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# Application of Impulse Momentum Theory to Vehicle Collisions

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**Abstract**

Collisions between two objects are classified into three groups : Linear Central Impacts, Oblique Central Impacts and Eccentric Impacts. In this paper, the effects of all three types of impacts by motor vehicles are studied using the theory of impulse and momentum.

The loss of energy caused by impacts is defined by the Coefficient of Restitution "e". Accordingly, Elastic, Elastoplastic and Plastic Collisions occur in all 3 types of impact. Oblique Central Impact is observed more often than Linear Central Impact, and Eccentric Collisions are observed more often than the other two types in highway collisions. Cases of these types of collision are examined, and examples of all 3 types of collision adapted to vehicle collisions are illustrated.

**Key Words:** Accidentology, Collisions, Coefficient of Restitution, Impulse, Momentum.

## İmpuls-Momentum Teorisinin Taşıt Çapışmalarına Uygulanması

**Özet**

Çarpışma noktaları dikkate alındığında, iki cisim arasında meydana gelen çarpışmaları, genelde 3 tipte toplamak olasıdır. Bunlar, Doğru Merkezsel Çarpışma, Eğik Merkezsel Çarpışma ve Eksantrik (Merkezden Kaçık) Çarpışmalardır. Bu makalede, her üç çarpışma tipinin karayolu taşıt çarpışmalarına uyarlaması yapılmış, zorunlu ve gerekli olan bazı kabul ve yaklaşımlarla, çarpışmalar teorisinin (İmpuls-Momentum Kanunları) pratikteki uygulaması etüd edilmiştir.

Çarpışmalar sonucu meydana gelen enerji kaybı, Çarpışma Katsayısı "e" ile belirlenmekte, buna göre de her 3 tipteki çarpışma için Elastik, Elastoplastik ve Plastik çarpışmalar söz konusu olmaktadır. Doğru Merkezsel Çarpışmaya kıyasla Eğik Merkezsel Çarpışma, her ikisine kıyasla da Eksantrik Çarpışma, karayolundaki trafik kazalarında daha çok görülmektedir. Sunulan örnekler, bu tipler için hazırlanmış, ayrıca her üç tipteki çarpışmanın taşıt çarpışmalarına uyarlanan örnekleri de şematik resimlerle belirtilmiştir. Kazabilim (Accidentologie) ve Karayolu Trafiğinde Güvenlik açısından, böylesi bir teoriden hareketle belirli sonuçlara ulaşılmış olmasının, daha ileride bu konuda yapılabilecek başka çalışmalara destek sağlayacağı umulmaktadır.

**Anahtar Sözcükler:** Kazabilim, Çarpışmalar, Çarpışma Katsayısı, İmpuls, Momentum.

## 1. Introduction

Types of vehicle collision on highways can generally be classified as head-on collisions (i.e. accidents due to dangerous overtaking), front-to-back collisions and eccentric collisions (collisions at crossroads). According to statistics published by the Ministry of Interior Affairs of Turkey, in 1997, 48883 overtaking collisions (17.92 % of total accidents), 74491 front-to-back collisions (27.31 % of total accidents) and 56922 intersection collisions (20.86 % of total accidents) took place on urban and rural roads in Turkey. The total of these three figures is 180296 collisions, which is 66 % of the total number of accidents (272774) which occurred in 1997. This is a significant figure 41.73 % of all these collisions were fatal and 42.06 % of them resulted only in injuries.

During head-on and front to back collisions, the symmetry axes of the vehicles are mostly congruent or parallel to each other. However, in the third case, eccentric impact, potential results are difficult to predict due to displacements caused by rotation. This is observed mostly in accidents at intersections. What is important is the changed directions of the vehicles after the collision, their displacements and their ultimate positions (overturned, rolled over, displaced, etc.,).

The intensity of the impact depends on the velocity ($v_i$) of the vehicle, its mass ($m_i$) and the material the vehicle is made of. Consequently, some accidents may result in greater damage whereas others may cause only minor damage. More detail can be obtained through analysis of the Impulse-Momentum Laws.

## 2. The Impulse-Momentum Theory

### 2.1. Linear Central Impact

Linear Central Impact between two objects' assumes the following :

a- Objects do not rotate within their axes during the crash.

b- The normal plane of an impact area passes through the two objects mass centers (Fig.1).

