Turk J Math
27 (2003) , 343 – 347.

Groups Whose Proper Subgroups are Hypercentral of Length at Most $\leq \omega$

Selami Ercan

Abstract

Groups, all proper subgroups of which are hypercentral of length at most ω and every proper subgroup of which is a $\mathbf{B_n}$-group for a natural number n depending on the subgroup, are studied in this article.

Key Words: hypercentral groups, locally nilpotent groups

1. Introduction

For $n \geq 0$, we denote by $\mathbf{B_n}$ the class of groups in which every subnormal subgroup has defect at most n. $\mathbf{B_n}$-groups are considered by many authors, both for special cases and in general. For results related to $\mathbf{B_1}$-groups see [10], [4], [11], for $\mathbf{B_2}$, $\mathbf{B_3}$, $\mathbf{B_4}$-groups see [5], [2] and for the general case see [6], [3]. It was shown in [14] that there exists a group G that is a hypercentral group of length exactly $\omega + 1$ and all of its subgroups are subnormal. The split extension G of a group of type C_{2^∞} by the inverting automorphism, is hypercentral of length $\omega + 1$ and every proper subgroup of G is nilpotent. A group G is locally graded if every non-trivial finitely generated subgroup of G has a finite non-trivial image. We denote by $\mathbf{N_0}$ class of groups in which every subgroup is subnormal.

The focus of this paper are those locally nilpotent groups whose every proper subgroup is a hypercentral of length at most ω; and where every proper subgroup of these

2001 *Math. Subject Classification:* 20E15

hypercentrals are B_n-groups in general, and prove that every such B_n-group is either soluble or a $\mathbf{N_0}$-group.

2. Main Results

Theorem 1 *Let G be a periodic hypercentral group and let every proper subgroup H of G be a $\mathbf{B_n}$-group for some natural number n depending on H. If G is hypercentral of length at most $\leq \omega$, then G is nilpotent.*

Proof. Suppose that G is not nilpotent. Then G is hypercentral of length ω and $G = \bigcup_{i=0}^{\infty} Z_i(G)$. For all $x \in G$, there exists $i \in N$ such that $x \in Z_i(G)$. Since $Z_i(G)$ is nilpotent for all natural numbers i, for all $x \in G$, $< x >$ is a subnormal subgroup of G. Thus G is a Baer group. Since G is hypercentral, $G' < G$ and also G' is nilpotent, by Lemma 6.1 of [6]. Since G/G' is abelian, G is soluble. Every proper subgroup of G is nilpotent, again by Lemma 6.1 of [6]. If G has no maximal subgroup, then every subgroups of G are subnormal by Theorem 3.1.(ii) of [15]. Thus G is nilpotent by Theorem 2.7 of [8]. If G has a maximal subgroup, then there is a maximal subgroup M such that $G =< x > M$ for some $x \in G$. Since G is Baer, $< x > M$ is nilpotent by Lemma 1 of [7]. □

Theorem 2 *Let G be a locally graded torsion-free group and let every proper subgroup H of G be a $\mathbf{B_n}$-group for some natural number n depending on H. If every proper subgroup of G is hypercentral of length at most $\leq \omega$, then G is nilpotent.*

Proof. Since every proper subgroup of G is hypercentral of length at most $\leq \omega$, $H = \bigcup_{i=0}^{\infty} Z_i(H)$ for all $H < G$; since $Z_i(H)$ is nilpotent, for all $i \geq 0$, $< x >$ is subnormal in H, for all $x \in H$. Thus H is a Baer group. By Lemma 6.1 of [6], H is nilpotent. Let F be a finitely generated non-trivial subgroup of G. If $F \neq G$ then F is nilpotent by the above. If $F = G$, then G is a finitely generated locally graded group and so G is nilpotent by Theorem 2 of [16]. Therefore G is locally nilpotent group. Finally, we conclude that G is nilpotent by Theorem 2.1 of [15] . □

Theorem 3 *Let G be a locally nilpotent group and let every proper subgroup H of G be a $\mathbf{B_n}$-group for some natural number n depending on H. If every proper subgroup of G is hypercentral of length at most $\leq \omega$, then G is soluble.*

Proof. Suppose that G is not soluble. Let T be the periodic part of G. T is a subgroup of G by 12.1.1 of [13]. If $T = 1$, then G is nilpotent by Theorem 2. Therefore G is soluble. If $G = T$, then every proper subgroup of G is nilpotent by Theorem 1. By Theorem 3.3.(i),(ii) of [15], G is a Fitting p-group. $G \neq G'$ by Theorem 1.1 of [1]. Therefore G is soluble. If $1 \neq T \neq G$, then T is hypercentral of length at most $\leq \omega$. Therefore T is nilpotent by Theorem 1. Since G/T is torsion-free, G/T is soluble by Theorem 1. Since T and G/T are soluble, G is soluble. This is a contradiction. □

