Turk J Math
27 (2003) , 389 – 405.

On L_p-Approximation by a Linear Combination of a New Sequence of Linear Positive Operators

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Abstract

In the present paper, we study some direct results in L_p-approximation by a linear combination of a new sequence of linear positive operators. The error in the approximation is estimated in terms of the higher order integral modulus of smoothness using some properties of the Steklov means.

Key words and phrases: Linear positive operators, Linear combination, Integral modulus of smoothness, Steklov means.

1. Introduction

For $f \in L_p[0,\infty)$, $(1 \leq p < \infty)$, we defined in [1] a new sequence of linear positive operators as:

$$M_n(f(t);x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)f(t)dt + (1+x)^{-n}f(0), \tag{1.1}$$

where

$$p_{n,\nu}(x) = \binom{n+\nu-1}{\nu} x^{\nu}(1+x)^{-n-\nu},\ x \in [0,\infty),$$

AMS Classification: 41A36, 41A25

and

$$q_{n,\nu}(t) = \frac{e^{-nt}(nt)^{\nu}}{\nu!}, \;\; t \in [0,\infty).$$

Alternatively, the operators (1.1) may be written as:

$$M_n(f(t);x) = \int_0^{\infty} W_n(t,x) f(t) dt,$$

where $W_n(t,x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\, q_{n,\nu-1}(t) + (1+x)^{-n}\delta(t)$, $\delta(t)$ being the Dirac-delta function.

It was observed that the order of approximation by operators (1.1) is, at best, $O(n^{-1})$ howsoever smooth the function may be. In [1] we showed that by taking a suitable linear combination of M_n, the order of approximation can be considerably improved. We applied the technique of linear combination introduced by May [5] and Rathore [6] for operators (1.1). The approximation process is defined as follows:

Following Agrawal and Thamer [2], the linear combination $M_n(f,k,x)$ of $M_{d_j n}(f(t);x)$, $j = 0, 1, \ldots, k$ is defined as:

$$M_n(f,k,x) = \frac{1}{\Delta}\begin{vmatrix} M_{d_0 n}(f;x) & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ M_{d_1 n}(f;x) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \ldots\ldots & \ldots & \ldots & \ldots & \ldots \\ M_{d_k n}(f;x) & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix}, \tag{1.2}$$

where $d_0, d_1, \ldots, d_k$ are $k+1$ arbitrary but fixed distinct positive integers and Δ is the Vandermonde determinant obtained by replacing the operators column of the above determinant with the entries 1. On simplification, (1.2) is reduced to

$$M_n(f,k,x) = \sum_{j=0}^{k} C(j,k)\, M_{d_j n}(f;x), \tag{1.3}$$

where

$$C(j,k)=\begin{cases}\displaystyle\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i} & , \quad k\neq 0\\ \quad 1 & , \quad k=0.\end{cases}$$

Let $m\in N$ (the set of positive integers) and $0<a<b<\infty$. For $f\in L_p[a,b]$, $1\leq p<\infty$, the m^{th} order integral modulus of smoothness of f is defined as:

$$\omega_m(f,\tau,p,[a,b])=\sup_{0<\delta\leq\tau}\|\Delta_\delta^m f(t)\|_{L_p[a,b-m\delta]},$$

where $\Delta_\delta^m f(t)$ is the m^{th} order forward difference of the function f with step length δ and $0<\tau\leq(b-a)/m$.

The spaces $AC[a,b]$ and $BV[a,b]$ are defined as the classes of absolutely continuous functions and functions of bounded variation over $[a,b]$, respectively. The seminorm $\|f\|_{BV[a,b]}$ is defined by the total variation of f on $[a,b]$.

Throughout this paper, we assume that:

$0<a_1<a_3<a_2<b_2<b_3<b_1<\infty, I_i=[a_i,b_i], i=1,2,3$ and C denotes a positive constant, not necessarily the same at all occurrence.

