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Elasticity solution for a functionally graded two-layer beam with simple supported edges

İsa ÇÖMEZ*
Department of Civil Engineering, Faculty of Engineering, Karadeniz Technical University, Trabzon, Turkey

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Abstract: An analytical solution for a functionally graded two-layer beam subjected to transverse loading is handled based on the theory of elasticity. The upper and lower layers are fully bonded to each other and simply supported at the edges. Poisson's ratios are taken as constant and Young's moduli are assumed to vary exponentially through the thickness of the layers. Numerical results for the normal stresses and shear stress are given as a solution and the effect of grading on the stress distributions is investigated. In the functionally graded beam solution it is required that one side of the beam is stiffer while the other side is softer when Young's modulus is assumed to vary exponentially through the thickness. The exponential variation of the elasticity modulus may be symmetrical about the mid-plane when the beam is designated as layered.

Key words: Layered beams, composite beams, functionally graded materials, elasticity solution

1. Introduction

Functionally graded materials (FGMs) are inhomogeneous composites whose material properties vary gradually in one or more directions. Due to their application to a great variety of structures in modern industries such as aerospace, automobile, biomedical, and nuclear energy, FGMs have been widely studied during the past two decades.

As the application of FGMs has increased, new methodologies have been developed to analyze the mechanical behavior of structural elements made of these materials. Bakirtas [1] investigated the contact problem of rigid punch on a nonhomogeneous elastic half space where the elasticity modulus was assumed to vary exponentially with depth. Delale and Erdogan [2] studied the crack problem for a nonhomogeneous plane with an exponentially varying shear modulus in the crack surface direction. Sankar [3] developed an elasticity solution and Zhu and Sankar [4] developed a combined Fourier series–Galerkin method for a functionally graded (FG) beam subjected to sinusoidal transverse loading. Zhong and Yu [5] developed a plane elasticity solution for a cantilever FG beam by means of the semiinverse method. Ding et al. [6] considered the plane stress problem of anisotropic FG beams with various end conditions and static loadings. Two dimensional elasticity solutions for the bending and free vibration of FG beams on a Winkler–Pasternak foundation were considered by Ying et al. [7]. Lü et al. [8] presented the semianalytical solutions of FG beams, with an exponentially varying elasticity modulus in two directions, using the state space-based differential quadrature method. Free vibration analyses of FG beams were given by Aydogdu and Taskin [9],Yang and Chen [10], and Sina et al. [11],

*Correspondence: isacomez@hotmail.com

and nonlinear vibration analyses of FGMs were given by Şimşek [12], Ke et al. [13], and Shooshtari and Rafiee [14]. Dag et al. [15] developed analytical and computational methods for the sliding contact problem between rigid punches and a FG half plane assuming that the shear modulus of the half-plane varies exponentially along the lateral direction.

Wang and Liu [16] investigated the bending problem of a bimaterial beam with a graded intermediate layer under various end conditions. Kapuria et al. [17] studied the static and free vibration response of layered FGM beams experimentally and theoretically. Morozov and Tovstik [18] considered the bending problem of two-layer beam strips with nonrigid contact between them.

In this study, an analytical solution for a FG two-layer beam subjected to transverse loading is handled based on the theory of elasticity. The upper and lower layers are fully bonded to each other and simply supported at the edges. Poisson's ratios are taken as constant and Young's moduli are assumed to vary exponentially through the thickness of the layers. Numerical results for the normal stresses and shear stress are given as a solution and the effect of grading on the stress distributions is investigated. In the FG beam solution it is required that one side of the beam is stiffer while the other side is softer when Young's modulus is assumed to vary exponentially through the thickness. This restriction is removed by designing the beam to be layered.

