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Connectedness in Isotonic Spaces

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Abstract

An isotonic space (X, cl) is a set X with isotonic operator $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies $cl(\emptyset) = \emptyset$ and $cl(A) \subseteq cl(B)$ whenever $A \subseteq B \subseteq X$. Many properties which hold in topological spaces hold in isotonic spaces as well.

The notion of connectedness that is familiar from topological spaces generalizes to isotonic spaces. We further extend the notions of Z-connectedness and strong connectedness to isotonic spaces, and we indicate the intimate relationship between these notions.

Key Words: generalized closure spaces, isotonic spaces, neighborhood spaces, connectedness, Z-connectedness, strong connectedness.

1. Generalized Closure Spaces

Closure spaces and (more generally) isotonic spaces have already been studied by Hausdorff [13], Day [3], Hammer [11, 12], Gnilka [6, 7, 8] and Stadler [15, 16]. In this paper we explore some meaningful topological concepts that can be defined for isotonic spaces, especially the connectedness.

Let X be a set, $\mathcal{P}(X)$ its power set and $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ be an arbitrary set-valued set-function, called a *closure function*. We call $cl(A)$, $A \subseteq X$, the *closure* of A, and we call the pair (X, cl) a *generalized closure space*. Consider the following axioms of the closure function for all $A, B, A_\lambda \in \mathcal{P}(X)$:

K0) $cl(\emptyset) = \emptyset$.

Kl) $A \subseteq B$ implies $cl(A) \subseteq cl(B)$ (isotonic).

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K2) $A \subseteq cl(A)$ (expanding).

K3) $cl(A \cup B) \subseteq cl(A) \cup cl(B)$ (sub-additive).

K4) $cl(cl(A)) = cl(A)$ (idempotent).

K5) $\bigcup_{\lambda \in \Lambda} cl(A_\lambda) = cl(\bigcup_{\lambda \in \Lambda}(A_\lambda))$ (additive).

The dual of a closure function is the *interior function* $int : \mathcal{P}(X) \to \mathcal{P}(X)$ which is defined by

$$int(A) := X \setminus cl(X \setminus A). \tag{1}$$

Given the interior function $int : \mathcal{P}(X) \to \mathcal{P}(X)$, the closure function is recovered by

$$cl(A) := X \setminus (int(X \setminus A)) \text{ for all } A \in \mathcal{P}(X). \tag{2}$$

A set $A \in \mathcal{P}(X)$ is *closed* in the generalized closure space (X, cl) if $cl(A)$=A holds. It is *open* if its complement $X \setminus A$ is closed or equivalently $A = int(A)$.

It should be noted that the open and closed sets will not play a central role in our discussion. From now on, (for short) the word *space* will mean a generalized closure space.

Definition 1.1 [15] Let cl and int be a closure and its dual interior function on X. Then the *neighborhood function* $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$ assigns to each $x \in X$ the collection

$$\mathcal{N}(x) := \{ N \in \mathcal{P}(X) \mid x \in int(N)\} \tag{3}$$

of its neighborhoods. A set V is a *neighborhood* of A, i.e. $V \in \mathcal{N}(A)$, if $V \in \mathcal{N}(x)\ \forall x \in A$.

The proof of the next lemma follows immediately from the definitions.

Lemma 1.1 [15, 16] *For any space (X, cl), $V \in \mathcal{N}(A)$ if and only if $A \subseteq int(V)$.*

The next theorem illustrates the intimate relationship between closures of sets and neighborhoods of points.

Theorem 1.1 [15, 16] *Let $\mathcal{N}$ be the neighborhood function defined in equ.(3). Then $x \in cl(A)$ if and only if $X \setminus A \notin \mathcal{N}(x)$.*

It should be noted that there are equivalent properties for (Ki), $i = 0, 1, ..., 5$, which can be expressed in terms of interior or neighborhood functions (see [15, 16, 10]).

2. Isotonic Spaces

Definition 2.1 [15, 16] An *isotonic space* is a pair (X, cl), where X is a set and $cl : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfies the axioms $(K0)$ and $(K1)$. An isotonic space (X, cl) that satisfies $(K2)$ is called a *neighborhood space*. A *closure space* is a neighborhood space that satisfies $(K4)$. A *topological space* is a closure space that satisfies $(K3)$.

Lemma 2.1 [12, Lemma10] The following conditions are equivalent for an arbitrary closure function $cl : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$:

(K1) $A \subseteq B \subseteq X$ implies $cl(A) \subseteq cl(B)$.

(K1^I) $cl(A) \cup cl(B) \subseteq cl(A \cup B)$ for all $A, B \in \mathcal{P}(X)$.

(K1^{II}) $cl(A \cap B) \subseteq cl(A) \cap cl(B)$ for all $A, B \in \mathcal{P}(X)$.