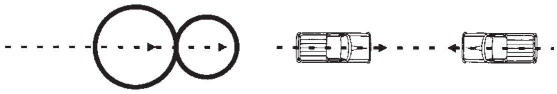

**Figure 1.** Linear Central Impact.

c- Velocity vectors at the mass center are on the collision normal line before the collision.

d- It is also assumed that both objects are rigid before and after the collision, but are able to change shape during the crash, and that the mass dispersion is not affected by the crash.

e- The impact time "t" is very short and, therefore, the location of the objects before and after impact is nearly the same.

$m_1$ and $m_2$ are the masses of the objects; their velocities before the impact are $v_1$ and $v_2$, and the velocities after the impact are $v_1'$ and $v_2'$; $F_1$ and $F_2$ forces exposed during the impact are equal (Newton's 3 rd Law), but their directions are opposite (Fig.2).

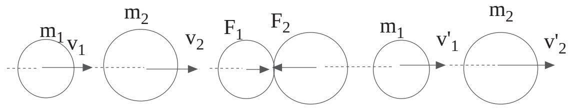


**Figure 2.** Linear Central Impact.

So:

$$\int_0^{t_i} F_1 dt + \int_0^{t_i} F_2 dt = 0 \quad (1)$$

This impulse integral is equal to momentum:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \quad (2)$$

In order to analyze this further, it is useful to divide the Impact Time $t$ into $t_1$ and $t_2$. ($t_1$) is the compression time and, during this time, the object starts to change shape and reaches its maximum level, i.e., Force and Reshaping are at a maximum; after this time, the relative velocity between the objects is zero. ($t_2$) is the restitution time and simultaneously the reshaping lessens and finally disappears (Fig. 3).

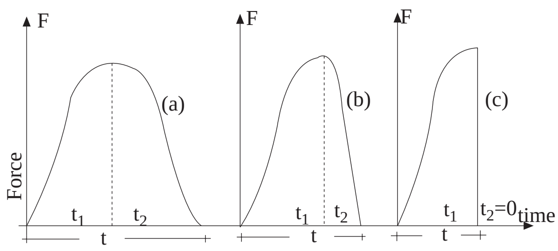


**Figure 3.** Relations between Force and Time during impact.

In the first diagram, $t_1 = t_2 = 1/2t$; it corresponds to an Elastic Collision. (Fig. 3a).

In the second diagram, $t_1 \neq t_2$, and this case is Inelastic, meaning that it is an Elastoplastic Collision (Fig. 3b).

In the third diagram, $t_2$ becomes zero ($t_2 = 0$), that means that the objects collide and cannot be separated. This impact is a Plastic Collision. (Fig. 3c).

The change in momentum is equal to the impulse integral and the common velocity $v_c$ at the beginning of the restitution time reaches the maximum level.

$$m_1(v_c - v_1) = \int_0^{t_1} F_2 dt$$
$$m_2(v_c - v_2) = \int_0^{t_1} F_1 dt = -\int_0^{t_1} F_2 dt$$
$$v_c = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_1} \tag{3}$$

During the restitution time $t_2$, the following is obtained :

$$m_1(v_1' - v_c) = \int_{t_1}^{t_2} F_2 dt$$
$$m_2(v_2' - v_c) = \int_{t_1}^{t_2} F_1 dt = -\int_{t_1}^{t_2} F_2 dt$$
$$v_c = \frac{m_1 v_1' + m_2 v_2'}{m_1 + m_2} \tag{4}$$

It is necessary to accept the existence of a correspondence between $\int_0^{t_1} F\ dt$ and $\int_{t_1}^{t_2} F\ dt$, which is called the Coefficient of Restitution, "e". So this relation can be formulated as :

$$\int_{t_1}^{t_2} F dt = e \int_0^{t_1} F dt \text{ (Newton Hypothesis).}$$

The Coefficient of Restitution (e) is related to objects, materials, masses and velocities.
$A_1$ = Area of Compression period.
$A_2$ = Area of Restitution period.
$\frac{A_2}{A_1} = e$ In Figure 4:
If $e = 1$, the impact is Elastic.
If $e = 0 \sim 1$, the impact is Elastoplastic.
If $e = 0$, the impact is Plastic.