2. Elasticity analysis for graded two-layer beam

Figure 1 shows a layered FG beam of thickness h and length L. The beam is assumed to be in a state of plane strain normal on the xy plane, and the width in the z-direction is taken as unity. While the bottom surface of the beam is free of tractions, the upper surface of the beam is subjected to symmetric normal tractions:

$$p_y(x) = p\cos\beta x, \tag{1}$$

where

$$\beta = \frac{\pi}{L}. \tag{2}$$

It is assumed that Poisson's ratios are taken as constant and Young's moduli are assumed to vary exponentially through the thickness of the layers as follows:

$$E_i(y) = E_{0i}\, e^{\gamma_i y}, \tag{3}$$

where E_{0i} denotes the elasticity modulus at the interface of the layers. The subscript $i (i = 1, 2)$ denotes the upper layer and lower layer respectively, and γ_i are the constants characterizing the gradual variation of the material properties of the layers along the thickness direction. Figure 2 shows the variation of Young's moduli in the thickness direction of the layered FG beam. As can be seen in the figure, many options can be obtained for the exponential variation of Young's moduli when the beam is designated as layered: for example, softer-softer, stiffer-stiffer, and softer-stiffer top and bottom faces.

The differential equations of equilibrium are:

$$\frac{\partial\sigma_{xi}}{\partial x} + \frac{\partial\tau_{xyi}}{\partial y} = 0,$$

$$\frac{\partial\tau_{xyi}}{\partial x} + \frac{\partial\sigma_{yi}}{\partial y} = 0. \tag{4}$$

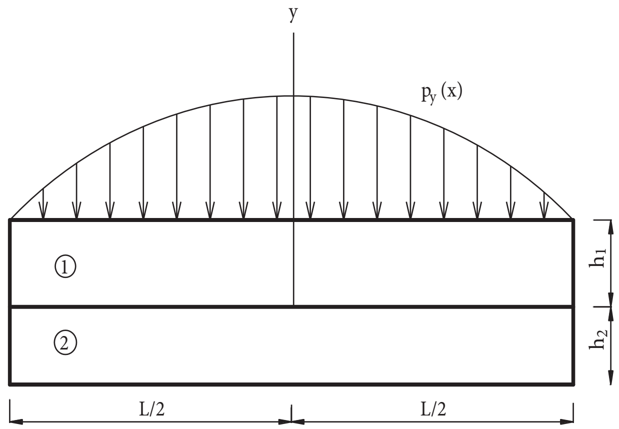

Figure 1. Geometry of a layered FG beam subjected to symmetric transverse loading.

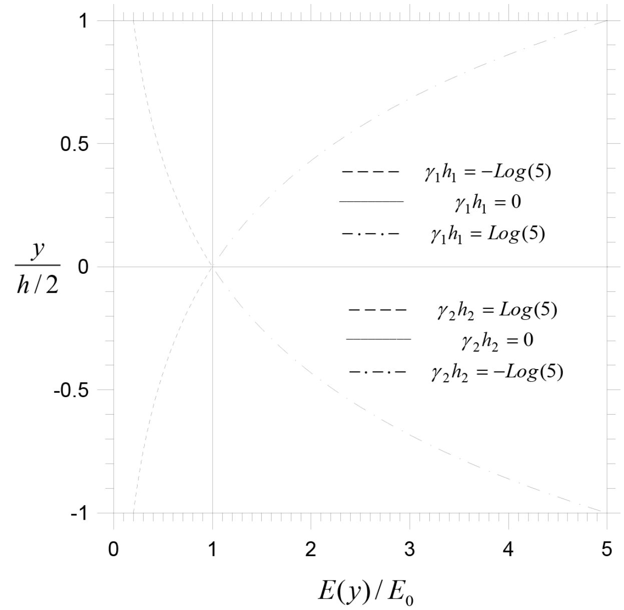

Figure 2. Variation of Young's moduli in the thickness direction, $(h_1 = h_2,\ E_{01} = E_{02} = E_0)$.

Assuming that the FGM is isotropic at every point, Hook's law can be written as

$$\sigma_{xi} = (\lambda_i + 2\mu_i)\frac{\partial u_i}{\partial x} + \lambda_i \frac{\partial v_i}{\partial y},$$

$$\sigma_{yi} = \lambda_i \frac{\partial u_i}{\partial x} + (\lambda_i + 2\mu_i)\frac{\partial v_i}{\partial y},$$

$$\tau_{xyi} = \mu_i\left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\right), \tag{5}$$

where u_i, v_i $(i = 1, 2)$ are the x- and y-components of the displacement vector and λ_i and μ_i are the Lamé constants.

