The Trace Formula for a Differential Operator of Fourth Order With Bounded Operator Coefficients and Two Terms

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Abstract

We investigate the spectrum of a differential operator of fourth order with bounded operator coefficients and find a formula for the trace of this operator.

Key Words: Hilbert Space, Self-adjoint operator, Kernel operator, Spectrum, Essential spectrum, Resolvent.

1. Introduction

Let $H$ be a separable Hilbert space of infinite dimension. Consider the operators $L_0$ and $L$ in the space $H_1 = L_2(H;[0,\pi])$ which are formed by differential expressions

$$l_0(y) = y''(x), \quad l(y) = y''(x) + Q(x)y(x)$$

with the same boundary conditions $y'(0) = y'(\pi) = 0$ and $y'''(0) = y'''(\pi) = 0$, respectively. Suppose that the operator function $Q(x)$ in the expression $l(y)$ satisfies the following conditions:

1. For every $x \in [0,\pi]$, $Q(x) : H \rightarrow H$ is a self adjoint kernel operator. Moreover, $Q(x)$ has weak derivative of second order in this interval and for $x \in [0,\pi]$, $Q^{(i)}(x) : H \rightarrow H$ are self-adjoint operators ($i = 1, 2$).

2. $||Q||_{H_1} < \frac{1}{2}$

3. There is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of the space $H$ such that

$$\sum_{n=1}^{\infty} ||Q(x)\varphi_n||_{H_1} < \infty.$$
4. The functions $|Q^{(i)}(x)||\sigma_1(H)$ are bounded and measurable functions in $[0, \pi]$, ($i = 0, 1, 2$).

Here $\sigma_1(H)$ is the space of kernel operators from $H$ to $H$ as in [16]. Moreover, we denote the norms by $\| \cdot \|$ and $\| \cdot \|_{H_1}$ and the inner products by $(\cdot, \cdot)$ and $(\cdot, \cdot)_{H_1}$ in $H$ and $H_1$, respectively and also denote the sum of eigenvalues of a kernel operator $A$ by $\text{tr}A = \text{trace}A$.

The spectrum of operator $L_0$ is the set $\{m^4\}_{m=0}^{\infty}$. Every point of this set is an eigenvalue of $L_0$ which has infinite multiplicity. The orthonormal eigenfunctions corresponding to eigenvalue $m^4$ are in the form

$$\psi_{mn}^0(x) = d_m \cos mx \cdot \varphi_n \quad (n = 1, 2, \ldots)$$

where

$$d_m = \begin{cases} \frac{\sqrt{2}}{\pi} & m = 0 \\ \frac{1}{2} & m = 1, 2, \ldots \end{cases}$$

In this work, we will firstly investigate the spectrum of operator $L$ and find a formula for the sum of the series

$$\sum_{m=0}^{\infty} \left[ \sum_{n=1}^{\infty} (\lambda_{mn} - m^4) - \frac{1}{\pi} \int_0^{\pi} \text{tr}Q(x)dx \right],$$

where $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of operator $L$ which belong to the interval $[m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}]$ ($m = 0, 1, 2, \ldots$).

Trace formulas for the scalar differential operators have been found by Gelfand and Levitan [1], Dikiy [2], Halberg and Kramer [3], Levitan [4], Lidskiy and Sadovnichiy [5], Guseynov and Levitan [6] and many others. A list of the works on this subject is presented by Levitan and Sargsyan [7] and Fulton and Pruess [8]. On the other hand, trace formulas for differential operators with operator coefficients has been investigated by Adigüzelov [9], Chalilova [10], Maksudov, Bayramoglu and Adigüzelov [11], Adigüzelov, Avci and Gül [12], Albayrak, Baykal and Gül [13] and Maksudov, Bairamoglu and Adigezalov [17]. A trace formula for higher order, including fourth order, differential operators with operator coefficients has been given in [17]. It is this latter problem we study in the present work, but with differential operators and boundary conditions different from those in [17].
2. The Spectrum of Operator $L$

Let $R^0_\lambda$ and $R_\lambda$ be resolvents of the operators $L_0$ and $L$, respectively.

**Lemma 1** If the operator function $Q(x)$ satisfies condition 3, and $\lambda \notin \{m^4\}_{m=0}^\infty = \sigma(L_0)$, then $QR^0_\lambda : H_1 \to H_1$ is a kernel operator, i.e. $QR^0_\lambda \in \sigma_1(H_1)$.

**Proof.** System (1) of the eigenfunctions of $L_0$ is an orthonormal basis of space $H_1$. As known in [16], to show that $QR^0_\lambda$ is a kernel operator, it is enough to see that the series

$$\sum_{m=0}^\infty \sum_{n=1}^\infty ||QR^0_\lambda \psi_{mn}||_{H_1}$$

is convergent. From (1) and (2), we find

$$\sum_{m=0}^\infty \sum_{n=1}^\infty ||QR^0_\lambda \psi_{mn}||_{H_1} = \sum_{m=0}^\infty \sum_{n=1}^\infty |m^4 - \lambda|^{-1} \cdot ||Q \psi_{mn}||_{H_1}$$

$$= \sum_{m=0}^\infty \sum_{n=1}^\infty |m^4 - \lambda|^{-1} \left[ \int_0^\pi (Q(x)d_m \cos mx \cdot \varphi_n, Q(x)d_m \cos mx \cdot \varphi_n) \right]^\frac{1}{2}$$

$$= \sum_{m=0}^\infty \sum_{n=1}^\infty |m^4 - \lambda|^{-1} \left[ \int_0^\pi d_m^2 \cos^2 mx ||Q(x)\varphi_n||^2 dx \right]^\frac{1}{2}$$

$$\leq \sum_{m=0}^\infty \sum_{n=1}^\infty |m^4 - \lambda|^{-1} \left[ \int_0^\pi ||Q(x)\varphi_n||^2 dx \right]^\frac{1}{2}$$

$$= \sum_{m=0}^\infty |m^4 - \lambda|^{-1} \sum_{n=1}^\infty ||Q(x)\varphi_n||_{H_1}. \quad (4)$$

In view of (4) and condition 3 we conclude that

$$\sum_{m=0}^\infty \sum_{n=1}^\infty ||QR^0_\lambda \psi_{mn}||_{H_1} < \infty.$$
This proves the lemma □

**Theorem 2** If $Q(x)$ satisfies conditions 2, and 3, then the spectrum of operator $L$ is a subset of the union of pairwise disjoint intervals $F_m = [m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}]$ ($m = 0, 1, 2, ...$); and the following conditions are satisfied:

1. Each point of spectrum of $L$ which is different from $m^4$ in $F_m$ is an isolated eigenvalue which has finite multiplicity.