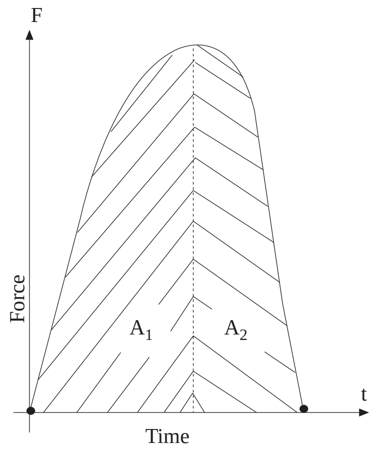


**Figure 4.** Relation between Force (F) and Time (t).

This can be written as:

$$v_1' - v_c = e(v_c - v_1)$$
$$v_2' - v_c = e(v_c - v_2)$$

So; $e = -\frac{v_1' - v_2'}{v_1 - v_2}$ and so we obtain :

$$v_1' = v_1 + \frac{m_2}{m_1 + m_2}(1 + e)(v_2 - v_1) \tag{5}$$
$$v_2' = v_2 + \frac{m_1}{m_1 + m_2}(1 + e)(v_1 - v_2) \tag{6}$$

$\Delta E_c$ = Kinetic Energy change (Loss).
If the general relation, which is

$$\Delta E_c = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{1}{2}m_1 v_1'^2 - \frac{1}{2}m_2 v_2'^2 \tag{7}$$

is changed with the new values, the following equation is obtained :

$$\frac{1}{2}m_1 v_1'^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}\frac{(1+e)^2 m_1 m_2^2 (v_2 - v_1)^2}{(m_1 + m_2)^2} + \frac{m_1(1+e) \cdot m_2 v_1 (v_2 - v_1)}{(m_1 + m_2)}$$
$$\frac{1}{2}m_2 v_2'^2 = \frac{1}{2}m_2 v_2^2 + \frac{1}{2}\frac{(1+e)^2 m_2 m_1^2 (v_1 - v_2)^2}{(m_1 + m_2)^2} + \frac{m_2(1+e) \cdot m_1 v_2 (v_1 - v_2)}{(m_1 + m_2)}$$

and finally: $$\Delta E_c = \frac{1}{2}(1 - e^2)\frac{m_1 m_2}{(m_1 + m_2)}(v_2 - v_1)^2 \tag{8}$$

If the impact is elastic, $e = 1$ and $\Delta E_c = 0$. So, after the impact, the kinetic energy remains the same; otherwise it lessens.

When the loss in kinetic energy reaches a maximum, then $e = 0$, and it corresponds to a Plastic Impact.

**Example 1:** A vehicle impact to a fixed body: (Fig. 5)

$v' = -e.v$, so $e = -\frac{v'}{v}$ If $e = 0$, the vehicle and body collide (Plastic Impact). If $e = 1$, the vehicle hits the body and goes back at the same velocity (Elastic Impact). If e is between 0 and 1, the vehicle hits the body and goes back at a lower velocity (Elastoplastic Impact).

**Example 2:** Front to back Collision (Fig. 6).

A 1100 kg. vehicle A, moving at a speed of 80 km/h (22.22 m/s), hits the back (nose to tail) of a vehicle B (950 kg.), which is not moving and has had its hand brake released. If, after the collision, vehicle B is observed to be moving to the right at a speed of 60 km/h (16.67 m/s), then we can determine the Coefficient of Restitution between the two cars.