It is easy to derive equivalent conditions for the associated interior function by repeated application of $int(A) = X \setminus cl(X \setminus A)$, as the following lemma shows.

Lemma 2.2 [15] The following conditions are equivalent for an arbitrary interior function: $int : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$:

(K1^{III}) $A \subseteq B \subseteq X$ implies $int(A) \subseteq int(B)$.

(K1^{IV}) $int(A) \cup int(B) \subseteq int(A \cup B)$ for all $A, B \in \mathcal{P}(X)$.

(K1^V) $int(A \cap B) \subseteq int(A) \cap int(B)$ for all $A, B \in \mathcal{P}(X)$.

An isotonic space can be described by means of interior and neighborhood functions, as the following two lemmas show. Their proofs follow immediately from the definitions and, therefore, are omitted.

Lemma 2.3 *A space (X, cl) is isotonic if and only if the interior function $int : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfies:*

I0) $int(X) = X$;

I1) $int(A) \subseteq int(B)$ $\forall A \subseteq B \subseteq X$.

Lemma 2.4 *A space (X, cl) is isotonic if and only if the neighborhood function $\mathcal{N} : X \rightarrow \mathcal{P}(\mathcal{P}(X))$ satisfies:*

N0) $X \in \mathcal{N}(x) \quad \forall x \in X$;

N1) $N \in \mathcal{N}(x), N \subseteq N_1$ *implies* $N_1 \in \mathcal{N}(x)$.

The next theorem shows that (N1)(or equivalently (K1)) is a necessary and sufficient condition for defining the closure function in terms of neighborhoods.

Theorem 2.1 [3, Theorem 3.1, Corollary 3.2] *Let* (X, cl) *be a space and* $c(A) := \{x \in X \mid \forall N \in \mathcal{N}(x) : A \cap N \neq \emptyset\}$ *for all* $A \subseteq X$. *Then*

(i) $c(A) \subseteq cl(A)$.

(ii) $c : \mathcal{P}(X) \to \mathcal{P}(X)$ *is isotonic (i.e., satisfies (K1)).*

(iii) $c(A) = cl(A)$ *if and only if* cl *is isotonic.*

The following brief study of the lower separation axiom T_1 for isotonic and neighborhood spaces is needed for the study of connectedness in section 5.

Definition 2.2 [15, 16] A space (X, cl) is a T_1-*space* if $\forall x, y \in X$, $x \neq y$, $\exists N^{'} \in \mathcal{N}(x)$ and $N^{''} \in \mathcal{N}(y)$ such that $x \notin N^{''}$, $y \notin N^{'}$.

Proposition 2.1 [15] *An isotonic space* (X, cl) *is a* T_1*-isotonic space if and only if* $cl(\{x\}) \subseteq \{x\} \, \forall x \in X$.

In general topology the definition of T_1-topological spaces has several equivalent forms ([5, Theorem V-1.2], [17, Theorem 13.4]), which can be generalized to neighborhood spaces, as in the next theorem.

Theorem 2.2 *Let* (X, cl) *be a neighborhood space. Then the following statements are equivalent:*

a) (X, cl) *is* T_1.

b) *Each one-point set in* X *is closed.*

c) *Each subset of* X *is the union of closed sets contained in it.*

Proof. Let (X, cl) be a neighborhood space.

(a $\Rightarrow$ b): If (X, cl) is T_1, then, by Proposition 2.1, $cl(\{x\}) \subseteq \{x\}$ for all $x \in X$. Since, by $(K-2)$, $\{x\} \subseteq cl(\{x\})$, we have $cl(\{x\}) = \{x\}$ $\forall x \in X$. Therefore, $\forall x \in X$, $\{x\}$ is a closed set.

(b $\Rightarrow$ c): It is trivial.

(c $\Rightarrow$ a): If each subset of X is the union of closed subsets contained in it, then it is trivial that $\{x\} = cl\{x\}$ $\forall x \in X$. Therefore, by Proposition 2.1, (X, cl) is T_1. $\square$

3. Continuous Functions

The purpose of this section is to define continuous functions on a space (X, cl) with arbitrary closure function, and establish their elementary properties.

Definition 3.1 [15, 16] Let (X, cl) and (Y, cl) be two spaces. A function $f : X \to Y$ is *continuous* if $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ $\forall B \in \mathcal{P}(Y)$.

Proposition 3.1 [16] *Let (X, cl) and (Y, cl) be two spaces and $f : X \to Y$. Then the following statements are equivalent:*

(1) *f is continuous.*

(2) $f^{-1}(int_Y(B)) \subseteq int_X(f^{-1}(B)) \qquad \forall B \in \mathcal{P}(Y)$.

