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A unified model for fuel spray penetration

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Abstract

This paper proposes a new unified (single) model that considers 3 flow regimes (viz: Stokes, Allen and Newton) in the initial stage of fuel spray penetration. An analytical result which takes into account of the flow regimes is derived via variational iteration method (VIM) of the unified model that marries 2 special limiting cases of the velocity of the gas (air) stream. The new results for fuel spray penetration under the limiting cases are the same when compared with previously obtained expressions via successive approximations in the available literature. Therefore, it has been shown that the VIM is very much easy, reliable and compatible with the nonlinear nature of the physical problem. Also, important features of the problem are discussed.

Key Words: Fuel spray model, spray penetration, successive approximation, variational iteration method

1. Introduction

Spray penetration is one of the key characteristics that influence vapour distribution, vapour mixing in air and combustion chamber gas turbulence. In particular, fuel spray penetration (FSP) has practical applications ranging from agricultural sprays to sprays in machineries such as boilers, diesel engines, gas turbines and space rockets. FSP has great effect on the efficiency and thrust power of engines. The prediction of the flow properties of fuel spray in a combustion engine requires the consideration of 2 phases in the flow field. This is because the turbulence inside the cylinder controls the mixing of the fuel with air. The combustion of FSP has been extensively studied experimentally and theoretically (see Borman and Ragland, 1998; Loth, 2000; Sazhin et al., 2001; Chung, 2002; Iyer et al., 2002; Pozorski et al., 2002). A detailed study on the fundamentals of engine sprays gains more insight when the various phenomena such as the formation of ligaments and their break-up, droplet break-up and evaporation, the entrainment of air and the effects of turbulence (Faeth et al., 1995; Ganji and Khalegi, 2005), just to mention a few, are take into consideration. On many occasions, therefore, it is far more important to establish a hierarchy of the importance of the various processes and develop simplified models suitable for practical applications. In this light, Sazhin et al. (2001) have developed simple analytical models describing the initial stage of spray penetration in 3 flow regimes viz: Stokes, Allen and Newton, respectively.

It is the objective of this paper, therefore, to construct a unified model that considers the 3 flow regimes in the initial stage of fuel spray penetration. An analytical result, which takes into account of the flow regimes is derived via variational iteration method of the unified model that marries 2 special limiting cases of the velocity of the gas (air) stream. These solutions give a wider applicability in understanding the basic physics of the problem, which are particularly important in industrial and technological fields.

The sections followed hereafter respectively are: the mathematical formulation of the problem, and hence the unified model; the construction of the solution via variational iteration method; and general concluding remarks of the results of the previous sections.

2. Mathematical formulation of the initial stage

The velocities of droplets injected from a nozzle are initially much greater than the velocity of the gas (air) stream, but are slowed down due to the drag force, while gas is accelerated. The most general equation describing the dynamics of an individual droplet can be written as

$$m_d \frac{dv_d}{dt} = -\frac{1}{2} C_D \rho_g \Big(v_d - v_g\Big)^2 A_d, \tag{1}$$

where m_d, v_d and A_d are droplet's mass, velocity and cross-sectional area, respectively, v_g and ρ_g are gas velocity and density respectively. In this study, only one-dimensional dynamics of gas and droplets are considered. C_D is the drag coefficient, which depends on the shape of the droplet and the Reynolds number: $Re = 2\rho_g(v_d - v_g)r_d/\mu_g$, r_d and μ_g are droplet's radius and gas dynamic viscosity, respectively.

For considerations that the droplets are perfect spheres, then equation (1) is simplified to

$$\frac{d^2 s}{dt^2} = -\frac{3}{8r_d} C_D \frac{\rho_g}{\rho_d}\Big(\frac{ds}{dt} - v_g\Big)^2, \tag{2}$$

where ρ_d is the droplet's density, s is the distance measured from the nozzle, $v_d = ds/dt$. Equation (2) is not easily amenable to analytical results, especially due to the drag coefficient C_D, which is a rather complicated function of the Reynolds number. Therefore, a number of approximations of C_D have been suggested in the literature (Sazhin et al., 2001). The most convenient approximation is the one found in Douglas et al. (1995), which considered 3 ranges of Reynolds numbers: $Re \le 0.2$ (Stokes flow), $0.2 < Re \le 500$ (Allen flow) and $500 < Re \le 10^5$ (Newton flow). The functions $C_D(Re)$ for these flows are given by the expressions: $C_D = 24/Re$ (Stokes flow), $C_D = 18.5/Re^{0.6}$ (Allen flow) and $C_D = 0.44$ (Newton flow). The expressions for C_D do not take into account effects of droplet acceleration, internal circulation, vaporisation, burning, non-spherical shape and vibrations and heating processes.

