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Finite Grid Analogy for Levy Plates on Generalized Foundations

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Abstract

In many engineering problems the need for accurate and rapid computation of stresses and deformations in Levy plates on 2-parameter foundations is encountered. The challenge may be met in a simplified way by resorting to a finite grid model of intersecting beams on generalized elastic foundations. The simplified formulation in this article is based on the discretized representation for plates composed of interlocking girders endowed with exact stiffness, geometric stiffness, and consistent mass matrices. These have been obtained through the use of exact shape functions. Sample problems of bending, buckling, and free vibration problems for rectangular Levy plates supported on elastic foundations are solved. Comparisons with known analytical solutions and other numerical solutions are presented.

Key words: Elastic foundation, Finite grid solution, Exact shape functions.

Introduction

Plates on elastic foundations represent a ubiquitous problem in applied mechanics that has received considerable attention in structural engineering. Since the interaction between foundations and supporting soil has a great importance in many engineering applications, a considerable amount of research has been conducted on plates on elastic foundations. The subject represents a palimpsest of approaches to solving the governing equations for plates altered to account for the presence of the underlying media. Much of this research has been conducted to deal with bending, buckling, and vibration problems of beams and plates on elastic foundations. The aim is to solve some engineering problem such as foundation analysis for buildings, pavements of highways, water tanks, airport runways and buried pipelines. The intent of this subsection is to give a brief synoptic overview of research accomplishments to date.

The ordinary approach in formulating closed solutions for beams, plates, and shells continuously supported by elastic media is based on the inclusion of the foundation reaction in the corresponding differential equation. In the 1-parameter model the soil underneath beams or plates (the Winkler model) leads to a discontinuity of the foundation deformation along the domain boundary. For purposes of satisfying the foundation displacement continuity, Hetényi (1946) suggested the use of an elastic plate at the top of the independent Winkler elements to represent an interaction between them. The principal artifice of the 2-parameter elastic foundation model is to provide a mechanical interaction between the individual spring elements. Many others have suggested such physical models of soil behavior. The second foundation parameter defined by Filonenko-Boroditch, Pasternak, and Kerr (1964) ensures in effect that the Winkler springs are interlinked by a thin elastic membrane, a layer of compressible vertical elements, and rotational springs, respectively. Depending on soil material properties and the thickness of the compressible soil layer Vlasov and Leont'ev (1966) provide guidance to determine both foundation parameters. Çelik and Saygun (1999) obtained the parameters by using the idealization of each layer of the foundation with a system of 1-dimensional vertical columns of interconnected shear springs. Since the second parameter for each model is constant, the nature of the governing equation will not change if the second parameter of the Filonenko-Boroditch model (T), Pasternak model (G), and Kerr model (k_θ) is replaced by a single second parameter denoted as k_2. Then the governing equation for transverse displacement $w(x,y)$ can be extended to plates on 2-parameter elastic foundation as:

$$D\left(\frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4}\right) + k_1 w(x,y) + k_2\left(\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}\right) = q(x,y) \quad (1)$$

where k_1 is the Winkler parameter with the unit of force per unit area/per unit length (force/length^3), k_2 is the second foundation parameter defined as the reaction moment proportional to the local angle of rotation in the generalized foundation model with unit of moment per unit length/unit length (force/length), and D is the flexural rigidity of the plate element.

There exist many different methods to solve Eq. (1) for transverse displacement and then internal forces. In comparison with 2-dimensional plate elements the solutions become more sophisticated and mathematically less familiar for most engineering applications. El-Zafrany and Fadhil (1996) utilized boundary integral equations and Wang et al. (1997) examined the buckling loads using classical Kirchhoff plate theory and shear deformable plate expressions. Lam et al. (2000) represented canonical exact solutions, based on Green's functions, for the elastic bending, buckling, and vibration problems of Levy-plates on elastic foundations. On the other hand, Sladek et al. (2002) observed that it is impossible to solve analytically the governing equation for plates with free edges. It would appear that for general plate problems the 2-parameter elastic foundation soil model cannot be solved analytically in readily understood format for general load and boundary conditions.

We may defer to gridwork models of plates for general applications. A differential part of a plate supported by a generalized foundation as shown in Figure 1 can be represented by 2 parallel sets of beam elements. The 2-dimensional plate element is reduced to intersecting 1-dimensional beam elements. The governing differential equation of a line element supported by the 2-parameter elastic foundation for transverse displacement $w(x)$ is a simplification of Eq. (1):

$$EI\frac{d^4 w(x)}{dx^4} + k_1 w(x) + k_2\frac{d^2 w(x)}{dx^2} = q(x) \quad (2)$$

In this case parameters $k_1$and $k_2$become force/length^2 and force/radians, respectively. EI is the flexural rigidity. It is useful to examine where we stand in relation to 1-dimensional beams on generalized foundations, i.e. Eq. (2). A broad range of the engineering problems have been solved by computer-based methods such as finite element and boundary element methods. Closed form solutions have been published for a limited number of cases. The formulations based on interpolation (shape) functions have been used in solution by the finite element method. Wang (1983) and Eisenberger and Clastornik (1987) have derived exact stiffness matrices for beams. Razaqpur and Shah (1991) derived a new finite element to eliminate the limitations of the solution, such as the requirement of certain combinations of beam and foundation parameters. Gülkan and Alemdar (1999) extended this to an analytical solution for the shape functions of a beam segment on a generalized 2-parameter elastic foundation, leading to exact element-level matrices.