(3) $B \in \mathcal{N}_Y(f(x))$ *implies* $f^{-1}(B) \in \mathcal{N}_X(x)$ $\forall B \in \mathcal{P}(Y)$.

Definition 3.2 [15] Let (X, cl) and (Y, cl) be two spaces. We say that $f : X \to Y$ is *continuous at $x \in X$* if $\forall B \in \mathcal{P}(Y)$ and $B \in \mathcal{N}_Y(f(x))$, we have $f^{-1}(B) \in \mathcal{N}_X(x)$.

Corollary 3.1 [15] *Let (X, cl) and (Y, cl) be two spaces. Then $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.*

Definition 3.3 [15] Let (X, cl) and (Y, cl) be two spaces. Then $f : X \to Y$ is *closure-preserving* if for all $A \in \mathcal{P}(X)$, $f(cl_X(A)) \subseteq cl_Y(f(A))$.

Theorem 3.1 [15] *Let (X, cl) and (Y, cl) be isotonic spaces. Then the following properties are equivalent:*

(i) *$f : X \to Y$ is continuous.*

(ii) *$f : X \to Y$ is closure-preserving.*

(iii) *$f(A) \subseteq B$ implies $f(cl_X(A)) \subseteq cl_Y(B)$ for all $A \in \mathcal{P}(X)$ and $B \in \mathcal{P}(Y)$.*

Lemma 3.1 *Let (X, cl) and (Y, cl) be neighborhood spaces and $f : X \to Y$. Then the following statements are equivalent:*

(i) *f is continuous.*

(ii) *$cl_X(f^{-1}(B)) = f^{-1}(cl_Y(B))$ for all closed sets $B \subseteq Y$; i.e., the inverse image of every closed set in Y is closed in X.*

(iii) *$int_X(f^{-1}(B)) = f^{-1}(int_Y(B))$ for all open sets $B \subseteq Y$; i.e., the inverse image of every open set in Y is open in X.*

Proof.

(i⇒ ii): Suppose f is continuous. Then for all closed sets B in Y, $cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))$. Since, by (K2), $f^{-1}(cl_Y(B)) \subseteq cl_X(f^{-1}(cl_Y(B)) = cl_X(f^{-1}(B)$, it follows that $cl_X(f^{-1}(B)) = f^{-1}(cl_Y(B))$.

(ii⇒ iii): Suppose (ii) holds. Then for all open sets B in Y, $int_X(f^{-1}(B)) = X \setminus cl_X(X \setminus f^{-1}(B)) = X \setminus cl_X(f^{-1}(Y \setminus B)) = X \setminus f^{-1}(cl_Y(Y \setminus B)) = f^{-1}(Y \setminus cl_Y(Y \setminus B)) = f^{-1}(int_Y(B))$.

(iii⇒ i): Immediate from Proposition 3.1. □

Lemma 3.2 *Let X, Y, Z be spaces and suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then $g \circ f : X \to Z$ is continuous.*

Proof. Let $A \subseteq Z$. Then $(g \circ f)^{-1}(cl_Z(A)) = f^{-1}(g^{-1}(cl_Z(A))) \supseteq f^{-1}(cl_Y(g^{-1}(A))) \supseteq cl_X(f^{-1}(g^{-1}(A)) = cl_X(g \circ f)^{-1}(A)$. Therefore $g \circ f : X \to Z$ is continuous. □

4. Subspaces

Definition 4.1 [16] Let (X, cl) be a space and $Y \subseteq X$. Then $c_Y : \mathcal{P}(Y) \to \mathcal{P}(Y)$, $A \longmapsto Y \cap cl(A)$ is the *relativization* of cl to Y. The pair (Y, c_Y) is called a *subspace* of (X, cl). If $A \subseteq Y$ then the *relative interior* of A is given by

$$int_Y(A) := Y \setminus c_Y(Y \setminus A) = Y \cap int(A \cup (X \setminus Y))$$

and the *relative neighborhoods* of A are

$$\mathcal{N}_Y(A) := \{N \cap Y | N \in \mathcal{N}(A)\}.$$

Definition 4.2 [16] A property $\mathbb{B}$ of a space (X, cl) is *hereditary* if every subspace (Y, c_Y) of (X, cl) also has the property $\mathbb{B}$.

The proof of the following lemma is obvious.

Lemma 4.1 *The properties (K0) and (K1) are hereditary in any space* (X, cl)*.*

Definition 4.3 Let (X, cl) and (Y, cl) be spaces, $f : X \to Y$ and $A \subseteq X$. We will use $f|A$ to denote the *restriction of* f *to* A which is defined by $(f|A)(x) := f(x)$ for each $x \in A$.