Two basic approximations were made to equation (2) in relation to v_g viz: (i) $v_g << v_d$, and (ii) $v_g = k\sqrt{s}$, where $k = \sqrt{K}/r_{g0}$, and where $K = \frac{3r_s^2\alpha_d}{8r_d} C_D \nu_{d0}^2 = \text{constant}$. Here, v_{d0} is the initial droplet velocity. The approximate analytical results emanating from these approximations and their physical interpretations are extensively discussed in Sazhin et al. (2001, 2003).

Now with the latter approximation for $v_g (= k\sqrt{s})$ in the equation (2), the governing equation for Stokes, Allen and Newton flows (Sazhin et al., 2001; Ebrahimian et al., 2008) are respectively written as follows:

$$\frac{d^2 s}{dt^2} + \alpha \frac{ds}{dt} - \alpha\, k\, \sqrt{s} = 0, \tag{3a}$$

$$\frac{d^2s}{dt^2} + \beta\left(\frac{ds}{dt}\right)^{1.4} - 1.4\,\beta\,k\,\sqrt{s}\left(\frac{ds}{dt}\right)^{0.4} = 0, \tag{3b}$$

$$\frac{d^2s}{dt^2} + \gamma\left(\frac{ds}{dt}\right)^{2} - 2\,\gamma\,k\,\sqrt{s}\left(\frac{ds}{dt}\right) = 0, \tag{3c}$$

where $\alpha = \frac{9\mu_g}{2r_d^2\rho_d}$, $\beta = \frac{4.577\mu_g^{0.6}\rho_g^{0.4}}{r_d^{1.6}\rho_d}$, and $\gamma = \frac{0.165\rho_g}{r_d\rho_d}$.

A careful examination of equations (3) revealed that they could be combined to give the model

$$\frac{d^2s}{dt^2} + \alpha_\eta\left(\frac{ds}{dt}\right)^{\eta+1} - (\eta+1)\,k\,\alpha_\eta\,\sqrt{s}\left(\frac{ds}{dt}\right)^{\eta} = 0, \tag{4}$$

where η is termed as flow regime parameter or identifier. Equation (4) is the unified model for the fuel spray penetration, and it has not been reported elsewhere in literature to the best of the author's knowledge. The equation is nonlinear and embraces all 3 regimes of flow in that the parameter η takes the values $0, 0.4$ and 1 such that $\alpha_0 = \alpha$ (Stokes flow), $\alpha_{0.4} = \beta$ (Allen flow), and $\alpha_1 = \gamma$ (Newton flow), respectively. This means that the specific applicable examples to the equations (3a, b, c) respectively are embedded in the model equation (4), as outlined in Sazhin et al. (2001, 2003) and Ebrahimian et al. (2008). It is pertinent to note that as $r_d \to \infty$, $K \to 0$, and hence $k \to 0$. This is, therefore, equivalent to $v_g << v_d$ (that is, v_g is so small compared with v_d that its contribution to the right hand side of equation (2) could be ignored altogether).

3. Variational iteration method (VIM)

As earlier noted, the unified fuel spray penetration model equation (4) is highly nonlinear. A powerful tool for solving effectively, easily and accurately various kinds of linear and nonlinear scientific and technological problems such as those encountered in chemical, ecological, biological, and engineering applications in recent times, is tackled by the VIM (He, 1999, 2006). The method is a modified general Lagranges multiplier method (Inokuti et al., 1976). The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as an initial approximation or trial function. Then a more highly precise approximation at some special point could be obtained. This approximation converges rapidly to an accurate solution. To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation:

$$Ls + Ns = g(t), \tag{5}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is an inhomogeneous term. According to the VIM, we can construct a correction functional as follows:

$$s_{n+1}(t) = s_n(t) + \int_0^t \lambda(\xi)\{Ls_n(\xi) + N\bar{s}_n(\xi) - g(\xi)\}d\xi, \tag{6}$$

where $\lambda(\xi)$ is a general Lagrangian multiplier (Inokuti et al., 1976), which can be identified optimally via the variational theory, the subscript n denotes the n^{th}-order approximation, $\bar{s}_n$ is considered as a restricted variation, that is, $\delta\bar{s}_n = 0$.