For $1\leq p<\infty$, let

$$L_p^{(2k+2)}(I_1):=\left\{f\in L_p[0,\infty):\ f^{(2k+1)}\in AC(I_1)\ and\, f^{(2k+2)}\in L_p(I_1)\right\}.$$

For $f\in L_p[a,b]$, $1\leq p<\infty$, the Hardy-Littlewood majorant [8, ch. 10, sec. 2, p. 244] of f is defined as:

$$h_f(x)=\sup_{\xi\neq x}\frac{1}{\xi-x}\int_x^{\xi}f(t)\,dt,\ \ (a\leq\xi\leq b).$$

Our object is to prove the following theorem:

Theorem. *Let $f\in L_p[0,\infty)$, $1\leq p<\infty$. Then, for all n sufficiently large*

$$\|M_n(f,k,.)-f\|_{L_p(I_2)}\leq M\left(\omega_{2k+2}(f,n^{-1/2},p,I_1)+n^{-(k+1)}\|f\|_{L_p[0,\infty)}\right),$$

where M is a constant that depends on k and p, but is independent of f and n.

2. Preliminary results

In order to prove the Theorem, we shall require the following results:

The following lemma gives a L_p-bound for the Hardy-Littlewood majorant h_f in terms of f.

Lemma 1. [8, ch. 10, sec. 2, p. 244]. *If $1 < p < \infty$ and $f \in L_p[a,b]$, then $h_f \in L_p[a,b]$ and*

$$\| h_f \|_{L_p[a,b]} \le 2^{1/p} \frac{p}{p-1} \| f \|_{L_p[a,b]}.$$

Let $f \in L_p[0,\infty)$, $1 \le p < \infty$. Then, for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of m^{th} order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \left(\int\limits_{-\eta/2}^{\eta/2} \right)^m \left\{ f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^m t_i} f(t) \right\} \prod_{i=1}^m dt_i\,, \quad t \in I_1.$$

Lemma 2. *For the function $f_{\eta,m}(t)$ defined above, we have*

(a) $f_{\eta,m}(t)$ has derivatives up to order m over I_1, $f^{(m-1)}_{\eta,m} \in AC(I_1)$ and $f^{(m)}_{\eta,m}$ exists almost everywhere (a.e.) and belongs to $L_p(I_1)$;

(b) $\left\| f^{(r)}_{\eta,m} \right\|_{L_p(I_2)} \le C_r\, \eta^{-r}\, \omega_r(f, \eta, p, I_1)\,,\ r = 1\,,\ 2\,,\ \ldots\,,\ m;$

(c) $\| f - f_{\eta,m} \|_{L_p(I_2)} \le C_{m+1}\ \omega_m(f, \eta, p, I_1)\,;$

(d) $\| f_{\eta,m} \|_{L_p(I_2)} \le C_{m+2}\, \| f \|_{L_p(I_1)};$

(e) $\left\| f^{(m)}_{\eta,m} \right\|_{L_p(I_2)} \le C_{m+3}\, \eta^{-m}\ \| f \|_{L_p(I_1)},$

where $C_i's$ are certain constants that depend on i but are independent of f and η.

Following [7, pp. 163–165] or [4, Theorem 18.17] the proof of the above lemma easily follows but to make the paper self contained we sketch below an outline of the proof.

Proof. Using [4, Theorem 18.17] recursively, we obtain (a).

Define $h_k(x) = \int\limits_{-\eta/2}^{\eta/2} \ldots \int\limits_{-\eta/2}^{\eta/2} f\left(x + k \sum\limits_{i=1}^m x_i \right) \prod\limits_{i=1}^m dx_i,\ k = 1\,,\ 2\,,\ \ldots\,,\ m.$

Applying [4, Theorem 18.17], we have

$$h_k'(x) = k^{-1} \int_{-\eta/2}^{\eta/2} \dots \int_{-\eta/2}^{\eta/2} \Delta_{k\eta} f \left(x - \frac{k\eta}{2} + k \sum_{i=1}^{m-1} x_i \right) \prod_{i=1}^{m-1} dx_i.$$

A repeated differentiation of the above expression yields

$$h_k^{(r)}(x) = k^{-r} \int_{\eta/2}^{-\eta/2} \cdots \int_{\eta/2}^{-\eta/2} \Delta_{k\eta} f \left(x - \frac{rk\eta}{2} + k \sum_{i=1}^{m-r} x_i \right) \prod_{i=1}^{m-r} dx_i, \tag{2.4}$$

$r = 1\,,\ 2\,,\ \dots\,,\ m-1$ and $h_k^{(m)}(x) = k^{-m} \Delta_{k\eta}^m f \left(x - \frac{mk\eta}{2} \right)$ a.e. .