The objective of this article is to develop an approximate but computationally manageable finite grid solution of plates on a generalized foundation. It is an extension of the discrete parameter approach where the physical domain is broken down into discrete sub-domains, each endowed with a property suitable for the purpose of mimicking the problem at hand. Conceptually, it is an application of the finite element method, except that each discrete element utilized is equipped with an exact solution for a beam. Its errors are attributable to the torsional constants of the grid members and the compromised effects of discretizing a continuous problem. In the method a plate edge is subdivided into a number of strips and each strip is characterized with the lumped characteristics of the corresponding width and plate depth. It offers several attractive advantages. First, orthotropic plates, for which we are aware of no analytical solutions, can be analyzed with no additional effort. The analyses are not confined to static deflection and internal force calculations, but may cover vibration and stability problems as well. Plates of any geometry, not only of the Levy type, also can be analyzed. Shear deformations can be easily considered.

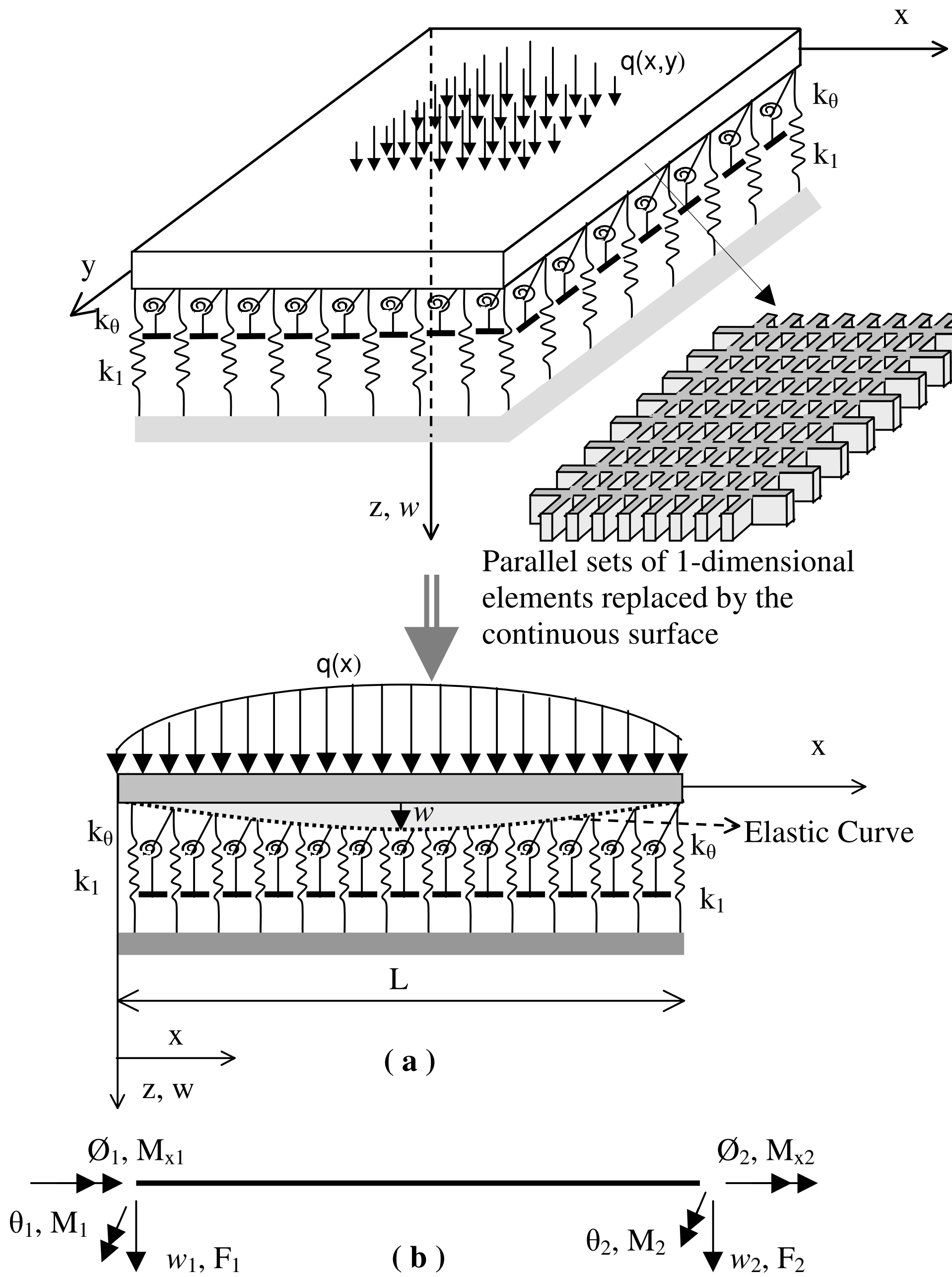


Figure 1. **a)** Model foundation representation of a 2-dimensional plate element by 1-dimensional beam elements on a 2-parameter (generalized) foundation, **b)** The local forces and displacements at nodes.

Representation of Plates as Grillages of Beam Elements

As Wilson (2002) has indicated, the structural behavior of a beam resembles that of a strip in a plate, and so replacing a continuous surface by an idealized discrete system can represent a 2-dimensional plate. The differential equation requires that the bond between the foundation and the plate be accounted for, and the soffit of the "equivalent" strip is not affected by the foundation in twisting. The torsional constant for the rectangular beam strips is adopted from Bowles (1988). The representation of a plate through the gridwork (or lattice) analogy at which the discrete elements are connected at finite nodal points is shown in Figure 2.