Theorem 4.1 *Let* (X, cl) *and* (Y, cl) *be isotonic spaces and* $A \subseteq X$*. Then* $(f|A) : A \to Y$ *is continuous, whenever* $f : X \to Y$ *is continuous.*

Proof. Let $B \subseteq Y$. We have $c_A((f|A)^{-1}(B)) = c_A(f^{-1}(B) \cap A)$. Since X is isotonic, by Lemma 2.1, $c_A(f^{-1}(B) \cap A) \subseteq c_A(f^{-1}(B)) \cap c_A(A) = cl_X(f^{-1}(B)) \cap A \cap c_A(A) \subseteq f^{-1}(cl_Y(B)) \cap A \cap c_A(A) = (f|A)^{-1}(cl_Y(B)) \cap c_A(A) \subseteq (f|A)^{-1}(cl_Y(B))$. Hence $(f|A) : A \to Y$ is continuous. □

5. Connectedness

Definition 5.1 In an isotonic space (X, cl), two subsets $A, B \subseteq X$ are called *semi-separated* if $cl(A) \cap B = A \cap cl(B) = \emptyset$.

Lemma 5.1 [16] *In an isotonic space* (X, cl)*, the following two conditions are equivalent for all* $A, B \subseteq X$*:*

(SS) *A, B are semi-separated.*

(SS') *There are $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ such that $A \cap V = U \cap B = \emptyset$.*

Lemma 5.2 [16] *Let (X, cl) be an isotonic space and let $A, B \subseteq Y \subseteq X$. Then A and B are semi-separated in (X, cl) if and only if A and B are semi-separated in (Y, c_Y).*

Lemma 5.3 [15] *Let (X, cl) and (Y, cl) be spaces and let $f : (X, cl) \to (Y, cl)$ be a continuous function. If $A, B \subseteq Y$ are semi-separated in Y, then $f^{-1}(A)$ and $f^{-1}(B)$ are semi-separated in X.*

Definition 5.2 [15] A set $Y \in \mathcal{P}(X)$ is *connected* in a space (X, cl) if it is not a disjoint union of a nontrivial semi-separated pair of sets $A, Y \setminus A$, $A \neq \emptyset, Y$. We say that a space (X, cl) is connected if X is connected in (X, cl).

Definition 5.3 Let (X, cl) be a space and $x \in X$. The *component* $C(x)$ of x in X is the union of all connected subsets of X containing x.

Definition 5.4 A space (X, cl) is *totally disconnected* if for each $x \in X$ the component $C(x) = \{x\}$.

Theorem 5.1 [15] *A set $Y \in \mathcal{P}(X)$ is connected in an isotonic space (X, cl) if and only if for each proper subset $A \subset Y$,*

$$[cl(A) \cap (Y \setminus A)] \cup [cl(Y \setminus A) \cap A] \neq \emptyset.$$

Topology books [5, 17, 9, 4] define connected spaces by means of open and closed sets. That definition is equivalent to our definition if and only if the space (X, cl) is a neighborhood space.

Theorem 5.2 *A neighborhood space (X, cl) is connected if and only if there are no nonempty disjoint open (closed) sets H and K in X with $X = H \cup K$.*

Proof. Suppose that X is disconnected. Then $X = H \cup K$, where H, K are semi-separated, disjoint sets. Since $H \cap cl(K) = \emptyset$, we have $cl(K) \subseteq X \setminus H \subseteq K$ and then, by (K2), $cl(K) = K$. Since $cl(H) \cap K = \emptyset$, a similar argument yields that H is a closed set.

Conversely, suppose that H and K are disjoint open sets such that $X = H \cup K$. Now $K = X \setminus H$, and H is an open set, hence K is a closed set. Thus $H \cap cl(K) = H \cap K = \emptyset$.

A similar argument yields that $cl(H) \cap K = \emptyset$. Thus H and K are semi-separated and therefore X is disconnected. $\square$

Dugundji [5, Chap.V 1.3] gave an equivalent definition of connectedness by proving that a topological space X is connected if and only if any continuous function from X to the discrete space $\{0, 1\}$ is constant. The next result shows that we can extend this definition to isotonic spaces by assuming that $\{0, 1\}$ is a T_1-isotonic space.

Theorem 5.3 *An isotonic space (X, cl) is connected if and only if for all T_1-isotonic doubleton spaces $Y = \{0, 1\}$, any continuous function $f : X \to Y$ is constant.*

Proof. Suppose X is connected, and let $f : X \to Y$ be continuous. To show that f is constant, assume not. Then there is a set $A \subseteq X$ such that $A = f^{-1}(\{0\})$ and $X \setminus A = f^{-1}(\{1\})$. Now by continuity and T_1, we have $cl_X(A) = cl_X(f^{-1}(\{0\})) \subseteq f^{-1}(cl_Y\{0\}) \subseteq f^{-1}(\{0\}) = A$, which implies that $cl_X(A) \cap (X \setminus A) = \emptyset$. A similar argument will give $A \cap cl_X(X \setminus A) = \emptyset$. Hence we have a contradiction, since X is connected. Therefore f must be a constant function.