2. $m^4$ can be an eigenvalue of $L$ which has finite or infinite multiplicity.

3. $\lim_{n \to \infty} \lambda_{mn} = m^4$ such that $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of $L$ in $F_m$.

**Proof.** If

$$\lambda \in R \setminus \bigcup_{m=0}^{\infty} [m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}],$$

then we get

$$|\lambda - m^4| > ||Q||_{H_1} \quad (m = 0, 1, 2, ...). \quad (5)$$

For the self adjoint operator $R_\lambda^0 = (L_0 - \lambda I)^{-1}$, since $||R_\lambda^0||_{H_1} = \max_m |\lambda - m^4|^{-1}$, then from (5) we can write

$$||R_\lambda^0||_{H_1} < ||Q||_{H_1}^{-1}.$$ 

Because of this, we have

$$||QR_\lambda^0||_{H_1} \leq ||Q||_{H_1} \cdot ||R_\lambda^0||_{H_1} < 1.$$ 

By considering this inequality, we conclude that

$$A(B) = R_\lambda^0 - BQR_\lambda^0$$

is a contraction operator from $L(H_1, H_1)$ to $L(H_1, H_1)$, where $B \in L(H_1, H_1)$. In this case, it is known that there exists a unique solution $B = B_0$ which belongs to the space $L(H_1, H_1)$ of the equation $R_\lambda^0 - BQR_\lambda^0 = B$. Moreover, since $R_\lambda^0 - R_\lambda Q R_\lambda^0 = R_\lambda$ we have $R_\lambda = B_0 \in L(H_1, H_1)$ and so $\lambda \in \rho(L)$ (resolvent set of $L$). Hence, the spectrum of $L$ is a subset of the union of the pairwise disjoint intervals $[m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}]$, 234
(m = 0, 1, 2, ...), i.e. \( \sigma(L) \subset \bigcup_{m=0}^{\infty} [m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}] \). From Lemma 1 and the equation \( R_\lambda = R^0_\lambda - R_\lambda QR^0_\lambda \), for every \( \lambda \in \rho(L_0) \cap \rho(L) \) we see that \( R_\lambda - R^0_\lambda \) is a kernel operator from \( H_1 \) to \( H_1 \). This means that, as known from [14], the essential spectra of \( L \) and \( L_0 \) coincide. According to this, and since \( L_0 \) has only the essential spectrum, the essential spectrum of \( L \) will be the set \( \{ m^4 \}_{m=0}^{\infty} \) and this shows that conditions 1, 2, 3, in the hypothesis of theorem 2 are satisfied. \( \square \)

3. A Formula for the Trace of \( L \)

In this section, we obtain a formula for the sum of series (3). The sum of this series is called the regularized trace of operator \( L \).

**Lemma 3** If \( Q(x) \) satisfies conditions 2, and 3., then operator function \( R_\lambda - R^0_\lambda \) is analytic in the region \( \rho(L) \) with respect to the norm in \( \sigma_1(H_1) \).

**Proof.** Since \( R_\lambda - R^0_\lambda = -R_\lambda QR^0_\lambda \), to prove this lemma we need to show that the operator function \( R_\lambda QR^0_\lambda \) is analytic in the region \( \rho(L) \). First, from Theorem 2, it follows that \( \rho(L) \subset \rho(L_0) \). Moreover, by using the relation \( R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu \) we have

\[
D(\lambda, \Delta \lambda) = \frac{R_\lambda + \Delta \lambda QR^0_\lambda + \Delta \lambda - R_\lambda QR^0_\lambda}{\Delta \lambda} - R^2_\lambda QR^0_\lambda - R_\lambda Q(R^0_\lambda)^2
\]

\[
= \frac{1}{\Delta \lambda}[(R_\lambda + \Delta \lambda QR^0_\lambda + \Delta \lambda - R_\lambda + \Delta \lambda QR^0_\lambda) + (R_\lambda + \Delta \lambda QR^0_\lambda)
- R_\lambda QR^0_\lambda)] - R^2_\lambda QR^0_\lambda - R_\lambda Q(R^0_\lambda)^2
\]

\[
= \frac{1}{\Delta \lambda} R_\lambda + \Delta \lambda Q(R^0_\lambda + \Delta \lambda - R^0_\lambda) + \frac{1}{\Delta \lambda}(R_\lambda + \Delta \lambda - R_\lambda)Q(R^0_\lambda)
- R^2_\lambda QR^0_\lambda - R_\lambda Q(R^0_\lambda)^2
\]

\[
= R_\lambda + \Delta \lambda QR^0_\lambda R^0_\lambda + \Delta \lambda + R_\lambda + \Delta \lambda R_\lambda QR^0_\lambda - R^2_\lambda QR^0_\lambda - R_\lambda Q(R^0_\lambda)^2
\]

\[
= [R_\lambda + \Delta \lambda QR^0_\lambda R^0_\lambda + \Delta \lambda - R_\lambda + \Delta \lambda Q(R^0_\lambda)^2] + [R_\lambda + \Delta \lambda (R^0_\lambda)^2]
- R_\lambda Q(R^0_\lambda)^2 + (R_\lambda + \Delta \lambda R_\lambda QR^0_\lambda - R^2_\lambda QR^0_\lambda)
\]

\[
= R_\lambda + \Delta \lambda QR^0_\lambda (R^0_\lambda + \Delta \lambda - R^0_\lambda) + (R_\lambda + \Delta \lambda - R_\lambda)Q(R^0_\lambda)^2
\]