Now constructing a correction functional to the unified model (4) in the form

$$s_{n+1}(t) = s_n(t) + \int_0^t \lambda(\xi) \left\{ \frac{d^2 s_n}{d\xi^2} + \alpha_\eta \left(\frac{d\bar{s}_n}{d\xi} \right)^{\eta+1} - (\eta+1)\, k\, \alpha_\eta \sqrt{\bar{s}_n} \left(\frac{d\bar{s}_n}{d\xi} \right)^{\eta} \right\} d\xi, \tag{7}$$

we obtain its stationary conditions as

$$\frac{d^2\lambda(\xi)}{d\xi^2} = 0, \tag{8a}$$

$$\lambda(t) = 0, \tag{8b}$$

$$\frac{d\lambda(t)}{dt} = 1. \tag{8c}$$

Therefore, the Lagrangian multiplier is obtained as

$$\lambda(\xi) = \xi - t. \tag{9}$$

Therefore, the equation for the respective iterative solutions now becomes

$$s_{n+1}(t) = s_n(t) + \int_0^t \left(\xi - t \right) \left\{ \frac{d^2 s_n}{d\xi^2} + \alpha_\eta \left(\frac{ds_n}{d\xi} \right)^{\eta+1} - (\eta+1)\, k\, \alpha_\eta \sqrt{s_n} \left(\frac{ds_n}{d\xi} \right)^{\eta} \right\} d\xi. \tag{10}$$

To start the iteration, we assume that v_{d0} is an initial droplet velocity such that $s(t=0)=0$, then the arbitrary approximation that satisfies the initial condition is

$$s_0(t) = v_{d0} t. \tag{11}$$

Thus, the first iteration to the equation (10) which satisfies the unified fuel spray penetration model equation (4) becomes

$$s = v_{d0}\, t - \frac{1}{2}\, \alpha_\eta\, v_{d0}{}^{\eta+1} t^2 + \frac{4}{15}\, (\eta+1)\, \alpha_\eta\, k\, v_{d0}^{\eta+\frac{1}{2}}\, t^{\frac{5}{2}}. \tag{12}$$

We can now look into the 2 limiting cases as outlined above using the solution (12). Firstly, the limiting case when $k \to 0$ (that is, v_g is small compared with v_d), equation (12) for the 3 flow regimes viz: Stokes flow ($\eta = 0$), Allen flow ($\eta = 0.4$) and Newton flow ($\eta = 1$) is, respectively, given by

$$s = v_{d0}\, t - \frac{1}{2}\, \alpha_0 v_{d0} t^2, \tag{13a}$$

$$s = v_{d0}\, t - \frac{1}{2}\, \alpha_{0.4} v_{d0}^{\frac{7}{5}}\, t^2, \tag{13b}$$

$$s = v_{d0}\, t - \frac{1}{2}\, \alpha_1 v_{d0}^2\, t^2. \tag{13c}$$

In this limit, using successive approximations, Sazhin et al. (2001) obtained the solutions

$$s = \frac{v_{d0}}{\alpha}\Big(1 - \exp(-\alpha t)\Big), \tag{14a}$$

$$s = \frac{1}{0.152\beta^{2.5}}\left(0.253 v_{d0}^{0.6}\beta^{1.5} - \Big(t + \frac{1}{0.4 v_{d0}^{0.4}\beta}\Big)^{-1.5}\right), \tag{14b}$$

$$s = \frac{1}{\gamma}\ln\left(1 + v_{d0}\gamma t\,\right), \tag{14c}$$

respectively, for Stokes flow, Allen flow and Newton flow (Sazhin et al., 2001; equations (3), (4), and (5), respectively). It is pertinent to note that the respective solutions (13) are equivalent to the solutions (14) provided series expansions of the latter are kept at a truncation order of $O(t^3)$ (that is, provided $\alpha t << 1\,, v_{d0}^{0.4}\beta << 1$ and $v_{d0}\gamma t << 1$, respectively).