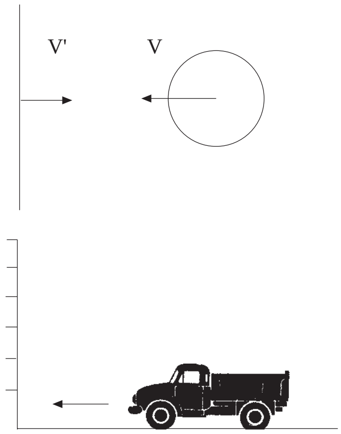


**Figure 5.** A vehicle impact to a fixed body ( wall ).

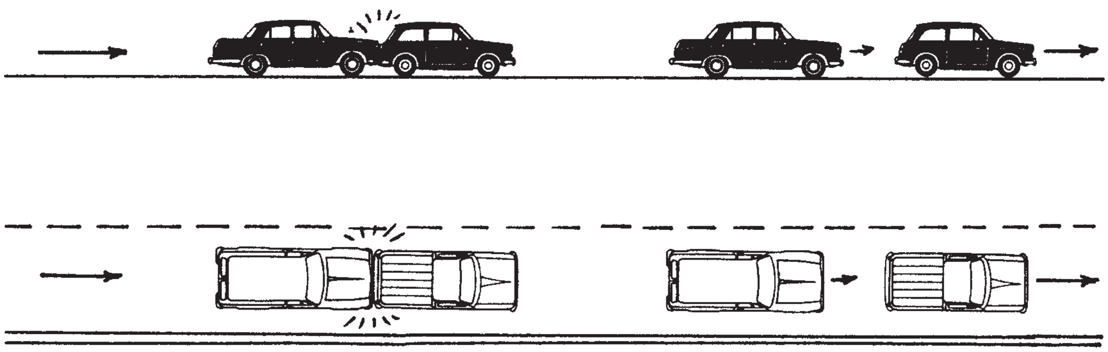

**Figure 6.** Front to back Collision.

The total momentum of the two cars is conserved; so:

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B$$

$$1100 \times 22.22 + 950 \times 0 = 1100 \times v_A + 950 \times 16.67$$

$$v'_A = 7.82 m/s (28.15 km/h)$$

$$e = \frac{v'_B - v'_A}{v_B - v_B} = \frac{16.67 - 7.82}{22.22 - 0} = 0.40$$

### 2.2. Oblique Central Impact

If the velocities of two objects are in different directions but their mass centers are on the plane normal, this method is still valid. The velocity vector is then the resultant of two components, one being on the impact plane, and the other vertical (Fig.7).

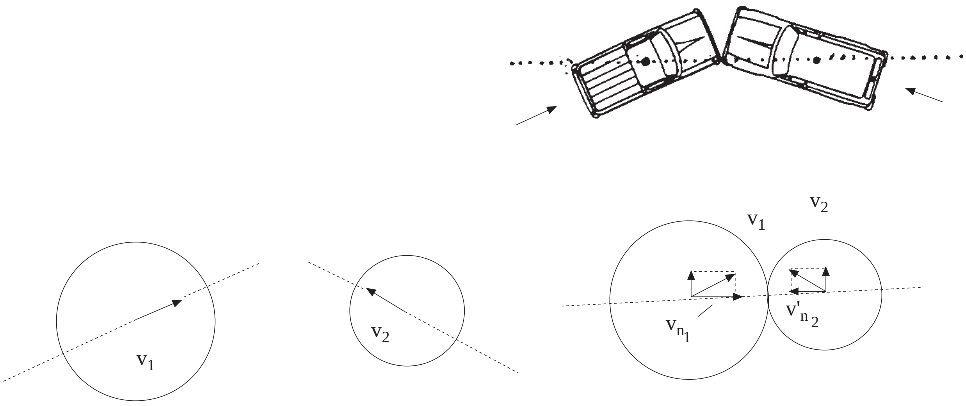


**Figure 7.** Oblique Central Impact.

In this case, vertical components remain the same; since the momentum at the plane perpendicular to the impact plane is preserved for each vehicle, velocities at the same plane also remain constant (the coefficient of friction is assumed to be zero); whereas for the components on the plane normal, it can be written as:

$$e = -\frac{(v'_{n_2} - v'_{n_1})}{(v_{n_2} - v_{n_1})} \tag{9}$$

If the Coefficient of Restitution “e” varying between zero (0) and one (1), is close to both sides, it creates an undesirable situation. In fact, “e” being one (1) or close to one (1) causes an Elastic Impact, meaning the energy absorbed by the vehicle is nearly zero (0). As a result, the vehicles do not suffer great damage. This case, which may seem to be less harmful at first, may cause injuries to the people inside the vehicles. In this case, the energy exerted through the impact is not absorbed by the vehicle’s body. However, this energy is still present, and will show its effect in one way or another. This effect is passed on to the people inside the vehicle instead of to the vehicle’s body, and in most cases people are injured by hitting their heads, necks, bodies, arms or legs on the interior of the vehicle. For this reason, cars are being made in such a way that they can absorb the harmful energy to some extent, so that the people in the vehicle are as safe as possible.

