. If $\bigcap_{n=1}^{\infty} J^n M = 0$, then $C$ is a finite cover of $M$.

**Proof.** If $C$ is not a cover of $M$, then there is an element $0 \neq x \in M$ such that $\text{ann}(x) \subsetneq (P : M)$ for all $P \in C$. Hence $\text{ann}(x) + (P : M) = R$. Let $1 = r + s$ with $s \in (P : M)$ and $r \in \text{ann}(x)$. Then for every $n \in N$, $1^n = (r + s)^n = r^n + s^n, r^n \in \text{ann}(x), s^n \in (P : M)^n$, so $x = r^n x + s^n x = s^n x$. Hence $Rx = (P : M)^n x$, for every $P \in C$, and so $J^n x = R x$. Hence $\bigcap_{n=1}^{\infty} J^n M \neq 0$, which is a contradiction. $\Box$

**Theorem 4.** Let $R$ be a Noetherian ring and $M$ a faithful finitely generated $R$-module. Then $M$ has a finite cover $C$ and $\bigcap_{n=1}^{\infty} J^n M = 0$, where $J = \bigcap_{P \in C} (P : M)$. In particular, if $M = R$, then $\bigcap_{n=1}^{\infty} J^n = 0$.

**Proof.** See [1. Theorem 6]. $\Box$

**Theorem 5.** Let $M$ be a finitely generated $R$-module and $C$ is a subset of $\text{Spec}(M)$. If for every prime ideal $P$ of $R$ and $N \in C, N \neq M_P$, then $C$ is a cover for $M$ over...
$R$ if and only if $C_P$ is a cover for $M_P$ over $R_P$, for every prime ideal $P$ of $R$, where $C_P = \{N_P | N \in C\}$.

**Proof.** Let $\frac{m}{s} \in M_P$. Since $m \in M$ and $C$ is a cover for $M$, there exists $N \in C$ such that $\text{ann}(m) \subset (N : M)$. Let $r/s \in \text{ann}(\frac{m}{s})$. Since $\text{ann}(mP) \subset (N_P : M_P), r/s \in (N_P : M_P)$ and $\text{ann}(\frac{m}{s}) \subset (N : M_P)$ so $C_P$ is a cover for $M_P$ over $R_P$. Let $m \in M$, then $\frac{m}{s} \in M_P$ so there exists $N_P \in C_P$ such that $\text{ann}(\frac{m}{s}) \subset (N_P : M_P)$. Now let $r \in \text{ann}(m)$. Then $\frac{r}{s}, \frac{r}{s} \in N_P$, where $\frac{r}{s} \in M_P$, so $\frac{r}{s} = \frac{r}{s}$ for some $n \in N$; and so there exists $s' \in R - P$ such that $rss'y = s'n \in N$. Hence $ss'(ry) \in N$, and since $ss' \notin (N : M), ry \in N$, so $rM \subset N$, and $\text{ann}(x) \subset (N : M)$.

**Theorem 6.** Let $R$ be a reduced ring and $C$ is a subset of $\text{Spec}(R)$. Then $C$ is a cover for $R$ as an $R$-module if and only if $C||x||$ is a cover for $R||x||$, where $C||x|| = \{P||x|| | P \in C\}$.

**Proof.** Let $C$ be a cover for $R$ and $g(x) \in \text{ann}(f(x))$ for $f(x), g(x) \in R||x||$. If $g(x) = \sum_{n=0}^{\infty} b_nx^n$ and $f(x) = \sum_{n=0}^{\infty} a_nx^n$, then for every $i, b_i, f(x) = 0$, so for every $i, b_i \in \text{ann}(a_0) \subset P$, for some $P \in C$ and hence $g(x) \in P||x||$. Conversely if $C||x||$ is a cover for $R||x||$ and let $a \in R, r \in R$ such that $r \in \text{ann}(a) \subset P||x||$, for some $P||x|| \in C||x||$. So $ra = 0$ and hence $r \in P||x|| \cap R$, so $r \in P$. Then $\text{ann}(a) \subset P$. Hence $C$ is a cover for $R$.