Since $f_{\eta,m}(x) = \dfrac{(-1)^{m-1}}{\eta^m} \sum_{k=1}^{m} \binom{m}{k} (-1)^{m-k} h_k(x)$, it follows that

$$f_{\eta,m}^{(r)}(x) = \frac{(-1)^{m-1}}{\eta^m} \sum_{k=1}^{m} \binom{m}{k} (-1)^{m-k} h_k^{(r)}(x). \tag{2.5}$$

Applying Jensen's inequality to (2.4) repeatedly, we are led to

$$\left| h_k^{(r)}(x) \right|^p = k^{-rp} \eta^{(m-r)(p-1)} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left| \Delta_{k\eta}^r f \left(x - \frac{rk\eta}{2} + k \sum_{i=1}^{m-r} x_i \right) \right|^p \prod_{i=1}^{m-r} dx_i.$$

Now, using Fubini's theorem

$$\int_{a_1}^{b_1} \left| h_k^{(r)}(x) \right|^p dx = k^{-rp} \eta^{(m-r)(p-1)} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \int_{a_1}^{b_1} \left| \Delta_{k\eta}^r f \left(x - \frac{rk\eta}{2} + k \sum_{i=1}^{m-r} x_i \right) \right|^p dx \prod_{i=1}^{m-r} dx_i$$

$$\leq C_r^p \, \eta^{(m-r)p} \ (\, \omega_r(f, \eta, p; [a_1, b_1]\,)\,)^p \,. \tag{2.6}$$

Combining (2.5) and (2.6), (b) follows.

Proceeding in similar manner, we can establish (c) to (e). □

The next lemma gives a bound for the intermediate derivatives in terms of the highest derivative and the function in L_p-norm $(1 \leq p < \infty)$. The proof is given in [3,p.5].

Lemma 3. *Let* $1 \leq p < \infty$, $f \in L_p[a,b]$, $f^{(k)} \in AC[a,b]$ *and* $f^{(k+1)} \in L_p[a,b]$. *Then*

$$\left\| f^{(j)} \right\|_{L_p[a,b]} \leq C_j \left\{ \left\| f^{(k+1)} \right\|_{L_p[a,b]} + \|f\|_{L_p[a,b]} \right\}, \quad j = 1\,,\ 2\,,\ \ldots\,,\ k,$$

where $C_j's$ are certain constants depending only on $j\,,\ k\,,\ p\,,\ a$ and b.

Lemma 4[1]. *Let* $m \in N^0$ *(the set of nonnegative integers), the* m^{th} *order moment for the operators (1.1) be defined by:*

$$T_{n,m}(x) = M_n((t-x)^m;x) = n\sum_{\nu=1}^{\infty} p_{n,\nu}(x)\int_0^{\infty} q_{n,\nu-1}(t)(t-x)^m dt + (-x)^m(1+x)^{-n}.$$

Then $T_{n,0}(x) = 1$, $T_{n,1}(x) = 0$ and

$$nT_{n,m+1}(x) = x(1+x)T'_{n,m}(x) + mT_{n,m}(x) + mx(x+2)T_{n,m-1}(x),\ m \geq 1.$$

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ is a polynomial in x of degree m, $m \neq 1$;

(ii) for every $x \in [0,\infty)$, $T_{n,m}(x) = O(n^{-[(m+1)/2]})$ where $[\beta]$ denotes the integer part of β.