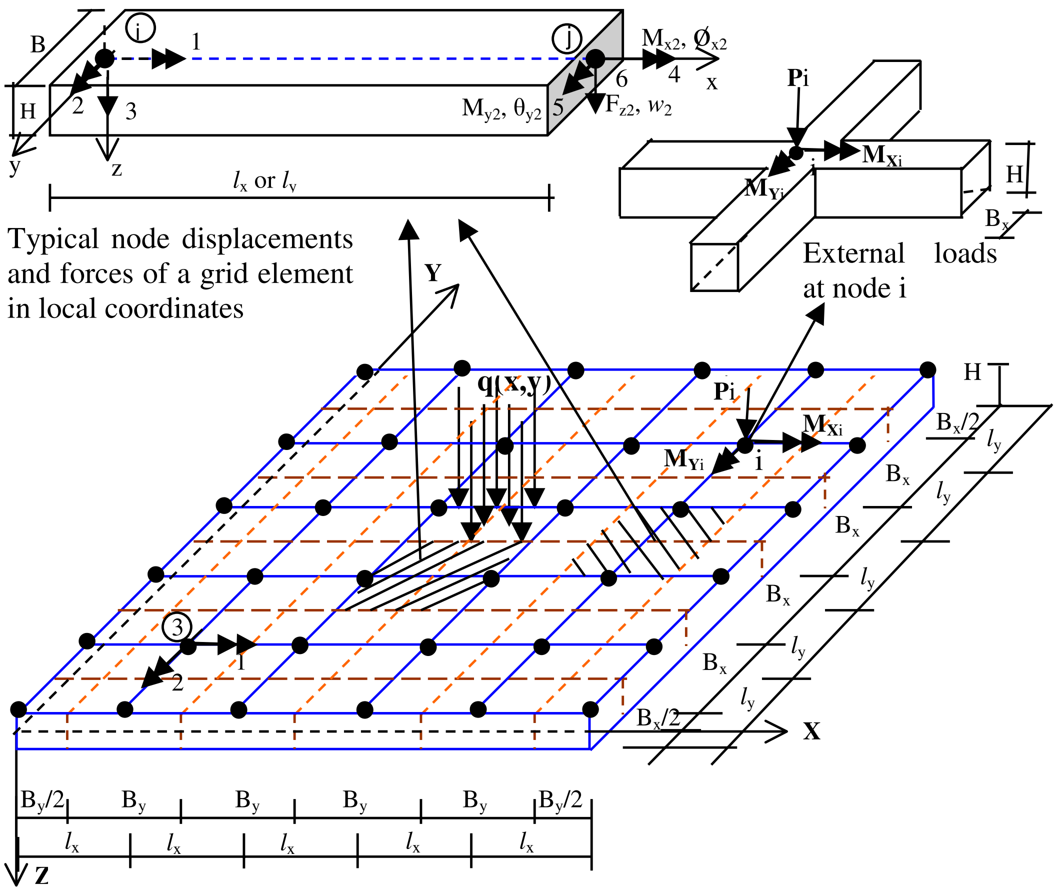


Figure 2. Idealized discrete system in which the elements are connected at nodal points of a rectangular plate.

Except for Levy plates it is not necessary to have the elements intersect at right angles. The replacement implies that there are rigid intersection joints between all sets of beam elements, ensuring slope and rotation continuity. Because of plane rigid intersection, the elements can resist torsion as well as bending moment and shear. Therefore, the idealized discrete element shown in Figure 2 can be replaced with a beam element that has 3 DOF at each node. With suitable stiffness coefficients and equivalent joint forces provided, the accuracy of the model will improve. We call this formulation the finite grid method (FGM). By this representation, plate problems including buckling and free vibration, with non-uniform thickness and foundation properties, arbitrary boundary and loading conditions and discontinuous surfaces, can be solved in a general form. The system cannot truly be equal to the continuous structure but solutions adequate for engineering purposes can be found with greater ease (Hrennikoff, 1949).

Matrices for Beam Elements on 2-Parameter Foundations

The 2-parameter foundation representation implies that at the end of each translational spring element there also exists a rotational spring to produce a reaction moment proportional to the local slope at that point. A display of the foundation with linear translational and rotational springs underlying a beam element is shown in Figure 1.

Shape functions

The homogeneous form of the governing differential equation for a 1-dimensional beam element on 2-parameter elastic foundation given in Eq. (2) is considered. The closed form of the solution depending on the foundation parameters k_1 and k_2 in terms of hyperbolic and trigonometric functions are:

$$w(x) = \begin{cases} c_1 \cos[\beta x]\cosh[\alpha x] + c_2 \cos[\beta x]\sinh[\alpha x] + \\ c_3 \sin[\beta x]\cosh[\alpha x] + c_4 \sin[\beta x]\sinh[\alpha x] & \text{for } k_2 < \sqrt{4k_1 EI} \\ c_1 \cosh[\beta x]\cosh[\alpha x] + c_2 \cosh[\alpha x]\sinh[\beta x] + \\ c_3 \cosh[\beta x]\sinh[\alpha x] + c_4 \sinh[\beta x]\sinh[\alpha x] & \text{for } k_2 > \sqrt{4k_1 EI} \end{cases} \tag{3}$$

where

$$\beta = \left\{ \begin{array}{l} \sqrt{\lambda^2 - \delta} \text{ for } k_2 < \sqrt{4k_1 EI} \\ \sqrt{\delta - \lambda^2} \text{ for } k_2 > \sqrt{4k_1 EI} \end{array} \right\}, \alpha = \sqrt{\lambda^2 + \delta} \text{ and } \lambda = \sqrt[4]{\frac{k_1}{4EI}}, \delta = \frac{k_2}{4EI}$$