**Proposition 7.** Let $R$ be a ring and $C$ is a subset of $\text{Spec}(R)$. Then $C$ is a cover for $R$ as an $R$-module if and only if $C[x] = \{P[x] | P \in C\}$ is a cover for $R[x]$ as an $R[x]$-module.

**Proof.** Let $C$ be a cover for $R$ and $f(x) \in R[x], g(x) \in \text{ann}(f(x))$, then $f(x)g(x) = 0$. If $g(x) = \sum_{i=0}^{k} b_i x^i$, and $f(x) = \sum_{i=1}^{m} a_i x^i$, then there is an element $a$ such that $ag(x) = 0$ so $b_i \in \text{ann}(a) \subset P$ for some $P \in C$. So $g(x) \in P[x]$ and hence $C[x]$ is a cover for $R[x]$.

Conversely, let $C[x]$ be a cover for $R[x]$, and let $a \in R, r \in R$, and $r \in \text{ann}(a)$. As $\text{ann}(a) \subset P[x]$ so $r \in P[x] \cap R = P$. Thus $C$ is a cover for $R$.  

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2. **C-injective Modules**

**Definition.** Let $R$ be a ring $M$, $X$ are $R$-modules, $C$ is a cover of $M$. We say that $X$ is $C$-injective provided every $R$-homomorphism $\phi : (N : M) \to X$, where $N \in C$ can be lifted to an $R$-homomorphism $\lambda : R \to X$. In the next results we shall be interested in ring $R$ with the following properties:

(P1) for every proper ideal $I$ there exists a finite set of prime ideals $P_1, P_2, \cdots, P_n$ such that $P_1P_2\cdots P_n \leq I \leq P_1 \cap P_2 \cap \cdots \cap P_n$.

(P2) The ascending chain condition on prime ideals.

**Proposition 8.** Let $R$ be a Noetherian ring. Then $R$ satisfies (P1) and (P2).

**Proof.** Since $R$ is Noetherian then $R$ satisfies (P2). Suppose $R$ does not satisfy (P1).

Let $S = \{ J \mid (P1) \text{ fails for } J \}$. Suppose $I$ be a maximal element of $S$. Then $I$ is not a prime ideal, so there exists ideal $I_1$ and $I_2$ properly containing $I$ such that $I_1I_2 \leq I$. By the choice of $I$, (P1) holds for each $I_1$ and $I_2$, and hence for $I$, which is a contradiction. $\square$

**Proposition 9.** Let $R$ be a ring which satisfies (P1) and (P2). Then every non-zero $R$-module contains a $D$-prime submodule.

**Proof.** Let $M$ be a non-zero $R$-module. Let $I = \text{ann}(M)$. The there exists prime ideal $P_1, P_2, \cdots, P_n$ such that $P_1P_2\cdots P_n \leq I \leq P_1 \cap P_2 \cap \cdots \cap P_n$. Suppose $P_1P_2\cdots P_n M = 0$ and it follows that there exists $P_k$ such that $P_km = 0$, for some $m \in M$. Suppose $B = \{ P : P$ is a prime ideal and $Px = 0$ for some $x \in M \}$. Let $Q$ be a maximal element of $B$ and let $y \in M$ such that $Qy = 0$. We show that $N = Ry$ is a $D$-prime submodule of $M$. let $K$ be a non-zero submodule of $N$. Then $Q \leq \text{ann}(K)$, we show that $Q = \text{ann}(K)$. Let $Q \neq \text{ann}(K)$. Then there exists prime ideal $q_1, q_2, \cdots, q_m$ such that $q_1q_2\cdots q_m \leq \text{ann}(K) \leq q_1 \cap q_2 \cap \cdots \cap q_m$. If follows that $q_1q_2\cdots q_m K = 0$, and there exists $x \in K$ such that $q_i x = 0$ for some $i$. But $Q < \text{ann}(K) \leq q_i$, and this contradicts the choice of $Q$. Hence $Q = \text{ann}(K)$, and so $N$ is $D$-prime submodule of $M$.