Lemma 5. *For* $m \in N^0$, *we define the function* $\mu_{n,m}(t)$ *as:*

$$\mu_{n,m}(t) = n\sum_{\nu=1}^{\infty} q_{n,\nu-1}(t)\int_0^{\infty} p_{n,\nu}(x)(x-t)^m dx.$$

Then $\mu_{n,0}(t) = \dfrac{n}{n-1}$, $\mu_{n,1}(t) = \frac{2n(1+t)}{(n-1)(n-2)}$ *and there holds the recurrence relation*

$$(n-m-2)\mu_{n,m+1}(t) = t\mu'_{n,m}(t) + (m+2mt+2t+2)\mu_{n,m}(t) + mt(t+2)\mu_{n,m-1}(t),$$

where $n > m+2$. *Consequently:*

(i) $\mu_{n,m}(t)$ is a polynomial in t of degree m;

(ii) for every $t \in [0,\infty)$, $\mu_{n,m}(t) = O(n^{-[(m+1)/2]})$.

The proof of this lemma follows on proceeding along the lines of the proof of Lemma 2 [1] and hence is omitted.

Lemma 6 [1]**.** *For $m \in N$ and n sufficiently large, there holds*

$$M_n((t-x)^m, k, x) = \begin{cases} 0\,,\ m = 1\,,\ 2\,,\ \ldots\,,\ k+1 \\ n^{-(k+1)}\{\,Q(m,k,x) + o(1)\}\,,\ m = k+2\,,\ k+3\,,\ \ldots\,,\ 2k+2 \\ o(n^{-(k+1)})\,,\ m = 2k+3\,,\ 2k+4\,,\ \ldots \end{cases}$$

where $Q(m,k,x)$ is a certain polynomial in x of degree m and $x \in [0,\infty)$ is arbitrary but fixed.

Lemma 7. *Let $f \in BV(I_1)$. The following inequality holds:*

$$\left\| M_n \left(\phi(t) \int_x^t (t-w)^{2k+1}\, df(w)\,; x \right) \right\|_{L_1(I_2)} \le C\, n^{-(k+1)}\, \|f\|_{BV(I_1)},$$

where $\phi(t)$ is the characteristic function of I_1 .

Proof. For each n there exists a nonnegative integer $r = r(n)$ such that $r\, n^{-1/2} \le \max\{b_1 - a_2, b_2 - a_1\} \le (r+1)\, n^{-1/2}$. Then,

$$K := \left\| M_n \left(\int_x^t (t-w)^{2k+1}\, df(w)\, \phi(t); x \right) \right\|_{L_1(I_2)}$$

$$\le \sum_{l=0}^{r} \int_{a_2}^{b_2} \left\{ \int_{x+l\, n^{-1/2}}^{x+(l+1)\, n^{-1/2}} \phi(t)\, W_n(t,x)\, |t-x|^{2k+1} \left[\int_x^{x+(l+1)\, n^{-1/2}} \phi(w)\, |df(w)| \right] dt \right.$$

$$\left. + \int_{x-(l+1)\, n^{-1/2}}^{x-l\, n^{-1/2}} \phi(t)\, W_n(t,x)\, |t-x|^{2k+1} \left[\int_{x-(l+1) n^{-1/2}}^{x} \phi(w)\, |df(w)| \right] dt \right\} dx.$$

Let $\phi_{x,c,d}(w)$ denote the characteristic function of the interval $[x - cn^{-1/2}, x + dn^{-1/2}]$ where c and d are nonnegative integers. Then, we have

$$K \leq \sum_{l=1}^{r} \left(n^2\, l^{-4} \int_{a_2}^{b_2} \left\{ \int_{x+l\, n^{-1/2}}^{x+(l+1)\, n^{-1/2}} \phi(t)\, W_n(t,x)\, |t-x|^{2k+5} \left[\int_{a_1}^{b_1} \phi_{x,0,l+1}(w)\, |df(w)| \right] dt \right.\right.$$

$$\left.\left. + \int_{x-(l+1)\, n^{-1/2}}^{x-l\, n^{-1/2}} \phi(t)\, W_n(t,x)\, |t-x|^{2k+5} \left[\int_{a_1}^{b_1} \phi_{x,l+1,0}(w)\, |df(w)| \right] dt \right\} dx \right)$$

$$+ \int_{a_2}^{b_2} \int_{a_2-n^{-1/2}}^{b_2+n^{-1/2}} \phi(t)\, W_n(t,x)\, |t-x|^{2k+1} \left[\int_{a_1}^{b_1} \phi_{x,1,1}(w)\, |df(w)| \right] dt\, dx.$$