The generalized displacement vector defined in Figure 1 is obtained by enforcing the boundary conditions at x = 0 and x = L. The arbitrary constants can be related to the end displacements. Then the closed form of the solution in terms of shape functions [N] and the generalized displacements, $[d]^T = \{w_1, \theta_1, w_2, \theta_2\}$, are obtained as:

$$w(x) = [\mathbf{N}]\,[\mathbf{d}] \tag{4}$$

After performing the necessary symbolic calculations, the 4 shape functions $\{\psi_2, \psi_3, \psi_5, \psi_6\}$ for flexural behavior are obtained. However, referring to Figure 1, the linear variation of the angular displacement $\varnothing_1 = a_1 + a_2 x$ by neglecting foundation effects can be used to derive the shape functions $\{\psi_1, \psi_4\}$ for torsion. After inserting the decoupled shape functions for torsion into the solution, the interpolation function [N] series is expanded to a 6 × 1 array.

Element stiffness matrix

The element stiffness matrix, $[K_e]$, for the prismatic beam element shown in Figure 1 that relates the nodal forces to the nodal displacements can be obtained from the minimization of strain energy functional U as follows:

$$[\mathbf{K}_e] = \frac{\partial U}{\partial \{\mathbf{d}\}} \tag{5}$$

where

$$U = \frac{EI}{2}\int_0^L \left\{ \frac{d^2 w(x)}{dx^2}\frac{d^2 w(x)}{dx^2}dx + \frac{k_1}{2}\int_0^L w(x)w(x)dx + \frac{k_2}{2}\int_0^L \frac{dw(x)}{dx}\frac{dw(x)}{dx}dx \right\} \tag{6}$$

Substituting w(x) and its derivatives from Eq. (2) into Eq. (6), the stiffness matrix can be written in the following form:

$$[\mathbf{K}_e] = EI\int_0^L \left\{ \frac{d^2\{\mathbf{N}\}^T}{dx^2}\frac{d^2\{\mathbf{N}\}}{dx^2}dx + k_1\int_0^L \{\mathbf{N}\}^T\{\mathbf{N}\}dx + k_2\int_0^L \frac{d\{\mathbf{N}\}^T}{dx}\frac{d\{\mathbf{N}\}}{dx}dx \right\} \tag{7}$$

N is a 4 × 1 sub-matrix of the exact shape functions for flexure. After carrying out the necessary integrals and procedures and assembling the torsion and bending terms with respect to Figure 1, 6 × 6 stiffness matrices are obtained. Since these effects are uncoupled the related influence coefficients are zero.

$$[\mathbf{K}_e] = \begin{bmatrix} k_{11} & 0 & 0 & k_{14} & 0 & 0 \\ 0 & k_{22} & k_{23} & 0 & k_{25} & k_{26} \\ 0 & k_{32} & k_{33} & 0 & k_{35} & k_{36} \\ k_{41} & 0 & 0 & k_{44} & 0 & 0 \\ 0 & k_{52} & k_{53} & 0 & k_{55} & k_{56} \\ 0 & k_{62} & k_{63} & 0 & k_{65} & k_{66} \end{bmatrix} \qquad (8)$$