$$\lambda_i(x) = \frac{\nu_i E_i(y)}{(1+\nu_i)(1-2\nu_i)}$$

$$\mu_i(x) = \frac{E_i(y)}{2(1+\nu_i)} \tag{6}$$

Substituting Eq. (5) into Eq. (4), the following equations in $u_i(x, y)$ and $v_i(x, y)$ are obtained.

$$\frac{\partial}{\partial x}\left((\lambda_i + 2\mu_i)\frac{\partial u_i}{\partial x} + \lambda_i\frac{\partial v_i}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu_i\left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\right)\right) = 0$$

$$\frac{\partial}{\partial x}\left(\mu_i\left(\frac{\partial u_i}{\partial y} + \frac{\partial v_i}{\partial x}\right)\right) + \frac{\partial}{\partial y}\left(\lambda_i\frac{\partial u_i}{\partial x} + (\lambda_i + 2\mu_i)\frac{\partial v_i}{\partial y}\right) = 0 \tag{7}$$

The solutions are assumed in a form such that satisfies the boundary conditions on the left and right end faces of the simple supported beam [3].

$$u_i(x, y) = \varphi_i(y)\,\sin\beta x$$

$$v_i(x,y) = \psi_i(y)\,\cos\beta\,x \tag{8}$$

Substituting Eq. (8) into Eq. (7) and applying Eqs. (3) and (6), the following ordinary differential equations are obtained.

$$(1-2\nu_i)\frac{d^2\varphi_i}{dy^2} + \gamma_i(1-2\nu_i)\frac{d\varphi_i}{dy} - 2\beta^2(1-\nu_i)\varphi_i - \beta\frac{d\psi_i}{dy} - \beta\gamma_i(1-2\nu_i)\psi_i = 0$$

$$\beta\frac{d\varphi_i}{dy} + 2\beta\gamma_i\nu_i\varphi_i + 2(1-\nu_i)\frac{d^2\psi_i}{dy^2} + 2\gamma_i(1-\nu_i)\frac{d\psi_i}{dy} - \beta^2(1-2\nu_i)\psi_i = 0 \tag{9}$$

The solution of Eq. (9) can be obtained as:

$$\varphi_i(y) = \sum_{j=1}^{4} A_{ij}e^{s_{ij}\,y},$$

$$\psi_i(y) = \sum_{j=1}^{4} A_{ij}m_{ij}e^{s_{ij}\,y}, \tag{10}$$

where $s_{ij}(i=1,2\,;j=1,...,4)$ satisfies the following characteristic equation.

$$s_i^4 + 2\gamma_i\,s_i^3 + (\gamma_i^2 - 2\beta^2)s_i^2 - 2\beta^2\gamma_i\,s_i + \beta^2(\beta^2 + \gamma_i^2\frac{\nu_i}{1-\nu_i}) = 0 \tag{11}$$

The roots of the characteristic equation are given by

$$s_{i1} = \frac{1}{2}\left(-\gamma_i - \sqrt{4\beta^2 + \gamma_i^2 + 4i\beta\,|\gamma_i|\sqrt{\frac{\nu_i}{1-\nu_i}}}\right),$$

$$s_{i2} = \frac{1}{2}\left(-\gamma_i - \sqrt{4\beta^2 + \gamma_i^2 - 4i\beta\,|\gamma_i|\sqrt{\frac{\nu_i}{1-\nu_i}}}\right),$$

$$s_{i3} = \frac{1}{2}\left(-\gamma_i + \sqrt{4\beta^2 + \gamma_i^2 + 4i\beta\,|\gamma_i|\sqrt{\frac{\nu_i}{1-\nu_i}}}\right),$$

$$s_{i4} = \frac{1}{2}\left(-\gamma_i + \sqrt{4\beta^2 + \gamma_i^2 - 4i\beta\,|\gamma_i|\sqrt{\frac{\nu_i}{1-\nu_i}}}\right), \tag{12}$$

and

$$m_{ij} = \frac{(s_{ij} + 2\gamma_i\nu_i)\left(2\,s_{ij}^2(1-\nu_i) + 2s_{ij}\,\gamma_i(1-\nu_i) - \beta^2(3-2\nu_i)\right)}{\beta\,(\beta^2 - 4\gamma_i^2(-1+\nu_i)\nu_i)}. \tag{13}$$