\[
+ (R_\lambda + \Delta \lambda - R_\lambda)R_\lambda QR^0_\lambda. \quad (6)
\]
From this, we obtain
\[
||D(\lambda, \Delta\lambda)||_{\sigma_1(H_1)} \leq \ ||R_{\lambda+\Delta\lambda} Q R_{\lambda}^0||_{\sigma_1(H_1)} ||R_{\lambda+\Delta\lambda} - R_{\lambda}^0||_{H_1} \\
+ ||R_{\lambda+\Delta\lambda} - R_{\lambda}||_{H_1} ||Q(R_{\lambda}^0)^2||_{\sigma_1(H_1)} \\
+ ||R_{\lambda} Q R_{\lambda}^0||_{\sigma_1(H_1)} \\
\leq \ ||R_{\lambda+\Delta\lambda}||_{H_1} ||Q R_{\lambda}^0||_{\sigma_1(H_1)} ||R_{\lambda+\Delta\lambda} - R_{\lambda}^0||_{H_1} \\
+ ||R_{\lambda+\Delta\lambda} - R_{\lambda}||_{H_1} ||Q R_{\lambda}^0||_{\sigma_1(H_1)} ||R_{\lambda}^0||_{H_1} \\
+ ||R_{\lambda}||_{H_1}. \tag{7}
\]

Since
\[
\lim_{\Delta\lambda \to \infty} ||R_{\lambda+\Delta\lambda} - R_{\lambda}||_{H_1} = \lim_{\Delta\lambda \to \infty} ||R_{\lambda+\Delta\lambda} - R_{\lambda}^0||_{H_1} = 0,
\]
and from (6) and (7), we find
\[
\lim_{\Delta\lambda \to \infty} \frac{||R_{\lambda+\Delta\lambda} Q R_{\lambda+\Delta\lambda} - R_{\lambda} Q R_{\lambda}^0 - R_{\lambda}^2 Q R_{\lambda} - R_{\lambda} Q(R_{\lambda}^0)^2||_{\sigma_1(H_1)}}{\Delta\lambda} = 0.
\]

This shows that the operator function \( R_{\lambda} - R_{\lambda}^0 = -R_{\lambda} Q R_{\lambda}^0 \) is analytic in the region \( \rho(L) \) with respect to the norm in \( \sigma_1(H_1) \), as desired. \( \square \)

Let \( \{\psi_{mn}(x)\}_{m,n=1}^{\infty} \) be orthonormal eigenfunctions corresponding to eigenvalues \( \{\lambda_{mn}\}_{m,n=1}^{\infty} \) of \( L \) and let
\[
\Gamma_p = \{\lambda : |\lambda - p^4| = \frac{1}{2}\}; \quad B_{mn}^0 = \langle \cdot, \psi_{mn}^0 \rangle_{H_1}; \quad B_{mn} = \langle \cdot, \psi_{mn} \rangle_{H_1};
\]
and
\[
L^{(r)}_{0m} = \sum_{n=1}^{\infty} m^{4r} B_{mn}^0; \quad L^{(r)}_m = \sum_{n=1}^{\infty} \lambda_{mn}^r B_{mn}, \quad (r = -1, 1).
\]

The spectra of operators \( L \) and \( L_0 \) only consist of eigenvalues and its limit points. Hence, from [15], we know that
\[
R_{\lambda}^0 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{B_{mn}^0}{m^4 - \lambda}; \quad R_{\lambda} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{B_{mn}}{\lambda_{mn} - \lambda}. \tag{8}
\]

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Theorem 4  If $Q(x)$ satisfies conditions 2 and 3, then the series
\[
\sum_{n=1}^{\infty} (\lambda_{pn} - p^4) \quad (p = 0, 1, \ldots)
\]
are absolutely convergent.

Proof. Since \( \{\lambda_{mn}\}_{n=1}^{\infty} \subseteq [m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}] \), and from the assumption
\( ||Q||_{H_1} < \frac{1}{2} \), for \( m < p \), we have
\[
\lambda_{mn} \leq m^4 + ||Q||_{H_1} < m^4 + \frac{1}{2} \leq (m + 1)^4 - \frac{1}{2} \leq p^4 - \frac{1}{2} \quad (n = 1, 2, \ldots).
\]
Thus we find
\[
\lambda_{mn} < p^4 - \frac{1}{2} \quad \text{or} \quad |\lambda_{mn} - p^4| > \frac{1}{2} \quad (m < p; n = 1, 2, \ldots). \quad (9)
\]
For \( p < m \),
\[
p^4 + \frac{1}{2} \leq (p + 1)^4 - \frac{1}{2} \leq m^4 - \frac{1}{2} < m^4 - ||Q||_{H_1} \leq \lambda_{mn} \quad (n = 1, 2, \ldots).
\]
and so we obtain
\[
\lambda_{mn} > p^4 + \frac{1}{2} \quad \text{or} \quad |\lambda_{mn} - p^4| > \frac{1}{2} \quad (m > p; n = 1, 2, \ldots). \quad (10)
\]
By using (8), (9) and (10), we have
\[
\frac{1}{2\pi i} \int_{\Gamma_p} \lambda (R_\lambda - R_\lambda^0) d\lambda = \frac{1}{2\pi i} \int_{\Gamma_p} \lambda [\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{B_{mn}}{\lambda_{mn} - \lambda} - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{B_{mn}^0}{m^4 - \lambda} d\lambda]
\]
\[
= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [B_{mn} \frac{1}{2\pi i} \int_{\Gamma_p} \frac{\lambda}{\lambda_{mn} - \lambda} d\lambda - B_{mn}^0 \frac{1}{2\pi i} \int_{\Gamma_p} \frac{\lambda}{m^4 - \lambda} d\lambda]
\]
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\[ \sum_{n=1}^{\infty} \left[ B_{pn}^0 \frac{1}{2\pi i} \int_{\Gamma_p} \frac{\lambda}{\lambda - p^4} d\lambda - B_{pn} \frac{1}{2\pi i} \int_{\Gamma_p} \frac{\lambda}{\lambda - \lambda_{pn}} d\lambda \right] \]

\[ = \sum_{n=1}^{\infty} \left( p^4 B_{pn}^0 - \lambda_{pn} B_{pn} \right) = L_{0p}^{(1)} - L_{p}^{(1)} \quad p = 0, 1, \ldots \]