Conversely, suppose X is disconnected. Then there are semi-separated sets $A, B \subseteq X$ such that $A \cup B = X$, and hence $cl(A) \subseteq A$ and $cl(B) \subseteq B$. Since $A \cup B = X$, we have $X \setminus A \subseteq B$, which implies, by (K1) and semi-separation of A, B, that $cl(X \setminus A) \subseteq cl(B) \subseteq X \setminus A$. Take the space (Y, cl) such that $Y := \{0, 1\}, cl(\emptyset) = \emptyset, cl(\{0\}) = \{0\}, cl(\{1\}) = \{1\}$ and $cl(X) = X$. It is clear that the space (Y, cl) is a T_1-isotonic space.

Claim: The function $f : X \to Y$ such that $f(A) = \{0\}$ and $f(X \setminus A) = \{1\}$ is continuous. To see this, let $\emptyset \neq C \subseteq Y$. Then we have three cases to consider.

Case 1. $C = Y$. In this case, $f^{-1}(C) = X$ and hence $cl_X(X) = cl_X(f^{-1}(C)) \subseteq X = f^{-1}(C) = f^{-1}(cl_y(C))$, since $Y = cl_Y(Y)$.

Case 2. $C = \{0\}$. In this case, $f^{-1}(C) = A$ and hence $cl_X(A) = cl(f^{-1}(C)) \subseteq A = f^{-1}(C) = f^{-1}(cl_Y(C))$

Case 3. $C = \{1\}$. In this case, $f^{-1}(C) = X \setminus A$ and hence $cl_X(X \setminus A) = cl(f^{-1}(C)) \subseteq X \setminus A = f^{-1}(C) = f^{-1}(cl_Y(C))$.

Thus the function f is continuous but not constant. This proves the claim, and hence the theorem.hfill$\square$

The importance of Theorem 5.3 lies in the fact that it can be used as a test for proving connectedness for subsets of isotonic spaces, as can be seen from the proofs of the following results.

Theorem 5.4 *If* $f : (X, cl) \to (Y, cl)$ *is a continuous function between isotonic spaces, and* X *is connected, then* $f(X)$ *is connected in* Y*.*

Proof. Let $g : f(X) \to \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is a T_1-isotonic space. Since f is continuous, then, by Lemma 3.2, $g \circ f : X \to \{0, 1\}$ is continuous and hence, by connectedness of X, it is constant. Thus g is constant and therefore, by Theorem 5.3, $f(X)$ is connected. □

Remark 5.1 Since every neighborhood space is an isotonic space, we see that our Theorem 5.4 generalizes Theorem 18 of [15] for neighborhood spaces.

Theorem 5.5 *If* (X, cl) *is an isotonic space and* $A \subseteq X$*, then* $cl(A)$ *is connected whenever* A *is connected and* $A \subseteq cl(A)$*.*

Proof. Let $f : cl(A) \to \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is a T_1-isotonic space. Then, by Theorem 4.1, $f|A : A \to \{0, 1\}$ is continuous. Since A is connected, then, by Theorem 5.3, $f|A$ is constant; say $f(A) = \{a\}$ for some $a \in \{0, 1\}$. Hence, by Theorem 3.1 and Proposition 2.1, we have $f(cl(A)) \subseteq cl_{\{0,1\}}(f(A)) = cl_{\{0,1\}}(\{a\}) \subseteq \{a\}$. Thus $f : cl(A) \to \{0, 1\}$ is constant, and therefore, by Theorem 5.3 again, $cl(A)$ is connected. □

Remark 5.2 It should be noted that our Theorem 5.5 yields Theorem 17 of [15] and a similar result in other cited references as an immediate corollary (see Corollary 5.1 below).

Corollary 5.1 [5, 17, 15, 9, 4] *If* (X, cl) *is a neighborhood space and* $Z \subseteq X$*, then* $cl(Z)$ *is connected whenever* Z *is connected.*

The following lemma appears in [15, Lemma 11]. We give a different proof which is based on Theorem 5.3.