Substituting Eqs. (8) and (10) into Eq. (5), the stress components for the beam can be written as follows:

$$\sigma_{xi}(x,y) = \frac{E_i(y)}{(1+\nu_i)(1-2\nu_i)}\sum_{j=1}^{4} A_{ij}\left[\beta(1-\nu_i) + \nu_i\,m_{ij}s_{ij}\right]\,e^{s_{ij}y}\cos\beta x,$$

$$\sigma_{y_i}(x,y)=\frac{E_i(y)}{(1+\nu_i)(1-2\nu_i)}\sum_{j=1}^{4}A_{ij}\left[\beta\nu_i+(1-\nu_i)m_{ij}s_{ij}\right]\,e^{s_{ij}y}\cos\beta x,$$

$$\tau_{xyi}(x,y)=\frac{E_i(y)}{2(1+\nu_i)}\sum_{j=1}^{4}A_{ij}\left[s_{ij}-\beta m_{ij}\right]\,e^{s_{ij}y}\sin\beta x, \tag{14}$$

where $A_{ij}(i=1,2\,;j=1,...,4)$ are the unknown constants that will be determined from the boundary conditions on the top and bottom surfaces of the beam and the interface continuity conditions.

The boundary and interface continuity conditions for the beam can be defined as follows.

$$\begin{gathered}
\sigma_{y1}(x,h_1)=-p\cos\beta x\\
\tau_{xy1}(x,h_1)=0\\
\sigma_{y1}(x,0)=\sigma_{y2}(x,0)\\
\tau_{xy1}(x,0)=\tau_{xy2}(x,0)\\
u_1(x,0)=u_2(x,0)\\
v_1(x,0)=v_2(x,0)\\
\sigma_{y2}(x,-h_2)=0\\
\tau_{xy2}(x,-h_2)=0
\end{gathered} \tag{15}$$

Using the boundary and interface continuity conditions given by Eq. (15), eight equations to find A_{ij} can be obtained.

$$\frac{E_{01}e^{\gamma_1 h_1}}{(1+\nu_1)(1-2\nu_1)}\sum_{j=1}^{4}A_{1j}\left[\beta\nu_1+(1-\nu_1)m_{1j}s_{1j}\right]\,e^{s_{1j}h_1}=-p$$

$$\sum_{j=1}^{4}A_{1j}\left[s_{1j}-\beta m_{1j}\right]\,e^{s_{1j}h_1}=0$$

$$\sum_{j=1}^{4}\left\{\left[\beta\nu_1+(1-\nu_1)m_{1j}s_{1j}\right]\,A_{1j}-\frac{E_{02}(1+\nu_1)(1-2\nu_1)}{E_{01}(1+\nu_2)(1-2\nu_2)}\left[\beta\nu_2+(1-\nu_2)m_{2j}s_{2j}\right]A_{2j}\right\}=0$$

$$\sum_{j=1}^{4}\left\{\left[s_{1j}-\beta m_{1j}\right]A_{1j}-\frac{E_{02}(1+\nu_1)}{E_{01}(1+\nu_2)}\left[s_{2j}-\beta m_{2j}\right]A_{2j}\right\}=0$$

$$\sum_{j=1}^{4}(A_{1j}-A_{2j})=0$$

$$\sum_{j=1}^{4}(A_{1j}m_{1j}-A_{2j}m_{2j})=0$$

$$\sum_{j=1}^{4}A_{2j}\left[\beta\nu_2+(1-\nu_2)m_{2j}s_{2j}\right]\,e^{-s_{2j}h_2}=0$$

$$\sum_{j=1}^{4} A_{2j}\left[s_{2j}-\beta m_{2j}\right] e^{-s_{2j}h_2}=0 \tag{16}$$

After determining $A_{ij}(i=1,2\,;j=1,...,4)$ from Eq. (16), the stresses and the displacements at any point in the layered beam can be evaluated.