From this last relation and Lemma 3 we obtain

\[ L_{0p}^{(1)} - L_{p}^{(1)} \in \sigma_1(H_1) \quad (p = 0, 1, 2, \ldots). \] \hfill (11)

This time, let us show that \( L_{p}^{(-1)} - L_{0p}^{(-1)} \in \sigma_1(H_1) \). Again if we use (8), (9) and (10) we find

\[ \frac{1}{2\pi i} \int_{\Gamma_p} \lambda^{-1} (R_{\lambda} - R_{\lambda}^0) d\lambda = \sum_{n=0}^{\infty} \sum_{n=1}^{\infty} \left[ B_{mn} \frac{1}{2\pi i} \int_{\Gamma_p} \frac{d\lambda}{\lambda (m^4 - \lambda)} \right. \]

\[ - B_{pn}^0 \frac{1}{2\pi i} \int_{\Gamma_p} \frac{d\lambda}{\lambda (m^4 - \lambda)} \]

\[ = \sum_{n=1}^{\infty} \left[ B_{pn}^0 \frac{1}{2\pi i} \int_{\Gamma_p} \frac{d\lambda}{\lambda (\lambda - p^4)} \right. \]

\[ - B_{pn} \frac{1}{2\pi i} \int_{\Gamma_p} \frac{d\lambda}{\lambda (\lambda - \lambda_{pn})} \]

\[ = \sum_{n=1}^{\infty} \left( p^{-4} B_{mn}^0 - \lambda_{pn}^{-1} B_{pn} \right) \]

\[ = L_{0p}^{(-1)} - L_{p}^{(-1)} \quad (p = 1, 2, \ldots). \] \hfill (12)

According to Lemma 3, the operator function \( \lambda^{-1} (R_{\lambda} - R_{\lambda}^0) \) is analytic in the region \( \rho(L) \) with respect to the norm in \( \sigma_1(H_1) \). Hence, from (12)

\[ L_{0p}^{(-1)} - L_{p}^{(-1)} \in \sigma_1(H_1) \quad (p = 1, 2, \ldots). \] \hfill (13)

Now, we can show that the series \( \sum_{n=1}^{\infty} (\lambda_{pn} - p^4) \quad (p = 0, 1, \ldots) \) are convergent. The spectrum of the operator \( L_{0p}^{(r)} \) only consist of the points 0 and \( p^4r \). In this case, from [15], we write

\[ p^{4r} \geq (L_{0p}^{(r)} \psi_{pn}, \psi_{pn})_{H_1} \quad (p = 1, 2, \ldots). \]

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On the other hand, 

\[ \lambda^n_{pn} = (L_p^{(r)} \psi_{pn}, \psi_{pn})_{H_1}. \]

By using last two relation, we find

\[ \sum_{n} (\lambda^n_{pn} - p^{4r})_{\lambda^n_{pn} > p^{4r}} \leq \sum_{n} ((L_p^{(r)} - L_{0p}^{(r)}) \psi_{pn}, \psi_{pn})_{H_1} \]

\[ \leq \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left| ((L_p^{(r)} - L_{0p}^{(r)}) \psi_{mn}, \psi_{mn})_{H_1} \right| \]

(14)

From (11) and (13) we have \( L_p^{(r)} - L_{0p}^{(r)} \in \sigma_1(H_1) \) \((r = 1, -1; p = 1, 2, \ldots)\). Hence, from [16] we write

\[ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left| ((L_p^{(r)} - L_{0p}^{(r)}) \psi_{mn}, \psi_{mn})_{H_1} \right| \leq \left\| L_p^{(r)} - L_{0p}^{(r)} \right\|_{\sigma_1(H_1)}. \]

(15)

From (14) and (15) we find

\[ \sum_{n} (\lambda^n_{pn} - p^{4r})_{\lambda^n_{pn} > p^{4r}} \leq \left\| L_p^{(r)} - L_{0p}^{(r)} \right\|_{\sigma_1(H_1)} \quad (p \geq 1) \]

(16)

\[ \sum_{n} (\lambda_{pn} - p^{4})_{\lambda_{pn} > p^{4}} < \infty \quad (p \geq 1) \]

and

\[ \sum_{n} (p^{4} - \lambda_{pn})_{\lambda_{pn} < p^{4}} \leq \text{const} \sum_{n} (p^{4} - \lambda_{pn})_{\lambda_{pn} < p^{4}} p^{-4} \lambda_{pn}^{-1} \quad (p \geq 1) \]

\[ = \text{const} \sum_{n} (\lambda_{pn}^{-1} - p^{-4})_{\lambda_{pn}^{-1} > p^{-4}} < \infty \quad (p \geq 1). \]

(17)

From (16) and (17) we have

\[ \sum_{n=1}^{\infty} |\lambda_{pn} - p^{4}| < \infty \quad (p \geq 1). \]
Moreover, by considering \(L_0^{(1)} = 0\) and (11) we obtain

\[
\sum_{n=1}^{\infty} |\lambda_{0n}| < \infty.
\]

This proves Theorem 4

\[\square\]

For every \(\lambda \in \rho(L)\), since \(R_\lambda - R_\lambda^0 \in \sigma_1(H_1)\) and from (8) and Theorem 4 we find

\[
\text{tr}(R_\lambda - R_\lambda^0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_{mn} - \lambda} - \frac{1}{m^4 - \lambda} \right).
\]

Let us multiply this equation by \(\frac{1}{2\pi i}\) and integrate on the circle \(|\lambda| = b_p = p^4 + 2p^3\), \((p \geq 1)\):

\[
\frac{1}{2\pi i} \int_{|\lambda| = b_p} \lambda \cdot \text{tr}(R_\lambda - R_\lambda^0) = \frac{1}{2\pi i} \int_{|\lambda| = b_p} \lambda \cdot \sum_{m=0}^{p} \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_{mn} - \lambda} - \frac{1}{m^4 - \lambda} \right) d\lambda
\]

\[
+ \frac{1}{2\pi i} \int_{|\lambda| = b_p} \lambda \cdot \sum_{m=p+1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_{mn} - \lambda} - \frac{1}{m^4 - \lambda} \right) d\lambda.
\]