Lemma 5.4 [15] *If* Y *and* Z *are connected sets in an isotonic space* (X, cl) *and* $Y \cap Z \neq \emptyset$*, then* $Y \cup Z$ *is connected.*

Proof. Let $f : Y \cup Z \to \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is a T_1-isotonic space. Then, by Theorem 4.1, $f|Y$ and $f|Z$ are continuous functions and hence, by Theorem 5.3, each is constant, since Y and Z are connected. Since $Y \cap Z \neq \emptyset$, then $f : Y \cup Z \to \{0, 1\}$ is constant, and therefore, by Theorem 5.3, $Y \cup Z$ is connected. □

Lemma 5.5 *Let A_1, A_2, ..., A_n be connected subsets of an isotonic space (X, cl) such that $A_i \cap A_{i+1} \neq \emptyset$ for all $i = 1$ to $n - 1$. Then $\bigcup A_i$ is connected.*

Proof. It follows easily from Lemma 5.4 and induction. □

Lemma 5.6 *Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a family of connected subsets of an isotonic space (X, cl) such that $\bigcap_{\lambda\in\Lambda} A_\lambda \neq \emptyset$. Then $\bigcup_{\lambda\in\Lambda} A_\lambda$ is connected.*

Proof. Let $x_0 \in \bigcap_{\lambda\in\Lambda} A_\lambda$ be fixed and let $f : \bigcup_{\lambda\in\Lambda} A_\lambda \to \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is a T_1-isotonic space. Then, by Theorem 4.1, $f|A_\lambda : A_\lambda \to \{0, 1\}$ is continuous for all $\lambda \in \Lambda$. Since A_λ is connected for all $\lambda \in \Lambda$, then, by Theorem 5.3, $f|A_\lambda$ is constant for all $\lambda \in \Lambda$, and hence $f(\{x_0\}) = f(A_\lambda)$ for all $\lambda \in \Lambda$. Thus the function $f : \bigcup_{\lambda\in\Lambda} A_\lambda \to \{0, 1\}$ is constant and therefore, by Theorem 5.3, $\bigcup_{\lambda\in\Lambda} A_\lambda$ is connected. □

Theorem 5.6 *If (X, cl) is an isotonic space with the property that for any distinct points $x, y \in X$ there exist connected subsets A_1, A_2, ..., A_n such that $x \in A_1$, $y \in A_n$ and $A_i \cap A_{i+1} \neq \emptyset$ for all $i = 1, 2, ..., n - 1$, then X is connected.*

Proof. Fix $x \in X$. Then, by hypothesis, for each $y \in X \setminus \{x\}$ there are connected subsets $A_{1y}, A_{2y}, ..., A_{ny}$ such that $x \in A_{1y}$, $y \in A_{ny}$ and $A_{iy} \cap A_{(i+1)y} \neq \emptyset$ for all $i = 1, 2, ..., n-1$. By Lemma 5.5, we have $B_y := \bigcup_{i=1}^{n} A_{iy}$ is connected. Since $x \in A_{1y}$ for all $y \in X \setminus \{x\}$, then $x \in B_y$ for all $y \in X \setminus \{x\}$. Therefore, by Lemma 5.6, $\bigcup_{y\neq x} B_y = X$ is connected. □

Remark 5.3 It should be noted that Lemma 5.6 generalizes Lemma 2 of [2] and Theorem 5.6 generalizes Theorem 1 of [2]

6. Z-Connected Isotonic Spaces

Definition 6.1 [1] Let (Z, cl) be an isotonic space with more than one element. An isotonic space (X, cl) is called *Z-connected* if and only if any continuous function $f : X \to Z$ is constant.

Lemma 6.1 *If Z is a neighborhood space with more than one element, then every Z-connected isotonic space is connected.*

Proof. Let (X, cl) be a Z-connected isotonic space, and let $f : X \to \{0, 1\}$ be a continuous function, where $\{0, 1\}$ is a T_1-isotonic space. Since Z is a neighborhood space and has more than one element, then there exists a continuous injection $g : \{0, 1\} \to Z$. By Lemma 3.2, $g \circ f : X \to Z$ is continuous. Hence $g \circ f$ is constant, since X is Z-connected. It follows that f is constant, and therefore, by Theorem 5.3, X is connected. □

Lemma 6.2 *Any continuous image of a Z-connected isotonic space is Z-connected.*

Proof. Let (X, cl) be a Z-connected isotonic space, (Y, cl) be an isotonic space, $f : X \to Y$ be a continuous surjection , and $g : Y \to Z$ be a continuous function. Then $g \circ f : X \to Z$ is continuous, and hence, by Z-connectedness, $g \circ f$ is constant. It follows that g is constant and therefore Y is Z-connected. □

Lemma 6.3 *Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a family of Z-connected subsets of an isotonic space (X, cl) such that $\bigcap_{\lambda\in\Lambda} A_\lambda \neq \emptyset$. Then $\bigcup_{\lambda\in\Lambda} A_\lambda$ is Z-connected.*

Proof. The proof is similar to the proof of Lemma 5.6. □

Theorem 6.1 *If (X, cl) is an isotonic space with the property that for any distinct points $x, y \in X$ there exist Z-connected subsets A_1, A_2, ..., A_n such that $x \in A_1$, $y \in A_n$ and $A_i \cap A_{i+1} \neq \emptyset$ for all $i = 1, 2, ..., n-1$, then X is Z-connected.*