Theorem 4 *Let G be a locally nilpotent group and let every proper H be a $\mathbf{B_n}$-group for some natural number n depending on H. If G is hypercentral of length at most $\leq \omega$, then G is nilpotent.*

Proof. Suppose that G is not nilpotent. G is soluble by Theorem 3. Every proper subgroup of G is nilpotent by the proof of Theorem 3. By hypothesis and Theorem 3.1.(i),(ii) of [15], every subgroup of G is subnormal. By Theorem 2.7 of [8] G is nilpotent. If G has a maximal subgroup, then G is a metabelian Chernikov p-group and G is hypercentral of length at most $\leq \omega + 1$ in [9]. This is a contradiction. □

Corollary 5 *Let G be a locally nilpotent group and let every proper subgroup H of G be a $\mathbf{B_n}$-group for some natural number n depending on H. If every proper subgroup of G is hypercentral of length at most $\leq \omega$, then either G is hypercentral or G is an N_0-group.*

Proof. Suppose that G is not hypercentral. Then G is not nilpotent. G is soluble by Theorem 3 and every proper subgroup of G is nilpotent by the proof of Theorem 3. If G has a maximal subgroup, then G is a metabelian Chernikov p-group and G is hypercentral of length at most $\leq \omega + 1$ in [9]. This is a contradiction. □

Theorem 6 *Let G be a locally soluble torsion-free group and let every proper subgroup H of G be a $\mathbf{B_n}$-group for some natural number n depending on H. Then either G is locally nilpotent or G is finitely generated.*

Proof. Suppose that G is not finitely generated. Let F be a finitely generated subgroup of G. Since $G \neq F$, F is nilpotent by Corollary 2 of Theorem 10.57 of [12]. Thus G is finitely generated. □

References

[1] A. O. Asar, *Locally nilpotent p-groups whose proper subgroups are hypercentral or nilpotent-by-Chernikov, J. London Math. Soc.***2** 61 , 412-422 (2001).

[2] C. Casolo, *Periodic soluble groups in which every subnormal subgroup has defect at most two, Arch. Math.* **46**, 1-7 (1986).

[3] C. Casolo, *Groups with subnormal subgroups of bounded defect, Rend. Sem. Mat. Univ. Padova* **77**, 177-187 (1987).

[4] W. Gaschütz. *Gruppen in denen das normalteilersein transitive ist, J. Reine Angew. Math.* **198**, 87-92 (1957).

[5] D. J. McCaughan and S. E. Stonehewer, *Finite soluble groups whose subnormal subgroups have defect at most two, Arch. Math.* **35**, 56-60 (1980).

[6] D. McDougall, *The subnormal structure of some classes of soluble groups, J. Austral. Mat. Soc.* **13**, 365-377 (1972).

[7] W. Möhres, *Torsionsgruppen, deren untergruppen alle subnormal Sind, Geom. Dedicata.***31**, 237-244 (1989).

[8] W. Möhres, *Hyperzentrale Torsionsgruppen, deren Untergruppen alle subnormal sind, Illinois J. Math.***35 (1)** , 147-157 (1991).

[9] F. M. Newman and J. Wiegold, *Groups with many nilpotent subgoups, Archiv der Mathematik* **XV**, 241-250 (1964).

[10] D. J. S. Robinson, *On groups in which normality is a transitive relation, Proc. Cambridge Phil. Soc.* **60**, 21-38 (1964).

[11] D. J. S. Robinson, *A note on finite groups in which normality is transitive, Proc. Amer.Math. Soc.* **19**, 933-937 (1968).

[12] D. J. S. Robinson, *Finiteness Condition and Generalized Soluble Groups,* vols. 1 and 2, Springer-Verlag, 1972.

[13] D. J. S. Robinson, *A Course in The Theory of Groups,* Springer-Verlag, Heidelberg-Berlin-New York, 1982

[14] H. Smith, *Hypercentral groups with all subgroups subnormal, Bull. London Math. Soc.* **15**, 229-234 (1983).

[15] H. Smith, *Groups with few non-nilpotent subgroups, Glasgow Journal of Math.* **39**, 141-151 (1997).

[16] H. Smith, Torsion-free groups with all non-nilpotent subgroups subnormal, Preprint.

Selami ERCAN
Gazi Üniversitesi
Gazi Eğitim Fakültesi Matematik
06500 Teknikokullar, Ankara-TURKEY
e-mail: ercans@gazi.edu.tr

Received 15.10.2002