On the other hand, for \(m \leq p\) and \(p \geq 1\) we have

\[
m^4 - ||Q||_{H_1} \leq \lambda_{mn} \leq m^4 + ||Q||_{H_1} \leq p^4 + ||Q||_{H_1} < p^4 + 2p^3 = b_p
\]

and so

\[
|\lambda_{mn}| < b_p, \quad m \leq p; \quad p \geq 1; \quad n = 1, 2, \ldots
\]

and for \(m > p\) we have

\[
\lambda_{mn} \geq m^4 - ||Q||_{H_1} \geq (p + 1)^4 - ||Q||_{H_1} > p^4 + 2p^3 = b_p
\]
or

\[ |\lambda_{mn}| > b_p, \quad m > p; \quad p \geq 1; \quad n = 1, 2, \ldots \]  \hspace{1cm} \text{(20)}

Hence, from (18), (19) and (20) we obtain

\[
\frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \cdot \text{tr}(R_\lambda - R_\lambda^0) d\lambda = \sum_{m=0}^{p} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{\lambda - m^4} d\lambda \\
- \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{\lambda - \lambda_{mn}} d\lambda \\
+ \sum_{m=p+1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{\lambda - m^4} d\lambda \\
- \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{\lambda - \lambda_{mn}} d\lambda \\
= \sum_{m=0}^{p} \sum_{n=1}^{\infty} \left( m^4 - \lambda_{mn} \right). \hspace{1cm} \text{(21)}
\]

By using the formula \( R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0 \), we find

\[ R_\lambda - R_\lambda^0 = \sum_{j=1}^{2} (-1)^j R_\lambda^0 (Q R_\lambda^0)^j - R_\lambda (Q R_\lambda^0)^3. \]

If we put this expression in equation (21), we have

\[
\sum_{m=0}^{p} \sum_{n=1}^{\infty} \left( m^4 - \lambda_{mn} \right) = \sum_{j=1}^{2} \frac{(-1)^j}{2\pi i} \int_{|\lambda|=b_p} \lambda \cdot \text{tr} [R_\lambda^0 (Q R_\lambda^0)^j] d\lambda \\
- \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \cdot \text{tr} [R_\lambda (Q R_\lambda^0)^3] d\lambda. \hspace{1cm} \text{(22)}
\]

Now, let

\[ M_{pj} = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_p} \lambda \cdot \text{tr} [R_\lambda^0 (Q R_\lambda^0)^j] d\lambda \quad (j = 1, 2) \hspace{1cm} \text{(23)}
\]
and

$$M_p = \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \cdot \text{tr}[R_\lambda(QR_\lambda^0)^3]d\lambda.$$  \hfill (24)

Then we can write equation (22) in the form

$$\sum_{m=0}^{P} \sum_{n=1}^{\infty} (\lambda_{mn} - m^4) = M_{p1} + M_{p2} + M_p$$ \hfill (25)

In a similar way to the proof of Lemma 3, we can prove that the operator function $QR_\lambda^0$ is analytic with respect to the norm in $\sigma_1(H_1)$ at every point $\lambda \neq m^4 (m = 0, 1, 2, \ldots)$; and so we can show that the formulas

$$M_{pj} = \frac{(-1)^j}{2\pi i} \int_{|\lambda|=b_p} \text{tr}[(QR_\lambda^0)^j]d\lambda \quad (j = 1, 2)$$ \hfill (26)

are satisfied.

**Lemma 5** If $Q(x)$ satisfies condition 3 and the function $||Q(x)||_{\sigma_1(H_1)}$ is integrable in the interval $[0, \pi]$, then the formula

$$M_{p1} = \frac{2p + 1}{2\pi} \int_0^\pi \text{tr}Q(x)dx + \frac{1}{2} \sum_{m=0}^{p} \sum_{n=1}^{\infty} d_m^2 \int_0^\pi (Q(x)\varphi_n, \varphi_n) \cos 2mxdx$$

holds.

**Proof.** Since the system (1) of eigenfunctions $\psi_{mn}^0 \quad (m = 0, 1, 2, \ldots; n = 1, 2, \ldots)$ corresponding to eigenvalue $m^4$ of operator $L_0$ is an orthonormal basis of space $H_1$, and by using the formula (26), we have

$$M_{p1} = -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \text{tr}(QR_\lambda^0)d\lambda$$

$$= -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (QR_\lambda^0\psi_{mn}^0, \psi_{mn}^0)_{H_1}d\lambda.$$ \hfill (27)
Taking advantage of (1) we can estimate the expression \(|(QR_\lambda^0 \psi_{mn}^0, \psi_{mn}^0)_{H_1}|\):

\[
|(QR_\lambda^0 \psi_{mn}^0, \psi_{mn}^0)_{H_1}| = \left| \int_0^\pi (QR_\lambda^0 \psi_{mn}^0(x), \psi_{mn}^0(x)) dx \right|
\]

\[
= |m^4 - \lambda|^{-1} \left| \int_0^\pi (Q(x)\psi_{mn}^0(x), \psi_{mn}^0(x)) dx \right|
\]

\[
= |m^4 - \lambda|^{-1} \cdot \left| \int_0^\pi (Q(x) d_m \cos mx \cdot \varphi_n, d_m \cos mx \cdot \varphi_n) dx \right|
\]

\[
\leq |m^4 - \lambda|^{-1} \left| \int_0^\pi (Q(x) \varphi_n, \varphi_n) dx \right|
\]

\[
\leq |m^4 - \lambda|^{-1} \int_0^\pi ||Q(x)\varphi_n|| dx
\]

\[
\leq \sqrt{\pi}|m^4 - \lambda|^{-1} \left( \int_0^\pi ||Q(x)\varphi_n||^2 dx \right)^{1/2}
\]

\[
= \sqrt{\pi}|m^4 - \lambda|^{-1} ||Q(x)\varphi_n||_{H_1}.
\]