3. Results and discussion

The effect of vertical grading on the normal stresses and shear stress is given in the figures. In the numerical solutions Poisson's ratio was taken as 0.25 ($\nu_i = 0.25$). In the figures, E_{h1} and E_{h2} denote the elasticity moduli on the top and bottom surfaces of the layered beam, respectively.

Figure 3 shows the axial stress $\sigma_x(0,y)$ distribution along the symmetry plane ($x = 0$) of the layered beam for various values of the inhomogeneity parameter (γ_i) in the thickness direction or E_{hi}/E_0 in the case of $\gamma_1 h_1 = \gamma_2 h_2$. The axial stress is compressive in the upper portion (near the loaded surface) and tensile in the lower portion (near the free surface) of the layered beam. The compressive stress increases on the loaded surface but the tensile stress decreases on the free surface when the loaded surface is stiffer than the free surface ($E_{h1} = 10E_0 = 100E_{h2}$). In contrast, the compressive stress decreases but the tensile stress increases when the loaded surface is softer. In the case of a homogeneous layered beam $\gamma_i h_i = 0$ ($E_{hi} = E_0$), the axial stress distribution becomes linear.

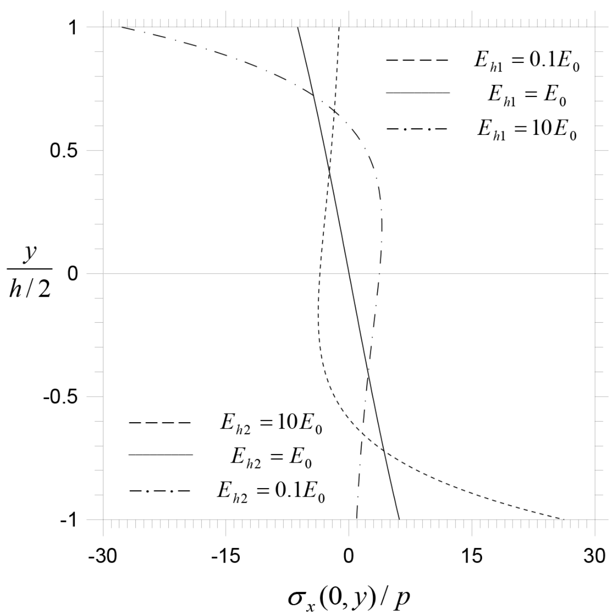

Figure 3. Axial stress $\sigma_x(0,y)$ distribution along the thickness of the layered beam for various values of E_{hi}/E_0: variation of Young's modulus is not symmetrical about the mid-plane, ($h_1 = h_2,\ h/L = 1/\pi,\ E_{01} = E_{02} = E_0$).

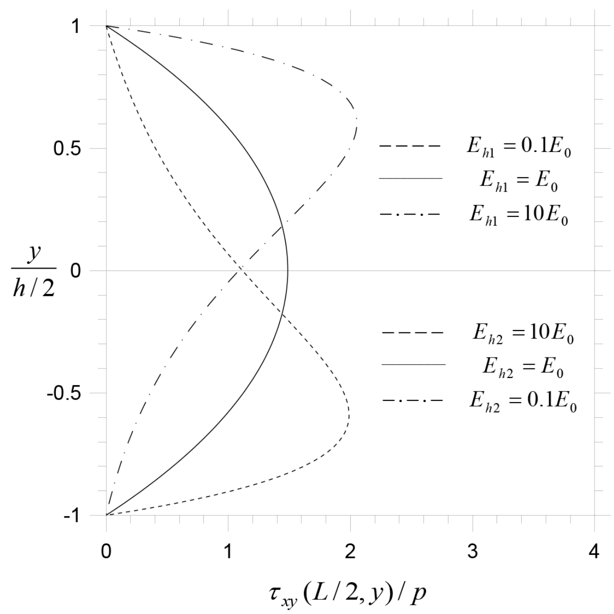

Figure 4. Transverse shear stress $\tau_{xy}(L/2,y)$ distribution along the right face of the layered beam for various values of E_{hi}/E_0: variation of Young's modulus is not symmetrical about the mid-plane, ($h_1 = h_2,\ h/L = 1/\pi,\ E_{01} = E_{02} = E_0$).