Using Lemma 4 and Fubini's theorem we get

$$K \leq C\, n^{-(2k+1)/2} \left\{ \sum_{l=1}^{r} l^{-4} \left[\int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \phi_{x,0,l+1}(w)\, dx \right) |df(w)| \right.\right.$$

$$\left.\left. + \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \phi_{x,l+1,0}(w)\, dx \right) |df(w)| \right] + \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \phi_{x,1,1}(w)\, dx \right) |df(w)| \right\}$$

$$\leq C\, n^{-(k+1)}\, \|f\|_{BV(I_1)}.$$

□

In order to prove our main result, we first discuss the approximation in the smooth subspace $L_p^{(2k+2)}(I_1)$ of $L_p[0, \infty)$.

Lemma 8. *Let $1 < p < \infty$ and $f \in L_p^{(2k+2)}(I_1)$, then for all n sufficiently large, the following inequality holds:*

$$\| M_n(f,k,.) - f\|_{L_p(I_2)} \le C_1\, n^{-(k+1)} \left\{ \left\| f^{(2k+2)} \right\|_{L_p(I_1)} + \|f\|_{L_p[0,\infty)} \right\}, \tag{2.7}$$

where $C_1 = C_1(k,p)$.

Let $f \in L_1[0,\infty)$. If f has $2k+1$ derivatives in I_1 with $f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then for all n sufficiently large, the following inequality holds:

$$\| M_n(f,k,.) - f\|_{L_1(I_2)} \le C_2\, n^{-(k+1)} \left\{ \left\| f^{(2k+1)} \right\|_{BV(I_1)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)} \right\}, \tag{2.8}$$

where $C_2 = C_2(k)$.

Proof. Let $p > 1$. With the given assumptions on f, for $x \in I_2$ and $t \in I_1$ we can write

$$f(t) = \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j\,!} f^{(j)}(x) + \frac{1}{(2k+1)\,!} \int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\, dw.$$

Hence, if $\phi(t)$ is the characteristic function of I_1, then

$$f(t) = \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j\,!} f^{(j)}(x) + \frac{1}{(2k+1)\,!} \int_x^t (t-w)^{2k+1} \phi(t)\, f^{(2k+2)}(w)\, dw$$

$$+F(t,x)(1-\phi(t)),$$

where $F(t,x) = f(t) - \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j\,!} f^{(j)}(x)$, for all $t \in [0,\infty)$ and $x \in I_2$.

In view of $M_n(1,k,x) = 1$, we obtain

$$M_n(f,k,x) - f(x) = \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j\,!} M_n\left((t-x)^j, k, x\right)$$

$$+\frac{1}{(2k+1)\,!}\,M_n\left(\phi(t)\int\limits_x^t (t-w)^{2k+1}\,f^{(2k+2)}(w)\,dw,k,x\right)+M_n\left(F(t,x)(1-\phi(t)),k,x\right)$$

$$:=\Sigma_1+\Sigma_2+\Sigma_3.$$

It follows from Lemmas 3 and 6 that

$$\|\,\Sigma_1\|_{L_p(I_2)}\le C\,n^{-(k+1)}\left(\|f\|_{L_p(I_2)}+\left\|f^{(2k+2)}\right\|_{L_p(I_2)}\right).$$