Since \(Q(x)\) satisfies condition (3), and from this last estimation we conclude that the series

\[
\alpha_m(\lambda) = \sum_{n=1}^{\infty} (QR_\lambda^0 \psi_{mn}^0, \psi_{mn}^0)_{H_1} (m = 0, 1, 2, \ldots); \quad \sum_{m=0}^{\infty} \alpha_m(\lambda)
\]

are absolutely and uniformly convergent with respect to \(\lambda\) on the circle \(|\lambda| = b_p\). And so, from (27) we find

\[
M_{p1} = -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (QR_\lambda^0 \psi_{mn}^0, \psi_{mn}^0)_{H_1} d\lambda
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (Q \psi_{mn}^0, \psi_{mn}^0)_{H_1} \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{d\lambda}{\lambda - m^4}.
\]
By using (1), (19), (20) and this last relation we obtain

\[
M_{p1} = \sum_{m=0}^{p} \sum_{n=1}^{\infty} (Q\psi_{mn}^0, \psi_{mn}^0)_{H_1}
\]

\[
= \sum_{m=0}^{p} \sum_{n=1}^{\infty} \int_{0}^{\pi} (Q(x)d_m \cos mx \cdot \varphi_n, d_m \cos mx \cdot \varphi_n) \, dx
\]

\[
= \sum_{m=0}^{p} \sum_{n=1}^{\infty} d_m^2 \int_{0}^{\pi} (Q(x)\varphi_n, \varphi_n) \cos^2 mx \, dx
\]

\[
= \frac{1}{2} \sum_{m=0}^{p} \sum_{n=1}^{\infty} d_m^2 \int_{0}^{\pi} (Q(x)\varphi_n, \varphi_n)(1 + \cos 2mx) \, dx.
\] (28)

Moreover, since

\[
\left| \sum_{n=1}^{q} (Q(x)\varphi_n, \varphi_n) \right| \leq \sum_{n=1}^{\infty} |(Q(x)\varphi_n, \varphi_n)| \leq ||Q(x)||_{\sigma_i(H)} (q = 1, 2, \ldots),
\]

and by assumption since

\[
\int_{0}^{\pi} ||Q(x)||_{\sigma_i(H)} < \infty,
\]

and also by applying the Lebesgue theorem, we find

\[
\sum_{n=1}^{\infty} \int_{0}^{\pi} (Q(x)\varphi_n, \varphi_n) \, dx = \int_{0}^{\pi} \left[ \sum_{n=1}^{\infty} (Q(x)\varphi_n, \varphi_n) \right] \, dx = \int_{0}^{\pi} trQ(x) \, dx.
\] (29)

From (2), (28) and (29) we obtain

\[
M_{p1} = \frac{2p + 1}{2\pi} \int_{0}^{\pi} trQ(x) \, dx + \frac{1}{2} \sum_{m=0}^{p} \sum_{n=1}^{\infty} d_m^2 \int_{0}^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2mx \, dx.
\]

This proves lemma. \(\Box\)

Now, we want to show that

\[
\lim_{p \to \infty} M_{p2} = 0.
\] (30)
From (26), we find

\[ M_{p2} = \frac{1}{4\pi i} \int_{|\lambda|=b_p} \text{tr}[(QR_0^\lambda)^2] d\lambda \]

\[ = \frac{1}{4\pi i} \int_{|\lambda|=b_p} \left( \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [(QR_0^\lambda)^2 \psi_{mn}^0, \psi_{mn}^0] H_1 \right) d\lambda. \]  

(31)

Moreover, we have

\[ QR_0^\lambda \psi_{mn}^0 = \frac{Q\psi_{mn}^0}{m^4 - \lambda} \]

\[ (QR_0^\lambda)^2 \psi_{mn}^0 = (m^4 - \lambda)^{-1} QR_0^\lambda Q\psi_{mn}^0 \]

\[ = (m^4 - \lambda)^{-1} QR_0^\lambda \left\{ \sum_{r=0}^{\infty} \sum_{q=1}^{\infty} (Q\psi_{mn}^0, \psi_{rq}^0) H_1 \right\} \]

\[ = (m^4 - \lambda)^{-1} \left\{ \sum_{r=0}^{\infty} \sum_{q=1}^{\infty} (r^4 - \lambda)^{-1} (Q\psi_{mn}^0, \psi_{rq}^0) H_1 Q\psi_{rq}^0 \right\}. \]

If we put this expression in (31), we obtain

\[ M_{p2} = \frac{1}{4\pi i} \int_{|\lambda|=b_p} \left[ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} \sum_{q=1}^{\infty} \frac{(Q\psi_{mn}^0, \psi_{rq}^0) H_1 (Q\psi_{rq}^0, \psi_{mn}^0) H_1}{(\lambda - m^4)(\lambda - r^4)} \right] d\lambda. \]  

(32)

On the other hand

\[ \int_{|\lambda|=b_p} \frac{d\lambda}{(\lambda - m^4)(\lambda - r^4)} = 0; \ m, r \leq p \]  

(33)

In fact, if \( m = r \) then

\[ \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{d\lambda}{(\lambda - m^4)(\lambda - m^4)} = 0. \]

If \( m \neq r \) then there exists a small number \( \varepsilon > 0 \) such that
\[
\frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{d\lambda}{(\lambda - m^4)(\lambda - r^4)} = \frac{1}{2\pi i} \int_{|\lambda-m^4|=\varepsilon} \frac{d\lambda}{(\lambda - m^4)(\lambda - r^4)} + \frac{1}{2\pi i} \int_{|\lambda-r^4|=\varepsilon} \frac{d\lambda}{(\lambda - m^4)(\lambda - r^4)} = \frac{1}{m^4 - r^4} + \frac{1}{r^4 - m^4} = 0.
\]