Figure 4 shows the shear stress $\tau_{xy}(L/2,y)$ distribution along the right face of the layered beam for various values of E_{hi}/E_0 in the case of $\gamma_1 h_1 = \gamma_2 h_2$. For the homogeneous layered beam the shear stress distribution resembles the typical parabolic profile as in beam theory and the maximum value occurs at the middle point of the face ($y = 0$). For an FG layered beam the maximum value of the shear stress moves to the softer layer. The shear stress is zero at the top and bottom point of the face, which is consistent with the boundary conditions.

When the inhomogeneity parameter of the lower layer is $\gamma_2 h_2 = -\gamma_1 h_1$, variation of Young's modulus of the layered beam becomes symmetrical about the mid-plane. That variation keeps the peak values of compressive stresses and tensile stresses equal in the layered beam (Figure 5). The maximum absolute value of $\sigma_x(0, y)$ decreases when the upper and lower surfaces of layered beams are softer than the mid-plane, and the point where the maximum stress occurs is close to the mid-plane ($E_{h1} = E_{h2} = 0.1E_0$).

As well as the homogeneous case, the shear stress $\tau_{xy}(L/2, y)$ distribution is symmetrical about the mid-plane when Young's modulus of the layered beam is symmetrical ($\gamma_2 h_2 = -\gamma_1 h_1$) in the FG case (Figure 6). The shear stress increases with the decreasing E_{hi}/E_0, andthe maximum value of the shear stress occurs at $y = 0$ in both cases.

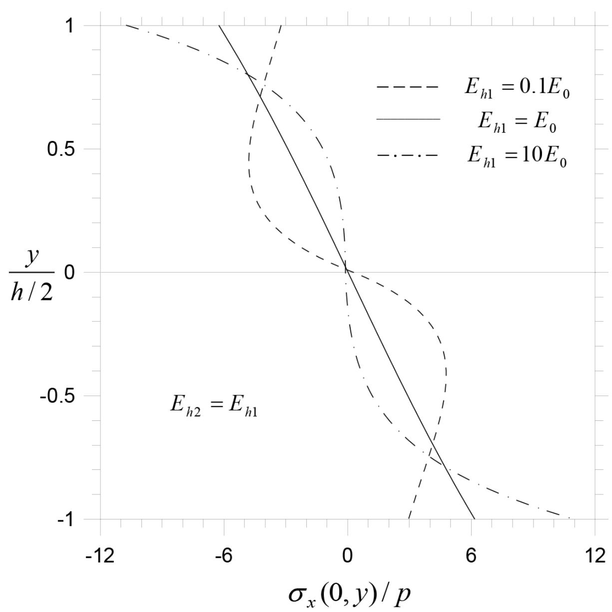

Figure 5. Axial stress $\sigma_x(0, y)$ distribution along the thickness of the layered beam for various values of E_{hi}/E_0: variation of Young's modulus is symmetrical about the mid-plane, ($h_1 = h_2$, $h/L = 1/\pi$, $E_{01} = E_{02} = E_0$).

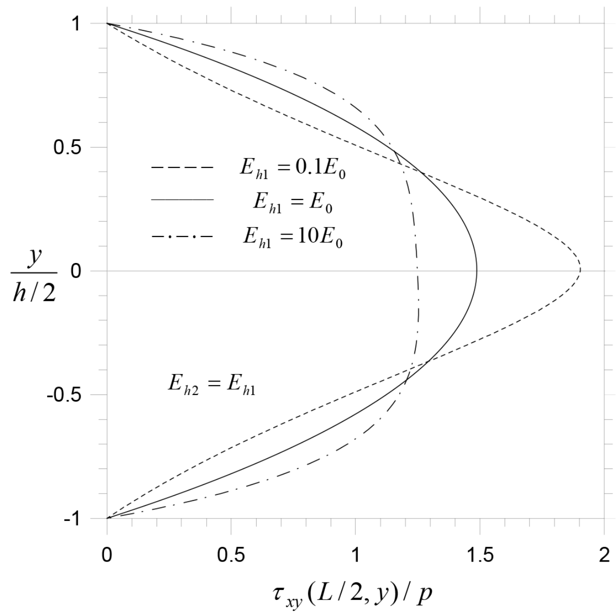

Figure 6. Transverse shear stress $\tau_{xy}(L/2, y)$ distribution along the right face of the layered beam for various values of E_{hi}/E_0: variation of Young's modulus is symmetrical about the mid-plane. ($h_1 = h_2$, $h/L = 1/\pi$, $E_{01} = E_{02} = E_0$).