In a case where “e” is zero (0) or close to zero (0), the vehicles collide and consequently the people in the vehicle are put in danger. In this situation, the people inside the car are squeezed or crushed. Because of this, modern vehicle bodies are produced so that they can crumple and absorb the impact energy, and, at the same time, allow the least possible injury to the people in the vehicle. This situation means that the coefficient of restitution ”e” is between zero (0) and one (1); that is, in light of the explanations above, a situation in which the people inside the car are harmed as little as possible, corresponding to neither an Elastic nor Plastic impact but an Elastoplastic impact. In Figure 8, some types of Oblique Central Impact are shown.

### 2.3. Eccentric Impact and Rotations

It is inevitable that vehicles undergo rotating displacements during the collision because the front wheels are pivotal. For this reason, it has to be accepted that collisions occurring in highway traffic cause rotating displacements after impact. The vehicle velocities, the type of collision, the parts of the vehicles at which the crash occurs, the first and last angles of the front wheels during the collision and material properties determine the new orbits and ultimate positions of the vehicles.

Now, the Eccentric Impact of two rigid bodies is analyzed below. Before collision, their velocities are $v_A$ and $v_B$ (Fig. 9a).

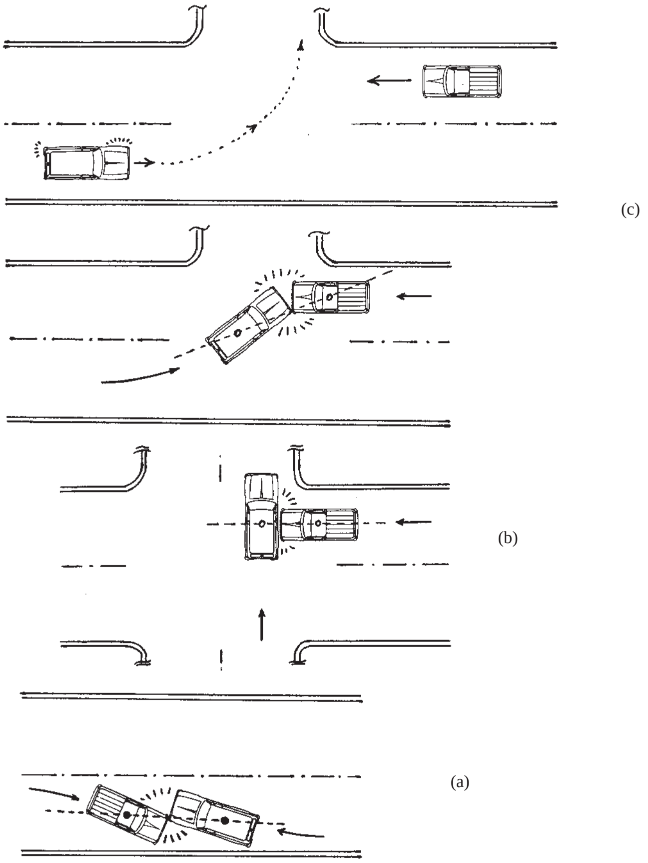


**Figure 8.** Examples of Oblique Central Impact at Crossroads.

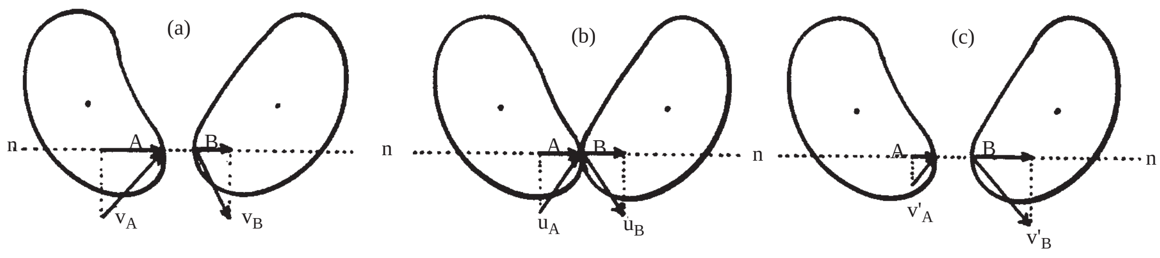


**Figure 9.** Eccentric Impact.

During impact, these two bodies will be deformed and when the impact is over, their velocities will change to $u_A$ and $u_B$ having equal components along the line of impact nn (Fig. 9b). A period of restitution will then take place, at the end of which A and B will have velocities $v'_A$ and $v'_B$ (Fig. 9c). Assuming that friction is negligible it is found that the forces they exert on each other are directed along the line of impact. Denoted respectively by $\int Pdt$ and $\int Rdt$, the magnitude of the impulse of one of these forces during the period of deformation and during the period of restitution, reference is made to the equation:

$$e = \frac{\int Rdt}{\int Pdt} \tag{10}$$

The relations established in section 2.1. and 2.2. between the relative velocities of two bodies before and after the impact are also valid between the components along the line of impact of the relative velocities at the two points of contact, A and B. To show this:

$$(v'_B)_n - (v'_A)_n = e\,[(v_A)_n - (v_B)_n] \tag{11}$$

It is first assumed that the motion of each of the colliding bodies is unconstrained. Thus, the impulses exerted on the bodies during impact are effective at A and B. By considering the body at point A, all three momentum and impulse diagrams corresponding to the period of deformation can be drawn (Fig. 10).

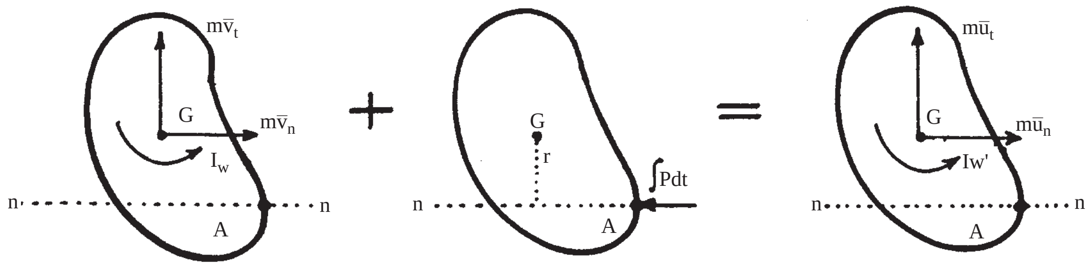


**Figure 10.** Analysis of an Eccentric Impact.

The velocity of the mass center at the beginning and at the end of the period of deformation is denoted by v and u respectively, and the angular velocity of the body at the same instants by $w$ and $w^0$. Adding and equating the components of the momenta and impulses along the line of impact nn, the following equation can be written:

$$m\overline{v}_n - \int Pdt = m\overline{u}_n \tag{12}$$

Adding and equating the moments about G:

$$\overline{I}w - r\int Pdt = \overline{I}w^\circ \tag{13}$$

where $r$ represents the perpendicular distance from G to the line of impact. Considering now the period of restitution, it is obtained in a similar way:

$$m\overline{u}_n - \int Rdt = m\overline{v}'_n \tag{14}$$

$$\overline{I}w^\circ - r\int Rdt = \overline{I}w' \tag{15}$$

where $\overline{v}'$ and $w'$ represent respectively the velocity of the mass center and the angular velocity of the body after impact. Solving (12) and (14) for the two impulses and substituting into (10), and then solving (13) and (15) for the same two impulses and substituting again into (10), the following two alternative expressions for the coefficient of restitution are obtained:

$$e = \frac{\overline{u}_n - \overline{v}'_n}{\overline{v}_n - \overline{u}_n} \qquad e = \frac{w^\circ - w'}{w - w^\circ} \tag{16}$$

Now, when considered that: $\dfrac{A}{B} = \dfrac{C}{D} = \dfrac{A + xC}{B + xD}$ (17)

the above formula (16) is reformulated as:

$$e = \frac{\overline{u}_n + rw^\circ - (\overline{v}'_n + rw')}{\overline{v}_n + rw - (\overline{u}_n + rw^\circ)} \tag{18}$$

Observing that $\overline{v}_n + rw$ represents the component $(v_A)_n$ along nn of the velocity of the point of contact A and that, similarly, $\overline{u}_n + rw^\circ$ and $\overline{v}'_n + rw'$ represent the components $(u_A)_n$ and $(v'_A)_n$, respectively:

$$e = \frac{(u_A)_n - (v'_A)_n}{(v_A)_n - (u_A)_n} \tag{19}$$

The analysis of the motion of the second body leads to a similar expression for "e" in terms of the components along nn of the successive velocities of point B. Recalling that $(u_A)_n = (u_B)_n$ and eliminating these two velocity components by manipulation similar to that used in 2.1. and 2.2., relation (11) is obtained.