So, for \(m, r > p\) since

\[
\int_{|\lambda|=b_p} \frac{d\lambda}{(\lambda - m^4)(\lambda - r^4)} = 0,
\]

and from (32) and (33), we find

\[
M_{p2} = \frac{1}{2\pi i} \sum_{m=0}^{p} \sum_{n=1}^{\infty} \sum_{r=p+1}^{\infty} \sum_{q=1}^{\infty} (Q_{\psi_{mn}, \psi_{rq}}^0) H_1 (Q_{\psi_{rq}, \psi_{mn}}^0) H_i 
\cdot \int_{|\lambda|=b_p} \frac{d\lambda}{(\lambda - m^4)(\lambda - r^4)}
= \sum_{m=0}^{p} \sum_{n=1}^{\infty} \sum_{r=p+1}^{\infty} \sum_{q=1}^{\infty} (Q_{\psi_{mn}, \psi_{rq}}^0) H_1 (Q_{\psi_{rq}, \psi_{mn}}^0) H_i
\cdot \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{1}{(\lambda - m^4 - r^4)} \left[ \frac{1}{(\lambda - m^4)} - \frac{1}{(\lambda - r^4)} \right] d\lambda
= \sum_{m=0}^{p} \sum_{n=1}^{\infty} \sum_{r=p+1}^{\infty} \sum_{q=1}^{\infty} (m^4 - r^4)^{-1} \left| (Q_{\psi_{mn}, \psi_{rq}}^0) H_i \right|^2.
\]

And from here we obtain
\[ |M_{p2}| = \sum_{m=0}^{p} \sum_{n=1}^{\infty} \sum_{r=p+1}^{\infty} \sum_{q=1}^{\infty} (r^4 - m^4)^{-1} |(Q\psi_{rq}^0, \psi_{mn}^0)_{H_1}|^2 \]

\[ \leq \sum_{r=p+1}^{\infty} \sum_{q=1}^{\infty} (r^4 - p^4)^{-1} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} |(Q\psi_{rq}^0, \psi_{mn}^0)_{H_1}|^2 \]

\[ = \sum_{r=p+1}^{\infty} (r^4 - p^4)^{-1} \sum_{q=1}^{\infty} ||Q\psi_{rq}^0||^2_{H_1}. \quad (34) \]

Since \( Q(x) \) satisfies condition (3), and by taking advantage of (1) and (34), we can give an estimation for the sum \( \sum_{q=1}^{\infty} ||Q\psi_{rq}^0||^2_{H_1} \) as

\[ \sum_{q=1}^{\infty} ||Q\psi_{rq}^0||^2_{H_1} = \sum_{q=1}^{\infty} \int_0^\pi ||Q(x)d_r \cos rx \cdot \varphi_q||^2 dx \]

\[ \leq \sum_{q=1}^{\infty} \int_0^\pi ||Q(x)\varphi_q||^2 dx \]

\[ = \sum_{q=1}^{\infty} ||Q(x)\varphi_q||^2_{H_1} < c, \quad (35) \]

where \( c \) is a positive constant. From (34) and (35) we find

\[ |M_{p2}| = c \cdot \sum_{r=p+1}^{\infty} (r^4 - p^4)^{-1}. \]

Here we can show that

\[ \sum_{r=p+1}^{\infty} (r^4 - p^4)^{-1} < p^{-\frac{5}{2}}. \quad (36) \]

Hence, we obtain

\[ |M_{p2}| < c \cdot p^{-\frac{5}{2}} \]

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and so
\[ \lim_{p \to \infty} M_{p2} = 0. \]

This time, let us show that
\[ \lim_{p \to \infty} M_p = 0. \]

To do this, we will first estimate \(|Q_{R_{\lambda}^0}^0|_{\sigma_1(H_1)}\) on the circle \(|\lambda| = b_p\). As known from [16],
\[ ||Q_{R_{\lambda}^0}^0||_{\sigma_1(H_1)} \leq \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} ||Q_{R_{\lambda}^0}^0\psi_{mn}||_{H_1}. \]

From (4) and since \(Q(x)\) satisfies condition 3, we find
\[ ||Q_{R_{\lambda}^0}^0||_{\sigma_1(H_1)} < c \cdot \sum_{m=0}^{\infty} |m^4 - \lambda|^{-1}. \] (37)

Moreover,
\[
\sum_{m=0}^{\infty} |m^4 - \lambda|^{-1} = \sum_{m=0}^{p} |m^4 - \lambda|^{-1} + \sum_{m=p+1}^{\infty} |m^4 - \lambda|^{-1} \\
\leq \sum_{m=0}^{p} (|\lambda| - m^4)^{-1} + \sum_{m=p+1}^{\infty} (m^4 - |\lambda|)^{-1} \\
= \sum_{m=0}^{p} (p^4 + 2p^3 - m^4)^{-1} + \sum_{m=p+1}^{\infty} (m^4 - p^4 - 2p^3)^{-1} \\
< \sum_{m=0}^{p} p^{-1} + \sum_{m=p+1}^{\infty} (m^4 - p^4 - 2p^3)^{-1} \\
= \frac{p+1}{p} + \sum_{m=p+1}^{\infty} \left[ \frac{1}{2}(m^4 - p^4) + \frac{1}{2}(p+1)^4 - p^4\right]^{-1} \\
< 2 + \sum_{m=p+1}^{\infty} \left[ \frac{1}{2}(m^4 - p^4) + \frac{1}{2}(p+1)^4 - p^4\right]^{-1} \\
< 2 + \sum_{m=p+1}^{\infty} \frac{2}{m^4 - p^4}. \\
\]
From (36) we have

\[ \sum_{m=0}^{\infty} |m^4 - \lambda|^{-1} < 4, \]

and by using this together with (37) we find

\[ ||Q R_\lambda^\theta||_{\sigma_1(H_1)} < c_1 ; \quad |\lambda| = b_p = p^3 + 2p^3, \quad c_1 > 0. \] (38)

Now, let us estimate \( ||R_\lambda^\theta||_{H_1} \) on the circle \( |\lambda| = b_p \). For \( m \leq p \) we have

\[ |m^4 - \lambda| \geq |\lambda| - m^4 = p^4 + 2p^3 - m^4 \geq 2p^3 > p^3, \]

and for \( m \geq p + 1 \) we have

\[ |m^4 - \lambda| \geq m^4 - |\lambda| = m^4 - p^4 - 2p^3 \geq (p + 1)^4 - p^4 - 2p^3 > 2p^3 > p^3. \]