To estimate $\sum_2$, let h_f be the Hardy-Littlewood majorant of $f^{(2k+2)}$ on I_1. Then using Hölder's inequality and Lemma 4, we get

$$J_1:=\left|\,M_n\left(\phi(t)\int\limits_x^t (t-w)^{2k+1}\,f^{(2k+2)}(w)\,dw\,;\,x\right)\right|$$

$$\le M_n\left(\phi(t)\,|t-x|^{2k+1}\left|\int\limits_x^t \left|f^{(2k+2)}(w)\right|dw\right|;\,x\right)$$

$$\le M_n\left(\phi(t)\,(t-x)^{2k+2}\,|h_f(t)|\,;\,x\right)$$

$$\le\left\{\,M_n\left(\phi(t)\,|t-x|^{(2k+2)q}\,;\,x\right)\right\}^{1/q}\left\{M_n\left(\phi(t)\,|h_f(t)|^p\,;\,x\right)\right\}^{1/p}$$

$$\le C\,n^{-(k+1)}\left\{\int\limits_{a_1}^{b_1} W_n(t,x)\;|h_f(t)|^p\,dt\right\}^{1/p}.$$

Hence, by Fubini's theorem, Lemmas 1 and 5 we have,

$$\| J_1\|^p_{L_p(I_2)} \le C\, n^{-(k+1)p} \int_{a_2}^{b_2} \int_{a_1}^{b_1} W_n(t,x)\ |h_f(t)|^p\, dt\, dx$$

$$\le C\, n^{-(k+1)p} \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} W_n(t,x)\, dx \right] |h_f(t)|^p\, dt$$

$$\le C\, n^{-(k+1)p}\, \frac{n}{n-1} \int_{a_1}^{b_1} |h_f(t)|^p\, dt$$

$\le C\, n^{-(k+1)p} \|h_f\|^p_{L_p(I_1)}$, since n is sufficiently large

$$\le C\, n^{-(k+1)p} \left\| f^{(2k+2)} \right\|^p_{L_p(I_1)}.$$

Consequently,

$$\| J_1\|_{L_p(I_2)} \le C\, n^{-(k+1)} \left\| f^{(2k+2)} \right\|_{L_p(I_1)}.$$

Thus, we have

$$\| \Sigma_2\| \le C\, n^{-(k+1)} \left\| f^{(2k+2)} \right\|_{L_p(I_1)}.$$

For $t \in [0,\infty)\backslash I_1$, $x \in I_2$ there exist a $\delta > 0$ such that $|t-x| \ge \delta$ which leads us to what follows:

$$| M_n(F(t,x)(1-\phi(t));x)|$$

$$\le \delta^{-(2k+2)} \left[M_n\left(|f(t)|\, (t-x)^{2k+2}; x\right) + \sum_{j=0}^{2k+1} \frac{f^{(j)}(x)}{j\,!} M_n\left(|t-x|^{2k+j+2}; x\right) \right] := J_2 + J_3.$$

It follows from Hölder's inequality and Lemma 4 that

$$| J_2| \leq C\, n^{-(k+1)} \{ M_n\left(|f(t)|^p ; x\right) \}^{1/p} .$$

Again, applying Fubini's theorem, we get

$$\| J_2\|_{L_p(I_2)} \leq C\, n^{-(k+1)}\, \|f\|_{L_p[0,\infty)}, \text{ for n sufficiently large.}$$

Moreover, using Lemmas 3 and 4 we have

$$\| J_3\|_{L_p(I_2)} \leq C\, n^{-(k+1)} \left(\|f\|_{L_p(I_2)} + \left\| f^{(2k+2)} \right\|_{L_p(I_2)} \right).$$

Hence

$$\| \Sigma_3\|_{L_p(I_2)} \leq C\, n^{-(k+1)} \left(\|f\|_{L_p[0,\infty)} + \left\| f^{(2k+2)} \right\|_{L_p(I_2)} \right).$$

Combining the estimates of $\sum_1 - \sum_3$, (2.7) follows.