The Eccentric Impact of two rigid bodies is defined as an impact in which the mass centers of the colliding bodies are not located on the line of impact. It has been shown that in such a situation a relation similar to that derived in Section 2 for the Central Impact of two bodies and involving the coefficient of restitution "e" still holds, but that the velocities at points A and B where contacts occur during the impact should be used. In equation (11), which is:

$$(v'_B)_n - (v'_A)_n = e\,[(v_A)_n - (v_B)_n]$$

$(v_A)_n$ and $(v_B)_n$ are the components along the line of impact of the velocities of A and B before impact, and $(v'_A)_n$ and $(v'_B)_n$ are their components after impact. This equation is applicable not only when colliding bodies move freely after impact but also when bodies are partially constrained in their motion. It should be used in conjunction with one or several of other equations obtained by applying the principle of impulse and momentum. It can also be considered in problems where the method of impulse and momentum and the method of work and energy may be combined.

Two vehicles collide; all velocities before impact are generally known; velocities $v'_i$ and angular velocities $w'_i$ after impact are to be determined.

**Example 3:** At an intersection, a 900 kg. vehicle A, moving along road a-a with a velocity of 60 km/h (16.67 m/s), strikes a 1000 kg. vehicle B moving along b-b, as seen in Figure 11.

Assuming that the coefficient of restitution between the two vehicles is 0.80, the angular velocity of the vehicle B and the velocity of the vehicle A immediately after impact can be determined.

It is also assumed that vehicle B rotates about point 0, and G is its mass center, and that the external impulse force is the impulsive reaction at 0. Therefore, the two vehicles can be considered as a single system and it can be expressed that the initial momenta of A and B and the impulses of external forces are together equivalent to the final momenta of the system.

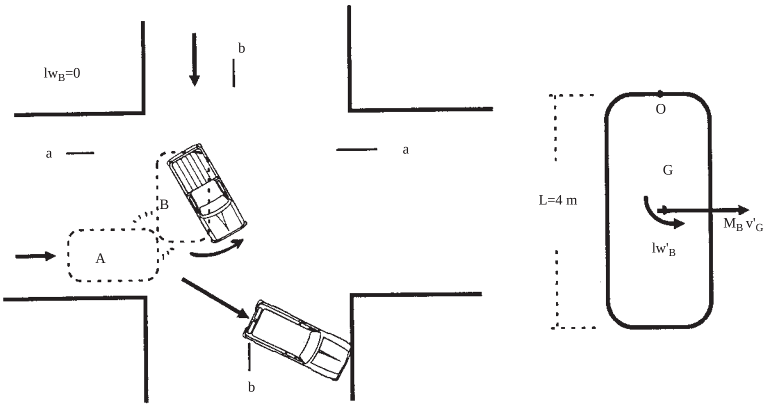


**Figure 11.** Eccentric Impact at an intersection.

The moments about 0 can be expressed as:

$$m_A \cdot v_A \cdot L = m_A \cdot v'_A \cdot L + m_B \cdot v'_G \cdot \frac{L}{2} + I_{B_0} \cdot w'_B$$

$$v'_B = \frac{L}{2}\cdot w'_B \text{ and } I_{B_0}\frac{1}{12}\cdot m_B \cdot L^2 = \frac{1}{12}\cdot 1000\cdot 4^2 = 1333 kgm^2$$

Choosing velocities positive to the right, the formula is then :

$$900\cdot\frac{60}{3.6}\cdot 4 = 900.v'_A.4+\cdot v'_A\cdot 4+1000\cdot\frac{4}{2}\cdot w'_B\cdot\frac{4}{2}+1333\cdot w'_B$$

$$60000 = 3600 \cdot v'_A + 5333 \cdot w'_B \qquad (20)$$

$$v'_B - v'_A = e(v_A - v_B) = 0.80(\frac{60}{3.6} - 0)$$

When B rotates about 0: $v'_B = 4 \cdot w'_B$
Then,

$$4w'_B - v'_A = 13.33 \quad (21)$$

Solving Equations (20) and (21), the following results are obtained:

$$w'_B = 5.47 rad/sec. \text{ and } v'_A = 8.55 m/sec = 31 km/h.$$

Some types of Eccentric Collision are shown in Fig. 12.