On the other hand, since

\[ ||R_\lambda^\theta||_{H_1} = \max_m \{|m^4 - \lambda|\}, \]

we obtain

\[ ||R_\lambda^\theta||_{H_1} \leq p^{-3}. \] (39)

From Theorem 2 we know that \( \{\lambda_{mn}\}_{n=1}^{\infty} \subset [m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}], \quad (m = 0, 1, 2, ...). \) Considering this and the assumption \( ||Q||_{H_1} < \frac{1}{2} \), we write

\[ |\lambda_{mn} - m^4| < \frac{1}{2} \quad (m = 0, 1, 2, ...; n = 1, 2, 3, ...). \]

In a similar way to the proof of (39), by using this inequality we can prove that

\[ ||R_\lambda|| < c_3 \cdot p^{-3} ; c_3 > 0 \] (40)

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on the circle $|\lambda| = b_p$ for the big value of $p$. From (24), (38), (39) and (40), we find

$$|M_p| = \frac{1}{2\pi} \left| \int_{|\lambda| = b_p} \lambda \cdot \text{tr}[R_\lambda(QR_\lambda^0)^3] d\lambda \right|$$

$$\leq \int_{|\lambda| = b_p} |\lambda| \cdot |\text{tr}[R_\lambda(QR_\lambda^0)^3]| \cdot |d\lambda|$$

$$\leq b_p \int_{|\lambda| = b_p} ||R_\lambda(QR_\lambda^0)^3||_{\sigma_1(H_1)} \cdot |d\lambda|$$

$$\leq b_p \int_{|\lambda| = b_p} ||R_\lambda||_{H_1} \cdot ||(QR_\lambda^0)^3||_{\sigma_1(H_1)} \cdot |d\lambda|$$

$$\leq c_3 b_p p^{-3} \int_{|\lambda| = b_p} ||QR_\lambda^0||_{H_1}^2 \cdot ||QR_\lambda^0||_{\sigma_1(H_1)} \cdot |d\lambda|$$

$$\leq c_3 b_p p^{-3} ||Q||^2 p^{-6} c_1 2\pi b_p \leq c_4 p^{-1}$$

and so we obtain

$$\lim_{p \to \infty} M_p = 0. \quad (41)$$

**Theorem 6** If $Q(x)$ satisfies conditions 1–4, then for the regularized trace of the operator $L$, the formula

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (\lambda_{mn} - m^4) - \frac{1}{\pi} \int_0^\pi \text{tr}Q(x)dx = \frac{1}{4} [\text{tr}Q(0) + \text{tr}Q(\pi)] - \frac{1}{2\pi} \int_0^\pi \text{tr}Q(x)dx$$

is satisfied.

**Proof.** From relations (25) and (41) and Lemma 5 we write

$$\lim_{p \to \infty} \left[ \sum_{m=0}^{p} \sum_{n=1}^{\infty} (\lambda_{mn} - m^4) - \frac{2p + 1}{2\pi} \int_0^\pi \text{tr}Q(x)dx \right] = $$. 250
$$I = \frac{1}{2} \lim_{p \to \infty} \sum_{m=0}^{p} \sum_{n=1}^{\infty} d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2m x dx$$

or

$$\lim_{p \to \infty} \left[ \sum_{m=0}^{p} \sum_{n=1}^{\infty} (\lambda_{mn} - m^4) - \sum_{m=0}^{p} \frac{1}{\pi} \int_0^{\pi} \text{tr}Q(x) dx \right] =$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} d_m^2 \cdot \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2m x dx - \frac{1}{2\pi} \int_0^{\pi} \text{tr}Q(x) dx$$

or

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (\lambda_{mn} - m^4) = \frac{1}{\pi} \int_0^{\pi} \text{tr}Q(x) dx$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2m x dx - \frac{1}{2\pi} \int_0^{\pi} \text{tr}Q(x) dx.$$

Now, let us evaluate the expression

$$I = \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2m x dx$$

in the right side of last equality. Since \(Q(x)\) satisfies the conditions (1)-(4), we have

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} d_m^2 \left| \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2m x dx \right| < \infty$$
and so we write

\[ I = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} d_m^2 \int_0^\pi (Q(x)\varphi_n, \varphi_n) \cos 2mxdx \]

\[ = \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} [d_m^2 \int_0^\pi (Q(x)\varphi_n, \varphi_n) \cos mx + d_m^2 (-1)^m \int_0^\pi (Q(x)\varphi_n, \varphi_n) \cos mx \cos m0] + \sum_{m=0}^{\infty} [d_m^2 \int_0^\pi (Q(x)\varphi_n, \varphi_n) \cos mx \cos m\pi] \].

Considering \( d_m \) as in (2), the sums with respect to \( m \) in this last relation are the values at the points 0 and \( \pi \) respectively of Fourier series with respect to the functions \( \cos mx \) in the interval \([0, \pi]\) of the function \( (Q(x)\varphi_n, \varphi_n) \) which has the derivative of second order. For this reason, we write

\[ I = \frac{1}{4} \sum_{n=1}^{\infty} \left[ (Q(0)\varphi_n, \varphi_n) + (Q(\pi)\varphi_n, \varphi_n) \right] \]

\[ = \frac{1}{4} [\text{tr}Q(0) + \text{tr}Q(\pi)]. \]

And hence we obtain

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\lambda_{nm} - m^4) - \frac{1}{\pi} \int_0^\pi \text{tr}Q(x)dx = \frac{1}{4} [\text{tr}Q(0) + \text{tr}Q(\pi)] - \frac{1}{2\pi} \int_0^\pi \text{tr}Q(x)dx \]

This completes the proof of Theorem 6. \( \square \)

If \( Q(x) \) satisfies the condition

\[ \int_0^\pi \text{tr}Q(x)dx = 0 \]
in addition to conditions (1)–(4), then the formula in the above takes the form

\[ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (\lambda_{mn} - m^4) = \frac{1}{4} \left[ \text{tr}Q(0) + \text{tr}Q(\pi) \right]. \]

References