Proof. The proof is similar to the proof of Theorem 5.6. □

Theorem 6.2 *If Z is a totally disconnected neighborhood space, then Z-connectedness for isotonic spaces is equivalent to connectedness.*

Proof. Let X be an isotonic space. By Lemma 6.1, if X is Z-connected, then it is connected. Conversely, if X is connected, then, by Theorem 5.4, for any continuous function $f : X \to Z$, $f(X)$ is connected. Since Z is totally disconnected, f must be constant, and therefore X is Z-connected. $\square$

7. Strongly Connected Isotonic Spaces

Definition 7.1 A space (X, cl) is *strongly connected* if there is no countable collection of pairwise semi-separated sets $\{A_i\}$ such that $X = \bigcup A_i$.

Lemma 7.1 *A strongly connected isotonic space is connected.*

Proof. It is immediate from Definition 5.2 and Definition 7.1. $\square$

It should be noted that the converse of Lemma 7.1 is false, as the following example shows.

Example 7.1 Let $X = \{a, b, c\}$ and define $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows: $cl(\emptyset) = \emptyset$, $cl(X) = X$, $cl(\{a\}) = \{a\}$, $cl(\{b\}) = \{b\}$, $cl(\{c\}) = \{c\}$, $cl(\{a, b\}) = X$, $cl(\{a, c\}) = X$, $cl(\{b, c\}) = X$. It is easy to check that X is a connected isotonic space. However, X is not strongly connected, since there are pairwise semi-separated sets $\{a\}$, $\{b\}$ and $\{c\}$ such that $X = \{a\} \cup \{b\} \cup \{c\}$.

Lemma 7.2 *A continuous image of a strongly connected isotonic space is strongly connected.*

Proof. Let $f : (X, cl) \to (Y, cl)$ be a continuous function between isotonic spaces and suppose $f(X)$ is not strongly connected. Then there exists a countable collection of pairwise semi-separated sets $\{A_i\}$ such that $f(X) = \bigcup A_i$. Since $f^{-1}(A_i) \cap cl_X(f^{-1}(A_j)) \subseteq f^{-1}(A_i) \cap f^{-1}(cl_Y(A_j)) = \emptyset$ for all $i \neq j$, then the collection $\{f^{-1}(A_i)\}$ is pairwise semi-separated. Thus we have a contradiction since X is strongly connected. Therefore $f(X)$ is strongly connected. $\square$

Theorem 7.1 *An isotonic space (X, cl) is strongly connected if and only if it is Z-connected, for any countable T_1-isotonic space (Z, cl).*

Proof. Suppose (X, cl) is strongly connected but not Z-connected for some countable T_1-isotonic space (Z, cl). Then there exists a continuous function $f : X \to Z$ which is not constant, and hence $B := f(X)$ is a countable set with more than one element. Now for each $b_i \in B$ there is $A_i \subseteq X$ such that $A_i = f^{-1}(\{b_i\})$, and hence $X = \bigcup A_i$. Since f is continuous and Z is (T_1), then for each $i \neq j$, $A_i \cap cl_X(A_j) = f^{-1}(\{b_i\}) \cap cl_X(f^{-1}(\{b_j\})) \subseteq f^{-1}(\{b_i\}) \cap f^{-1}(cl_Z(\{b_j\})) \subseteq f^{-1}(\{b_i\}) \cap f^{-1}(\{b_j\}) = \emptyset$, contradicting the strong connectedness of X. Therefore X is Z-connected.

Conversely, Suppose X is Z-connected for any countable T_1-isotonic space (Z, cl) , but X is not strongly connected. Then there exists a countable collection of pairwise semi-separated sets $\{A_i\}$ such that $X = \bigcup A_i$. Now consider the space $(\mathbb{Z}, cl)$, where $\mathbb{Z}$ is the set of integers and $cl : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$is defined by $cl_{\mathbb{Z}}(B) = B$ for each $B \subseteq \mathbb{Z}$. Clearly $(\mathbb{Z}, cl)$ is a countable T_1-isotonic space. Fix $A_0 \in \{A_i\}$.

Claim: The function $f : X \to \mathbb{Z}$ such that $f(A_0) = \{a\}$ and $f(X \setminus A_0) = \{b\}$ where $a, b \in \mathbb{Z}$ and $a \neq b$ is continuous.