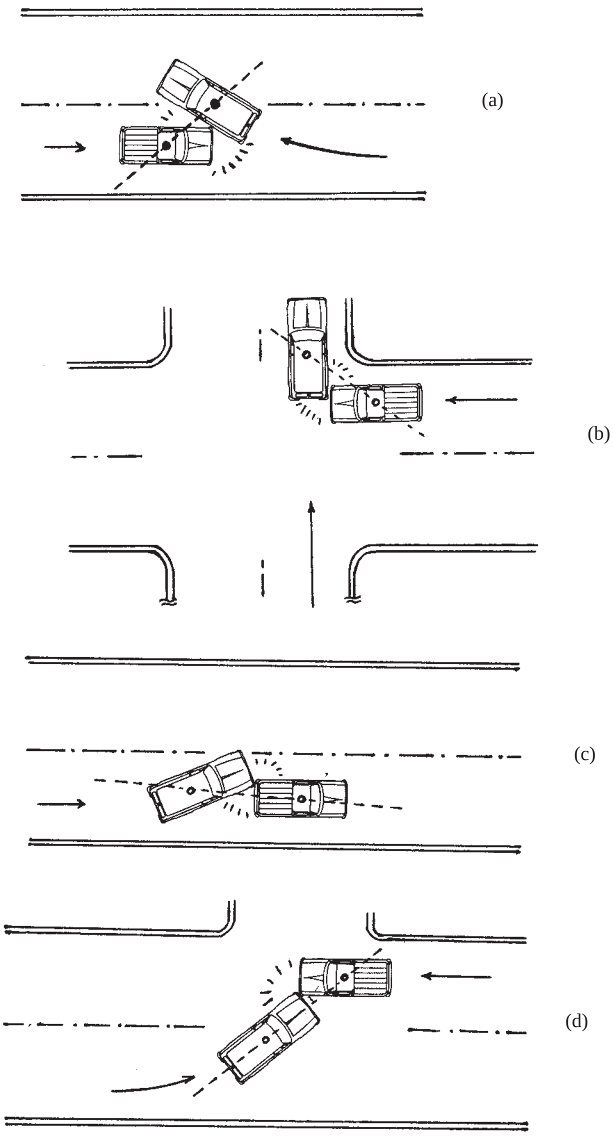


**Figure 12.** Some types of Eccentric Collision.

## 3. Conclusions

In this study, besides well-known and important parameters such as velocity and mass dimensions, some specific parameters, such as the intensity of impact, the impact areas and their trajectories, etc. have also been analyzed.

In general, the theory analyses the subject step by step. As is known, vehicles have four wheels and the impact starts at one point but quickly affects a certain area. Because of this, a certain amount of energy is not only lost at a specific point but in a wider area. In addition, there is another loss of energy due to the friction force between the wheels and the road surface. These losses are evaluated in the Coefficient of Restitution "e", which defines the Intensity of the Impact.

Except in minor collisions, practically no vehicle can remain undamaged after a crash. Since the impact energy is absorbed by damage to the vehicle's body, the people inside a car are protected to a limited extent. However, during crashes at high speeds, the energy released is absorbed both by the vehicles and the people inside. Therefore, severe injuries may happen in such a case. The theories developed in this paper may help the search for practical solutions by simulation in the design of passenger cars to increase human safety, but only to a limited extent because certain assumptions are made and some parameters are neglected. A certain range of errors must be accepted and the theory has to be improved.

The accuracy of the results is related to the correct and appropriate determination of the hypothesis and assumptions. The hypothesis and assumptions have to be appropriate.

Finally, it is stated that every type of collision theory can be applied to traffic accidents. It is hoped that this study can help to develop further the science of accidentology by providing a basis for other studies in the future.

In respect of accidentology and highway traffic safety, it is hoped that, by obtaining certain practical results using such a theory, further studies in the future can be facilitated.

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