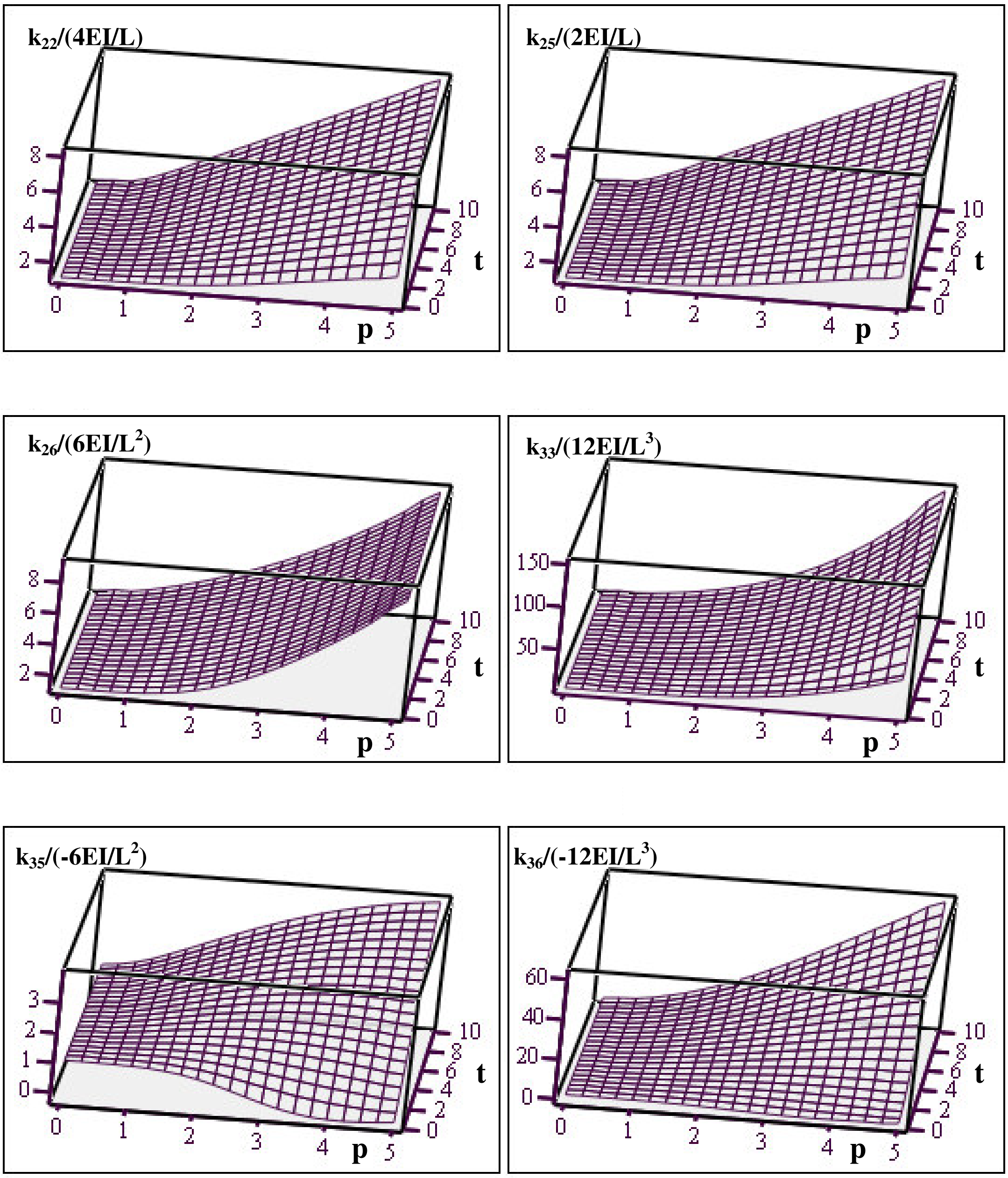


Figure 3. The normalized stiffness terms for the 2-parameter foundation.

The stiffness terms normalized with respect to those for an ordinary beam element are plotted in 3-dimensional view as in Figure 3 to display the influence of the foundation parameters. The dimensionless p and t terms given in the figures represent effects of the first and the second foundation parameters, respectively, as follows:

$$p = L\lambda \text{ and } t = \frac{\delta}{\lambda^2} \tag{9}$$

Work equivalent nodal loads

The work equivalent nodal loads of a beam element are formed by the exact shape functions. Many types of loading can be represented with uniformly distributed loads or point loads applied at the nodes. Here, the nodal load vector will be derived only for the uniform load case q(x) = q_0. The equivalent forces at nodes 1 and 2 for uniformly distributed loading q_0 on the span L referring to Figure 1 are given by:

$$\{\underline{P}\} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \int_0^L q_0 \begin{Bmatrix} N_2 \\ N_3 \\ N_5 \\ N_6 \end{Bmatrix} dx \tag{10}$$

Here [N] is the array of the shape functions corresponding to flexure. Inserting the corresponding expressions into Eq. (10), the nodal loads are obtained.

Normalizing nodal loads with the conventional terms can be used to observe the effect of the foundation parameters. The normalized terms are shown in Figure 4.

Consistent mass and geometric stiffness matrices

It is possible to calculate consistent mass and geometric stiffness matrix coefficients of a structural element with procedures similar to that for obtaining the element stiffness matrix. The degrees of freedom of the element are the torsional rotation, vertical translation, and bending rotation at each end. In the interest of consistency it can be assumed that the displacements within the span are defined again by the same interpolation functions as those already derived. This is not strictly correct because displacement interpolation functions under nodal displacements do not apply when inertia or axial forces are involved. Using the principle of virtual displacements the mass influence coefficients associated with the accelerations causing the nodal inertial forces and the element geometric stiffness terms associated with a constant axial force can be evaluated, respectively, by evaluating the following integrals:

$$m_{ij} = \mu \int_0^L \psi_i(x)\psi_j(x)dx \tag{11}$$

$$k_{Gij} = N \int_0^L \frac{d\psi_i(x)}{dx}\frac{d\psi_j(x)}{dx}dx \tag{12}$$

Here μ is uniform mass per unit length, $N$is axial force, and ψ_i and ψ_j are the shape functions associated with bending. Recalling the corresponding shape functions for both cases and substituting them into Eqs. (11) and (12) enable us to evaluate the consistent mass and geometric stiffness matrices (Karaşin, 2004).

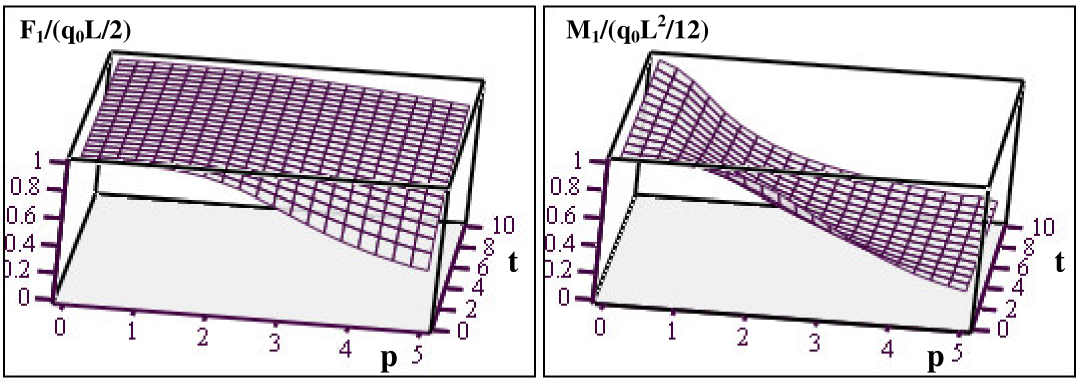


Figure 4. The normalized nodal forces for the 2-parameter foundation.