Figures 7 and 8 show the axial stress $\sigma_x(0, y)$ and the shear stress $\tau_{xy}(L/2, y)$ distributions along the thickness of the layered beam for various values of E_{02}/E_{01} with the homogeneous lower layer, respectively. In the case of a stiffer lower layer ($E_{02} = 10E_{01}$) the axial stress $\sigma_x(0, y)$ and the shear stress $\tau_{xy}(L/2, y)$ decrease in the upper layer but increase in the stiffer lower layer. It may be observed in Figure 7 that the axial stress $\sigma_x(0, y)$ that occurred in the layers takes the same value in the interface for $E_{02} = E_{01}$.

Figures 9 and 10 show the axial stress $\sigma_x(0, y)$ and the shear stress $\tau_{xy}(L/2, y)$ distributions along the thickness of the layered beam for various values of h/L, respectively.Small values of h/L correspond to long, thin beams while larger values of h/L correspond to short, thick beams. When decreasing the value ofh/L both the axial stress $\sigma_x(0,y)$ and the shear stress $\tau_{xy}(L/2,y)$ increase, as expected from beam theory.

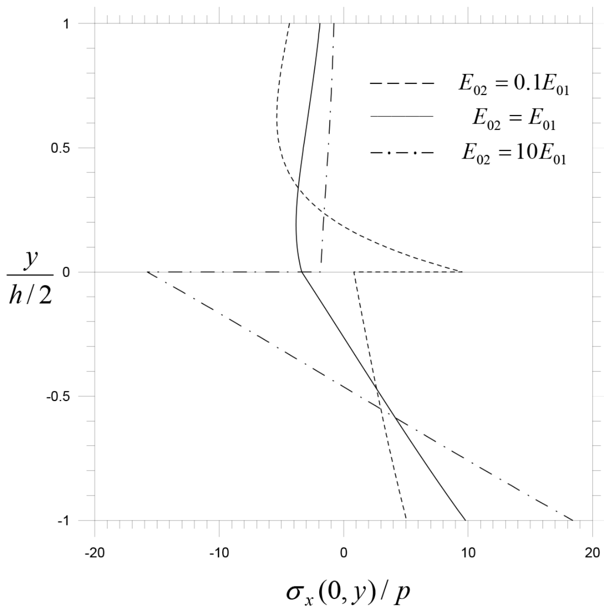

Figure 7. Axial stress $\sigma_x(0,y)$ distribution along the thickness of the layered beam for various values of E_{02}/E_{01}, in the case of homogeneous lower layer. ($h_1 = h_2,\ h/L = 1/\pi,\ E_{h1} = 0.1E_{01},\ E_{h2} = E_{02}$).

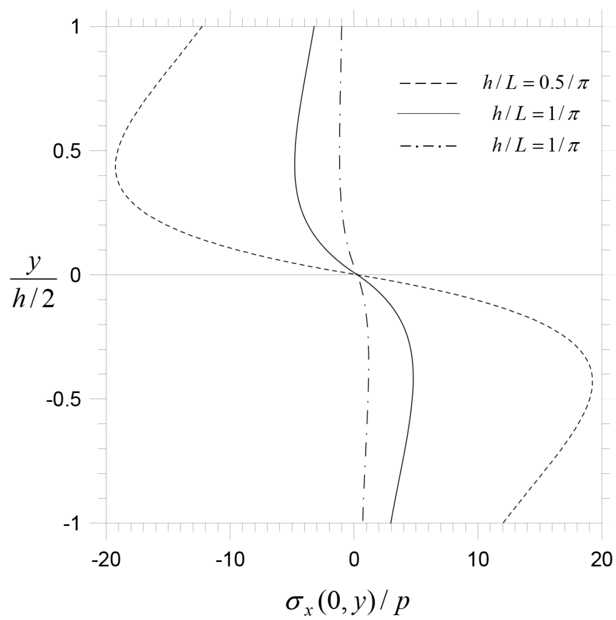

Figure 8. Transverse shear stress $\tau_{xy}(L/2,y)$ distribution along the right face of the layered beam for various values ofE_{02}/E_{01} in the case of homogeneous lower layer, ($h_1 = h_2,\ h/L = 1/\pi,\ E_{h1} = 0.1E_{01},\ E_{h2} = E_{02}$).