Secondly, for the limiting case $v_g = k\sqrt{s}$, we obtain the solutions:

$$s = v_{d0}\,t - \frac{1}{2}\,\alpha_0 v_{d0} t^2 + \frac{4}{15}\,\alpha_0 k v_{d0}^{\frac{1}{2}} t^{\frac{5}{2}}, \tag{15a}$$

$$s = v_{d0}\,t - \frac{1}{2}\,\alpha_{0.4} v_{d0}^{\frac{7}{5}}\,t^2 + 0.3733333334\,\alpha_{0.4}\,k\,v_{d0}^{\frac{9}{10}} t^{\frac{5}{2}}, \tag{15b}$$

$$s = v_{d0}\,t - \frac{1}{2}\,\alpha_1 v_{d0}^2\,t^2 + \frac{8}{15}\,\alpha_1 k v_{d0}^{\frac{3}{2}}\,t^{\frac{5}{2}}, \tag{15c}$$

respectively from the solution (12) for the 3 flow regimes viz: Stokes flow ($\eta = 0$), Allen flow ($\eta = 0.4$) and Newton flow ($\eta = 1$). These solutions are exact and consistent with those obtained by Sazhin et al. (2001) (equations (15), (20), and (25), respectively) using successive approximations. According to Sazhin et al. (2001), the first term in each of the equations (15) gives the zeroth approximation for s. The second term gives the correction to s due to the drag force acting on the droplet, while the third term takes into account the reduction of this drag force due to the gas acceleration. We also remark here that Ebrahimian et al. (2008) solved the individual governing equations (3) by the use of VIM. The results clearly demonstrated that the solutions (15b, c) of the respective equations for the Allen and Newton flows are exact and consistent with the solutions from the unified model. On the other hand, the Stokes' equation solution is higher than the unified model solution by the terms: $\frac{1}{6}v_{d0}\alpha^2 t^3 - \frac{8}{105}k v_{d0}^{\frac{1}{2}}\alpha^4 t^{\frac{11}{2}} - \frac{1}{24}\alpha^3 v_{d0} t^4 + \frac{16}{945}k\alpha^3 v_{d0}^{\frac{1}{2}} t^{\frac{9}{2}} + \frac{1}{120}v_{d0}\alpha^4 t^5 - \frac{32}{10395}k v_{d0}^{\frac{1}{2}}\alpha^4 t^{\frac{11}{2}}$. This means that these are additional terms in the Stokes equation (3a) solution that reduces the drag force due to the gas acceleration, which were not captured either by the successive approximations solution nor that of the unified model (4) solution. However, Ebrahimian et al. (2008) validated the VIM results by using fourth order Runge-Kutta numerical solutions together with the successive approximations. It was generally observed that the results for the flow regimes demonstrated high degree of agreement between the 3 methods. In particular, there was no discernible difference between the solutions of the Stokes equation (15a) due to the VIM and that of the Runge-Kutta numerical solution. This means that the additional terms resulting from the VIM solution makes little or no contribution after all.

4. Concluding remarks

In this paper, a unified fuel spray model that accounts for 3 regimes of flow is proposed. The model equation, which is highly nonlinear, is solved via variational iteration method that gives a general analytical solution. The main conclusions are the following:

1. The unified model gives a simple analytical result that accounts for 3 flow regimes including any arbitrary regime.
2. The unified model has great advantage in that there is no need for solving separate equations for the different flow regimes.
3. The unified model vis-a-vis the analytical solution is derived under 1 limiting case of the air (stream) velocity that actually accounts for 2 limiting cases after all.
4. The unified model is a generalization of fuel spray penetration equations for the 3 flow regimes earlier derived in the literature.
5. The 3 flow regime solutions via the new unified model correspond exactly to the previously reported expressions, which are obtained using successive approximations and VIM by solving the individual equations governing Stokes flow, Allen flow and Newton flow.
6. The unified model vis-a-vis the analytical solution via the VIM can minimize time wasting and complicated calculations of numerical methods and many successive approximations. This is because the unified model via the VIM makes use of just one iteration and converges to the solutions.

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Symbols

A_d	droplet cross sectional area (m^2);
C_D	drag coefficient;
r_s	radius of the spray (mm);
$S(t)$	distance from the nozzle (mm);
$g(t)$	Iinhomogeneous term;
L	linear operator;
m_d	droplet mass (kg);
N	nonlinear operator;
Re	Reynolds number;
r_d	droplet radius (mm);
r_{g0}	radius of the air jet near the nozzle (mm);
v_d	droplet velocity (m/s);
v_{d0}	droplet velocity near the nozzle (m/s);
v_g	gas velocity (m/s);
ρ_d	droplet density (kg/m^3);
ρ_g	gas density (kg/m^3);
λ	general Lagrangian multiplier;
η	low regime parameter or identifier;
μ_g	gas dynamic viscosity (m^2/s).

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