Assembling the System Matrices for Discretized Elements

In Levy-type gridwork systems elements are connected along external or internal lines. At interior nodes 4 typical discrete individual beam elements as shown in Figure 2 intersect. Matrix displacement is the standard tool for arbitrary load and boundary conditions. The applied loads are usually normal to the plane of the plate as limited by the degrees of freedom. For a typical member from the grid work assembly with the ends denoted by i and j a simple transformation is indicated. By using any convenient numbering scheme to collect all displacements for each nodal point in a convenient sequence the stiffness matrix for rectangular grids can be generated as follows:

$$[k_{sys}] = \sum_{i=1}^{n} [a_i]^T [k_i]^{[} a_i] \tag{13}$$

In Eq. (13) i is loops over n elements, $[a_i]$ is the individual coordinate transformation matrix, $[k_i]$ is the element stiffness matrix for a beam element on a 2-parameter elastic foundation, and $[k_{sys}]$ is the assembled stiffness matrix of the system. Similarly a geometric stiffness matrix $[k_{Gsys}]$ and consistent mass matrix $[M_{sys}]$ can be assembled to handle buckling vibration problems.

Numerical Tests

The validity of the solution technique is demonstrated through examples for a wide range of plates. The examples given for comparison in this article will cover bending, buckling, and free vibration problems examined by Lam et al. (2000) and Sladek et al. (2002).

Plate bending problems

Firstly, a comparison of FGM with LBIE (meshless local boundary integral equation) method solutions by Sladek et al. (2002) for simply supported and clamped square plates on a 2-parameter foundation is given. In the case study the thickness t, side length a, the flexural rigidity D, and Poisson ratio υ were chosen as 0.1 m, 8 m, 1000 N.m, and 0.3, respectively. The uniformly distributed load q was taken as 1 N/mm^2. Note that edges with free ends present no difficulties in FGM, but abruptly ending ends require an artificial extension of the beam to account for the proper shear force boundary conditions at the free end. This artifice is not performed here.

For the simply supported case (ssss), Winkler and Pasternak foundations are considered. The comparison of the results along the centerline of the plate for 3 different Winkler coefficients is plotted in Figure 5. From here one can see that the maximum relative error for deflections of points located on the axis passing through the center of the plate is about 1%.

The simply supported plate on a 2-parameter foundation case is considered for the central deflection. Next, the same plate with all edges clamped under the uniformly distributed load is considered to rest on a Winkler foundation. The comparisons of FGM with LBIE for the maximum deflections w_{max} at the center of the plates for both cases are plotted in Figure 6 and 7. From the figures it is noted that the maximum relative error of the central deflections is less than 3% for both cases.

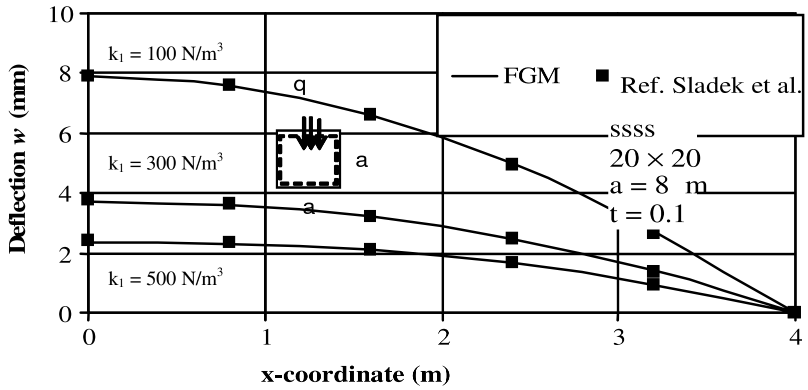


Figure 5. Comparison of the deflections at the centerline for a simply supported square plate resting on a Winkler foundation with Sladek et al. (plate represented by a mesh 20 × 20 beams on a side).

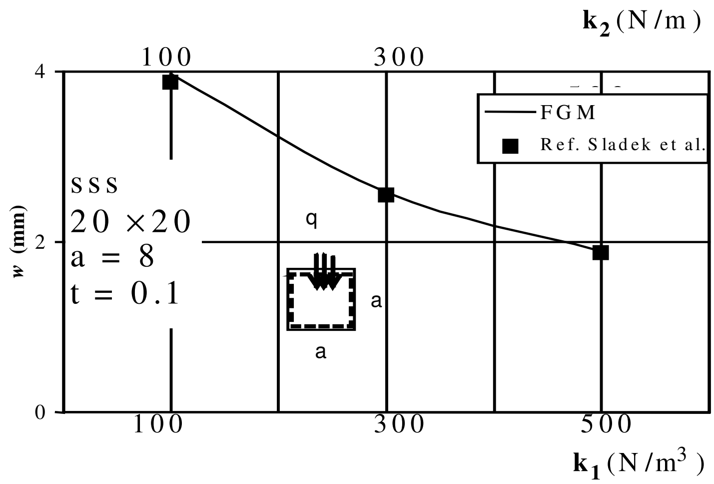


Figure 6. Comparison of maximum deflections of simple supported for the variation of foundation parameters under uniformly distributed load with Sladek et al. results (plate represented by a mesh 20 × 20 beams on a side).