**Theorem 10.** Let $M$ be an $R$-module and $R$ satisfies (P1) and (P2), $C$ a cover of $M$. Then $M$ is $C$-injective if and only if $M$ is an injective $R$-module.

**Proof.** Let $M$ be a $C$-injective and $I$ be an ideal of $R$ and $\phi : I \to M$ an $R$-
homomorphism. By zorn lemma there exists an ideal $J$ containing $I$ maximal with respect to the property that $\phi$ can be lifted to a homomorphism $\lambda: J \to M$. We show that $J = R$.

Suppose $J \neq R$. Thus $R/J$ is a non-zero $R$-module and so $R/J$ has a $D$-prime submodule. \square

Let $K$ be an ideal containing $J$ such that $R/J$ is a $D$-prime module. Let $P = \{r \in R | rk \in J\}$. Then $P \simeq (R/K + J)/J$ and hence $P$ is a prime ideal of $R$. As $P = (N : M)$, where $N \in C$, define $\gamma : P \to M$ by $\gamma(x) = \lambda(kx)$. Then $\gamma$ is a homomorphism, and because $P = (N : M)$ for $N \in C$, there exists $m \in M$ such that $\gamma(x) = mx$. Now define $\theta : kR + J \to M$ by $\theta(rk + j) = rm + \lambda(j)$, so $\theta$ is well-defined, $\theta$ is a homomorphism and $\theta$ extends $\lambda$ and hence $\phi$. This contradiction shows that $J = R$. it follows that $M$ is injective.

3. Quasi-Injective Modules

**Definition.** An $R$-module $M$ is said to be quasi-injective if every $R$-homomorphism $\phi : N \to M, N$ a submodule of $M,$ is induced by an $R$-endomorphism of $M$.

**Notation.** Let $C$ be a cover for $R$-module $M$, denote $C(M) = \{x \in M | (N : M) \subset \text{ann}(x), \text{for some } N \in C\}$.

**Lemma 11.** $C(M)$ is a submodule of $M$.

**Proof.** It is obvious. \hfill \square

**Theorem 12.** An $R$-module $M$ is quasi-injective if and only if $M = E[C(M)]$, where $E[C(M)]$ is injective hull of $C(M)$.

**Proof.** If $M$ is quasi-injective. Then $M \leq E[C(M)]$, we show that $E[C(M)] \leq M$. Let $y \in E[C(M)]$, then there exists $N \in C$ such that $(N : M) \subset \text{ann}(y)$; and since $C$ is a cover for $M$ there exists $x \in M$ such that $\text{ann}(x) \subset (N : M)$. We define $\alpha : Rx \to Ry$ by $\alpha(x) = y$. Let $E = E[M]$, so we have the mapping
\[
\begin{array}{ccc}
0 & \to & Rx \\
\alpha & \downarrow & \lambda \\
& & E
\end{array}
\]
Now $\phi = \lambda/M$ maps $x$ onto $y$; and since $M$ is quasi-injective, it is fully invariant in $E$, then $y \in M$ so that $E[C(M)] \leq M$, and equality holds. Conversely, suppose that
$M = E[C(M)]$, since $E[C(M)]$ is a injective $R$-module so is $M$, and since every injective $R$-module is quasi-injective. Hence $M$ is quasi-injective $R$-modules.

\[ \square \]

**Corollary 13.** Let $C$ be a cover for an $R$-module $M$. Then the following are equivalent.

1. $M$ is quasi-injective $R$-module.
2. $M$ is a injective $R$-module.
3. $M = E[C(M)]$.

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