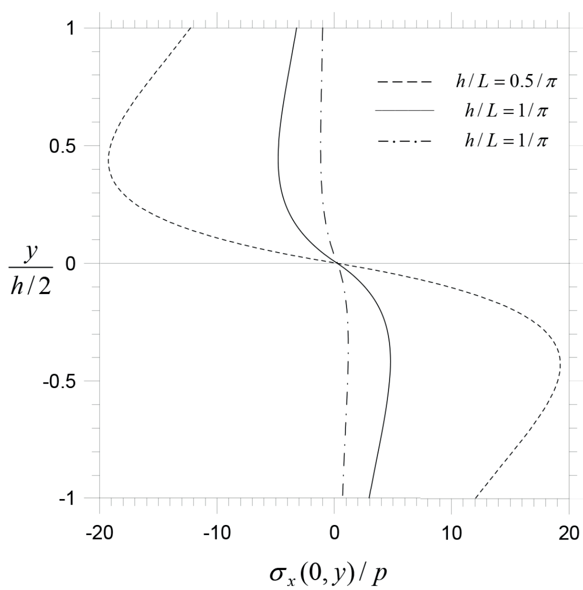

Figure 9. Axial stress $\sigma_x(0,y)$ distribution along the thickness of the layered beam for various values of h/L, ($h_1 = h_2,\ E_{h1} = E_{h2} = 0.1E_0$).

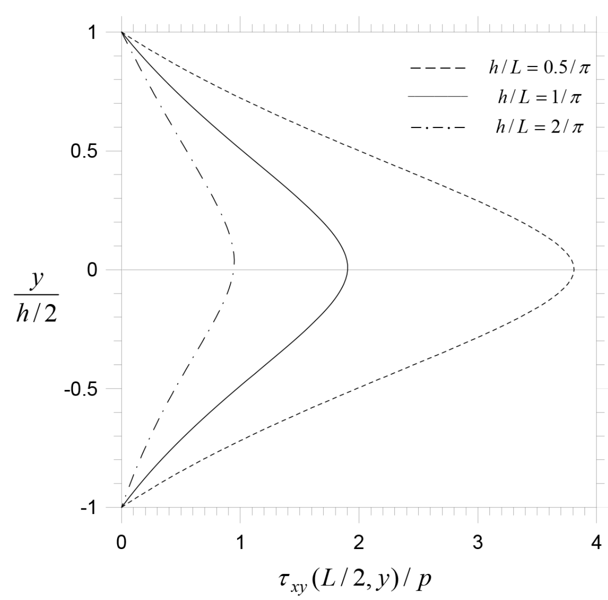

Figure 10. Transverse shear stress $\tau_{xy}(L/2,y)$ distribution along the right face of the layered beam for various values of h/L. ($h_1 = h_2,\ E_{h1} = E_{h2} = 0.1E_0$).

4. Conclusions

An exact solution is obtained for the FG two-layer beam subjected to cosinusoidal transverse loading, based on the theory of elasticity. The exponential variation of the elasticity modulus may be symmetrical about the mid-plane when the beam is designated as layered.

In the case of a simple beam when the stiffer side is loaded vertically, grading has a stress-reducing effect on the unloaded softer side but the stress on the loaded stiffer side increases, and the reverse is true when the softer side is loaded.

The symmetrical variation of the elasticity modulus about the mid-plane keeps the peak values of compressive stresses and tensile stresses equal in the layered beam. The maximum absolute value of $\sigma_x(0, y)$ decreases but the shear stress $\tau_{xy}(L/2, y)$ increases when the top and bottom surfaces of layered beams are softer than the mid-plane.

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