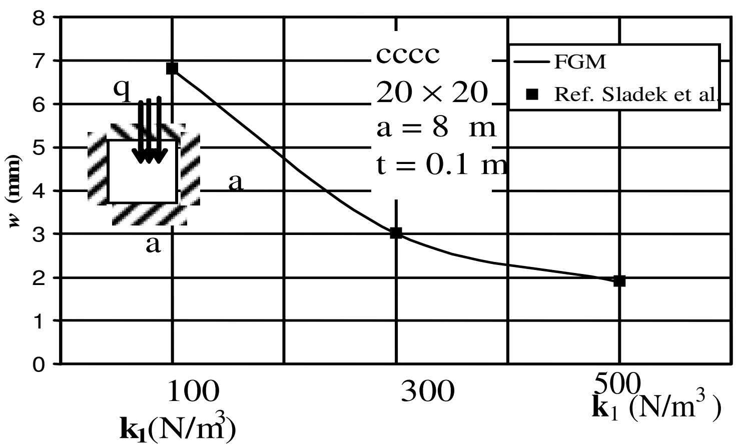


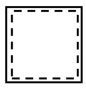
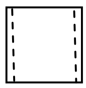
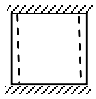
Figure 7. Comparison of maximum deflections of the clamped plate (cccc) on a Winkler foundation under uniformly distributed load with Sladek et al. results (plate represented by a mesh 20 × 20 beams on a side).

FGM is compared next with the canonical exact solutions derived by Lam et al. (2002) for Levy plates on a 2-parameter foundation. In the article it is noted that the solutions for Levy plates derived through the use of Green's functions can be accepted as benchmark results to check the convergence, validity, and accuracy of other numerical solutions. For convenience and generality the following non-dimensional parameters are defined for bending, free vibration, and buckling problems. These are examined in detail in Tables 1-3.

$$\underline{k}_1 = \frac{k_1 L_y^4}{D}; \underline{k}_2 = \frac{k_2 L_y^2}{D}; \underline{N}_x = \frac{N_x L_y^2}{D}; \quad \underline{N}_y = \frac{N_y L_y^2}{D}; \underline{\omega}^2 = \frac{\rho H L_y^2}{D}\omega^2 \tag{14}$$

Here L_x and L_y are the length dimensions, H is its thickness, ρ is the mass density, ω is the frequency and N_x, N_y are the in-plane loads along the x- and y-axes, respectively. The comparison of the central deflections carried out for central point loaded, all edges simple supported (SSSS), 2 opposite edges simple supported the others clamped (SCSC) and 2 opposite edges simple supported the others free (SFSF) rectangular plates on 2-parameter elastic foundations are given in Table 1 for different combinations of the foundation parameters and plate aspect ratios. From the table it can be seen that the results are quite acceptable.

Table 1. Comparison of central deflections of Levy-plates under concentrated load (P) at the centre on 2-parameter elastic foundation ($\nu = 0.3$).

Mesh:20x20, D=1		Boundary conditions	SSSS  Ly Lx			SCSC  Ly Lx			SCSC  Ly Lx		
Ly/Lx	Non-Dimensional Foundation Coefficients		Central Deflection Parameter ($Dw_{(0.5,0.5)} / (PL_x^2)$) ($\times 10^{-3}$)								
	k_1	k_2	This Study	Lam et al. (2000)	Relative Error %	This Study	Lam et al. (2000)	Relative Error %	This Study	Lam et al. (2000)	Relative Error %
0.5	1	1	3.86	3.84	0.42	29.21	27.67	5.58	1.83	1.77	3.40
	1	81	0.80	0.79	0.60	1.67	1.66	1.07	0.69	0.68	1.68
	1	625	0.16	0.16	0.55	0.28	0.28	0.16	0.16	0.16	0.10
	81	1	2.83	2.82	0.07	4.14	4.00	3.39	1.66	1.62	2.91
	81	81	0.76	0.76	0.65	1.33	1.31	1.11	0.67	0.66	1.76
	81	625	0.16	0.16	0.69	0.27	0.27	0.12	0.16	0.16	0.32
	625	1	1.27	1.26	0.31	1.30	1.28	1.70	1.11	1.09	1.88
	625	81	0.62	0.61	0.81	0.72	0.71	1.43	0.58	0.57	1.58
	625	625	0.16	0.16	0.11	0.23	0.23	0.22	0.15	0.15	0.15
1.0	1	1	10.62	11.07	4.08	23.09	21.10	9.43	6.82	6.87	0.73
	1	81	2.86	2.88	0.52	3.60	3.56	1.01	2.61	2.61	0.00
	1	625	0.63	0.63	0.00	0.72	0.72	0.14	0.62	0.62	0.16
	81	1	9.15	9.47	3.42	14.05	13.29	5.71	6.34	6.38	0.67
	81	81	2.78	2.80	0.47	3.41	3.37	0.95	2.55	2.55	0.04
	81	625	0.63	0.63	0.14	0.72	0.72	0.28	0.62	0.62	0.19
	625	1	5.07	5.14	1.44	5.22	5.21	0.19	4.41	4.42	0.38
	625	81	2.37	2.38	0.23	2.60	2.59	0.63	2.24	2.24	0.09
	625	625	0.62	0.62	0.02	0.69	0.69	0.19	0.61	0.61	0.09
2.0	1	1	16.01	16.24	1.40	17.76	17.11	3.82	14.79	15.13	2.29
	1	81	7.34	7.40	0.72	7.54	7.52	0.23	7.20	7.27	0.93
	1	625	1.98	1.98	0.01	2.00	2.00	0.10	1.98	1.98	0.01
	81	1	15.60	15.82	1.39	17.13	16.57	3.40	14.48	14.81	2.23
	81	81	7.28	7.33	0.71	7.47	7.45	0.20	7.14	7.21	0.93
	81	625	1.98	1.98	0.01	2.00	2.00	0.10	1.98	1.98	0.01
	625	1	13.39	13.58	1.35	14.04	13.85	1.39	12.72	12.98	2.00
	625	81	6.88	6.93	0.68	7.01	7.01	0.03	6.77	6.83	0.86

Buckling problems

For the square plates defined formerly in the vibration example with different values of $\underline{k}_1$ and $\underline{k}_2$ the buckling load parameters ($\underline{N}_{cr}$) are compared in Table 2 with the benchmark results given by Lam et al. (2002). The non-dimensional buckling load parameters, defined in Eq. (14), due to uniaxial and biaxial in-plane loads for square Levy-plates on 2-parameter elastic foundations results are tabulated with the corresponding canonical exact solutions as shown in Table 2.