Now, let $p = 1$. With the given assumptions on f, for almost all $x \in I_2$ and for $t \in I_1$, we can write

$$f(t) = \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j\,!}\, f^{(j)}(x) + \frac{1}{(2k+1)\,!} \int_x^t (t-w)^{2k+1}\, df^{(2k+1)}(w) .$$

Hence, if $\phi(t)$ is the characteristic function of I_1 then

$$f(t) = \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j\,!}\, f^{(j)}(x) + \frac{1}{(2k+1)\,!} \int_x^t (t-w)^{2k+1}\, df^{(2k+1)}(w)\, \phi(t)$$

$$+F(t,x)(1-\phi(t)),$$

where $F(t,x) = f(t) - \sum_{j=0}^{2k+1} \frac{(t-x)^j}{j\,!}\, f^{(j)}(x)$, for almost all $x \in I_2$ and for all $t \in [0,\infty)$.

By operating M_n on the last equation, we get

$$M_n(f,k,x) - f(x) = \sum_{j=1}^{2k+1} \frac{f^{(j)}(x)}{j\,!}\, M_n\left((t-x)^j, k, x\right)$$

$$+\frac{1}{(2k+1)!}\,M_n\left(\int_x^t (t-w)^{2k+1}\,df^{(2k+1)}(w)\,\phi(t)\,,k,x\right)+M_n\left(F(t,x)(1-\phi(t)),k,x\right)$$

$$:=\Sigma_1+\Sigma_2+\Sigma_3.$$

Applying Lemmas 3 and 6, we have

$$\|\,\Sigma_1\|_{L_1(I_2)}\le C\,n^{-(k+1)}\left(\|f\|_{L_1(I_2)}+\left\|f^{(2k+1)}\right\|_{L_1(I_2)}\right).$$

Next, using Lemma 7, we obtain

$$\|\,\Sigma_2\|_{L_1(I_2)}\le C\,n^{-(k+1)}\left\|f^{(2k+1)}\right\|_{BV(I_1)}.$$

For all $t\in[0,\infty)\backslash I_1$, $x\in I_2$, we can choose a $\delta>0$ such that $|t-x|\ge\delta$. Then

$$\|\,M_n\left(F(t,x)(1-\phi(t));x\right)\|_{L_1(I_2)}\le\int_{a_2}^{b_2}\int_0^\infty W_n(t,x)\,|f(t)|\,(1-\phi(t))\,dt\,dx$$

$$+\sum_{j=0}^{2k+1}\frac{1}{j!}\int_{a_2}^{b_2}\int_0^\infty W_n(t,x)\left|f^{(j)}(x)\right|\,|t-x|^j\,(1-\phi(t))\,dt\,dx:=J_1+J_2.$$

For sufficiently large t, we can find positive constants M and C' such that

$$\frac{(t-x)^{2k+2}}{t^{2k+2}+1}>C'\text{ for all }t\ge M,x\in I_2.$$

By Fubini's theorem,

$$J_1=\left(\int_0^M\int_{a_2}^{b_2}+\int_M^\infty\int_{a_2}^{b_2}\right)\,W_n(t,x)\,|f(t)|\,(1-\phi(t))\,dx\,dt:=J_3+J_4.$$

Now, using Lemma 5 we have

$$J_3 \leq \delta^{-(2k+2)} \int_0^M \int_{a_2}^{b_2} W_n(t,x)\,|f(t)|\,(t-x)^{2k+2}\,dx\,dt$$

$$\leq C\,n^{-(k+1)} \int_0^M |f(t)|\,dt, \text{ and}$$

$$J_4 \leq \frac{1}{C'} \int_M^\infty \int_{a_2}^{b_2} W_n(t,x)\,\frac{(t-x)^{2k+2}}{t^{2k+2}+1}\,|f(t)|\,dx\,dt$$

$$\leq C\,n^{-(k+1)} \int_M^\infty |f(t)|\,dt, \text{ since } t \text{ is sufficiently large.}$$

Combining the estimates of J_3 and J_4, we get

$$J_1 \leq C\,n^{-(k+1)}\,\|f\|_{L_1[0,\infty)}.$$

Further, using Lemmas 3 and 4 we obtain

$$J_2 \leq C\,n^{-(k+1)} \left(\|f\|_{L_1(I_2)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right).$$