To see this, note that since $cl_X(A_0) \cap A_i = \emptyset$ for all $i \neq 0$, we have $cl_X(A_0) \cap \bigcup_{i \neq 0} A_i = \emptyset$, and hence $cl_X(A_0) \subseteq A_0$. Let $\emptyset \neq B \subseteq \mathbb{Z}$. Then we have three cases:

Case 1. $a, b \in B$. In this case, $f^{-1}(B) = X$ and hence $cl_X(f^{-1}(B)) = cl_X(X) \subseteq X = f^{-1}(B) = f^{-1}(cl_{\mathbb{Z}}(B))$.

Case 2. $a \in B$ and $b \notin B$. In this case, $f^{-1}(B) = A_0$ and hence $cl_X(f^{-1}(B)) = cl_X(A_0) \subseteq A_0 = f^{-1}(B) = f^{-1}(cl_{\mathbb{Z}}(B))$.

Case 3. $b \in B$ and $a \notin B$. In this case, $f^{-1}(B) = X \setminus A_0$. Since $cl_Z(B) = B$ for each $B \subseteq \mathbb{Z}$, we have $int_{\mathbb{Z}}(B) = B$ for each $B \subseteq \mathbb{Z}$. Now $X \setminus A_0 \subseteq \bigcup_{i \neq 0} A_i \subseteq X \setminus cl_X(A_0) = int_X(X \setminus A_0)$. Hence, $f^{-1}(int_{\mathbb{Z}}(B)) = X \setminus A_0 = f^{-1}(B) \subseteq int_X(X \setminus A_0) = int_X(f^{-1}(B))$.

Thus, by Proposition 3.1, f is continuous. This proves the claim. Since f is not constant, we have a contradiction to the $\mathbb{Z}$-connectedness of X. Therefore X is strongly connected.□

Lemma 7.3 *Let* $\{A_\lambda \mid \lambda \in \Lambda\}$ *be a family of strongly connected subsets of an isotonic space* (X, cl) *such that* $\bigcap_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. *Then* $\bigcup_{\lambda \in \Lambda} A_\lambda$ *is strongly connected.*

Proof. The proof is similar to the proof of Lemma 5.6. □

Theorem 7.2 *If* (X, cl) *is an isotonic space with the property that for any distinct points* $x, y \in X$ *there exist strongly connected subsets* $A_1, A_2, ..., A_n$ *such that* $x \in A_1$, $y \in A_n$ *and* $A_i \cap A_{i+1} \neq \emptyset$ *for all* $i = 1, 2, ..., n-1$, *then* X *is strongly connected.*

Proof. The proof is similar to the proof of Theorem 5.6. □

References

[1] Bo, D., Yan-loi, W.: *Strongly Connected Spaces.* Department of Mathematics, National University of Singapore, 1999.

[2] Brandsma, H.: *Connectedness I.* Topology Atlas, Topology Explained, pp. 1, November 2003.

[3] Day, M. M.: *Convergence, closure an neighborhood.* Duke Math. J., **11**, 181–199 (1994).

[4] Deshpands, J. V.: *Introduction to Toplogy.* Tata McGraw-hill Publishing company Limited, 1988.

[5] Dugundji, J.: *Topology.* Wm. C. Brown Publishers, 1989.

[6] Gnilka, S.: *On extended topologies I: Closure operators.* Ann. Soc. Math. Pol., Ser. I, commented. Math. 34, 81-94 (1994).

[7] Gnilka, S.: *On extended topologies II: Compactness, quasi-metrizability, symmetry.* Ann. Soc. Math. Pol., Ser. I, commented. Math. 35, 147-162 (1995).

[8] Gnilka, S. : *On continuity in extended topologies.* Ann. Soc. Math. Pol., Ser. I, commented. Math. 37, 99-108 (1997).

[9] Kelley, J: *General Topology.* Springer-Verlag NewYork Heidelberg Berlin, 1955.

[10] Kent D. C., Won-Keum Min: *Neighborhood spaces.* IJMMS **32:7**, 387-399 (2002).

[11] Hammer, P. C.: *Extended topology: Continuity I.* Portug. Math. 25,77-93 (1964).

[12] Hammer, P. C.: *Extended topology: Set-valued set functions.* Nieuw Arch. Wisk. III. 10, 55-77 (1962).

[13] Hausdorff, F.: *Gestufte raume.* Fund. Math. 25, 486-502 (1935).

[14] Pisani, C.: *Convergence in exponentiable spaces.* Theory and Applications of Categories 5. No. 6, 148162 (1999).

[15] Stadler, B. M. R., Stadler, P. F.: *Basic properties of closure spaces.* J. Chem. Inf. Comput. Sci. 42, 577-585 (2002).

[16] Stadler, B. M. R., Stadler, P. F.: *Higher separation axioms in generalized closure spaces.* Commentationes Mathematicae Warszawa, Ser. I. 43, 257-273 (2003).

[17] Willard, S.: *General Topology.* Addsion-Wesley Publishing Company, Inc, 1970.

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