Hence

$$\|\, M_n\,(F(t,x)(1-\phi(t));x)\,\|_{L_1(I_2)} \leq C\,n^{-(k+1)} \left(\|f\|_{L_1[0,\infty)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right).$$

Consequently,

$$\|\,\Sigma_3\|_{L_1(I_2)} \leq C\,n^{-(k+1)} \left(\|f\|_{L_1[0,\infty)} + \left\| f^{(2k+1)} \right\|_{L_1(I_2)} \right).$$

Finally, combining the estimates of $\sum_1 - \sum_3$, we get (2.8). □

Now, we shall prove the Theorem (the object of this paper).

3. Proof of Theorem

Let $f_{\eta,2k+2}(t)$ be the Steklov mean of $(2k+2)^{\text{th}}$ order corresponding to $f(t)$ over I_1, where $\eta > 0$ is sufficiently small and $f_{\eta,2k+2}(t)$ is defined as zero outside I_1. Then we have

$$\| M_n(f,k,\,.\,) - f\|_{L_p(I_2)} \leq \| M_n(f - f_{\eta,2k+2}, k,\,.\,)\|_{L_p(I_2)}$$

$$+\| M_n(f_{\eta,2k+2}, k,\,.\,) - f_{\eta,2k+2}\|_{L_p(I_2)} + \|f_{\eta,2k+2} - f\|_{L_p(I_2)} := \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Letting $\phi(t)$ to be the characteristic function of I_3, we get

$$M_n((f - f_{\eta,2k+2})(t);x) = M_n(\phi(t)(f - f_{\eta,2k+2})(t);x)$$

$$+M_n((1-\phi(t))(f - f_{\eta,2k+2})(t);x) := J_1 + J_2.$$

Clearly, the following inequality holds for $p = 1$, for $p > 1$, it follows from Hölder's inequality

$$\int_{a_2}^{b_2} |\, J_1|^p\, dx \leq \int_{a_2}^{b_2}\int_{a_3}^{b_3} W_n(t,x)\, |(f - f_{\eta,2k+2})\,(t)\,|^p\, dt\, dx.$$

Using Fubini's theorem and Lemma 5 we get

$$\| J_1\|_{L_p(I_2)} \leq 2\, \|f - f_{\eta,2k+2}\|_{L_p(I_3)}.$$

Proceeding in a similar manner, for all $p \geq 1$

$$\| J_2\|_{L_p(I_2)} \leq C\, n^{-(k+1)} \|f - f_{\eta,2k+2}\|_{L_p[0,\infty)}.$$

Consequently, by the property (c) of Steklov means we get

$$\Sigma_1 \leq C\left(\omega_{2k+2}(f,\eta,p,I_1) + n^{-(k+1)}\, \|f\|_{L_p[0,\infty)}\right).$$

Since $\left\|f_{\eta,2k+2}^{(2k+1)}\right\|_{BV(I_3)} = \left\|f_{\eta,2k+2}^{(2k+1)}\right\|_{L_1(I_3)}$, by Lemma 8, for all $p \geq 1$ there follows

$$\Sigma_2 \leq C\, n^{-(k+1)} \left(\left\| f_{\eta,2k+2}^{(2k+2)} \right\|_{L_p(I_3)} + \| f_{\eta,2k+2} \|_{L_p[0,\infty)} \right)$$

$$\leq C\, n^{-(k+1)} \left(\eta^{-(2k+2)}\, \omega_{2k+2}(f,\eta,p,I_1) + \|f\|_{L_p[0,\infty)} \right),$$

in view of the properties (b) and (d) of Steklov means.
Finally, by the property (c) of Steklov means

$$\Sigma_3 \leq C\omega_{2k+2}(f,\eta,p,I_1).$$

Choosing $\eta = n^{-1/2}$, and combining the estimates of $\sum_1 - \sum_3$, the required result follows.
$\square$

Acknowledgement

We are extremely grateful to the referee for his careful reading of the manuscript and making valuable suggestions leading to a better presentation of the paper.

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Received 11.07.2002