The comparison in this table is the least satisfactory if the benchmark results in a number of cases are accepted as being absolutely correct, as a benchmark result is expected to be. The greatest discrepancies occur in those cases when uniform forces are applied in the vertical y-direction. If the normalized critical load coefficient is indeed 4 for the SSSS case (for which our result is 3.99), then it cannot be the same number for SCSC (second group), SSSC (fourth group), SCSF (fifth group), or SSSF (sixth group). Furthermore, the purported benchmark results appear to repeat cyclically for the last 3 sets of plate boundary conditions (groups 4-6), which clearly contravene the physics of the problem. We must therefore conclude that the comparisons with the benchmark results in Table 2 are not reliable because of a printing error in the original article that has remained uncorrected. We have no way of knowing which of our values is compared with the correct result from Lam et al. (2002), and so commenting on these deviations is not possible.

Vibration problems

Free vibration analyses of square Levy-plates on 2-parameter elastic foundation studied by Lam et al. (2002) are compared next with the present study. For the square plates with different combinations for values of the non-dimensional foundation parameters $\underline{k}_1$ and $\underline{k}_2$ defined in Eq. (14), the fundamental frequencies are compared with the reference results in Table 3 for various boundary conditions. Again the results show that the method produces very good results for the computational effort that goes into it.

Summary and Conclusions

Easily understood engineering approaches for analysis of plates on elastic foundations have not been covered sufficiently in the literature. For particular plate problems, closed form solutions have been obtained. However, even for conventional plate analysis these solutions can only be applied to the problems with simple geometry, load and boundary conditions. For plates supported by the 2-parameter elastic foundations the solution is usually much too complex and there is apparently no analytical solution other than for simple cases. The objective of this article has been to develop a more general simplified numerical approach for plates on elastic foundations. A gridwork analogy called the finite grid method involving discretized plate properties mapped onto equivalent beams with adjusted parameters and matrix displacement analysis is used. In this solution the plate is modeled as an assemblage of individual beam elements interconnected at joints. The exact fixed end forces, stiffness, consistent mass, and geometric stiffness matrices for beam elements resting on 1 or 2 parameter foundation are the tools to solve plate bending, vibration, and buckling problems.

It is inferred that the presence of the second foundation parameter k_2 in the analysis is a remarkably dominant trend that is known from analytical solutions. Its presence leads to diminished displacements, smaller internal stresses, larger buckling loads, and larger free vibration frequencies. This might have been anticipated because the strain energy density function includes an additional term in the case of the 2-parameter foundation as compared with the Winkler foundation. Comparisons with known analytical or numerical solutions yield generally accurate results for this approximate method. An acute exception occurs in comparing our results with those of a benchmark study for problems involving critical loads for plates. It appears that printing errors in the earlier article may be the primary cause for the large discrepancies we list here. The simplicity of this method that extends existing exact solutions for beams to plates outweighs the small errors it may produce.

Table 2. Comparison of buckling load parameter for square Levy-plate under uniaxial and biaxial compressive loading on 2-parameter foundation with the benchmark results ($\nu = 0.3$).

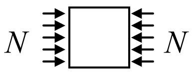
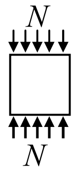
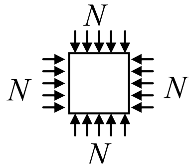
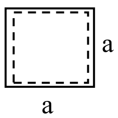
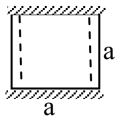
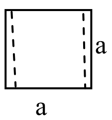
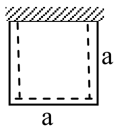
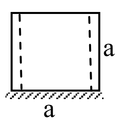
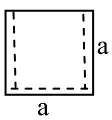
Mesh:16x16, D=1			N_x=0 and N_y=N			N_x=N and N_y=0			N_x=N and N_y=N		
Boundary Conditions of plate	Loading Conditions										
	Foundation Coefficients		Buckling load parameter ($\underline{N}_{cr}/\pi^2$)								
	$\underline{k}_1$	$\underline{k}_2$	This Study	Lam et al. (2000)	Rel. Error %	This Study	Lam et al. (2000)	Rel. Error %	This Study	Lam et al. (2000)	Rel. Error %
SSSS 	0	0	3.99	4.00	0.32	3.99	4.00	0.32	2.01	2.00	0.70
	100	0	5.01	5.03	0.31	5.01	5.03	0.31	2.53	2.51	0.72
	0	100	18.51	18.92	2.18	18.51	18.92	2.18	12.15	12.13	0.13
	100	100	18.93	19.17	1.27	18.93	19.17	1.27	12.64	12.65	0.08
SCSC 	0	0	7.60	7.69	1.20	6.62	4.00	65.44	3.76	3.83	1.84
	100	0	7.79	7.95	1.95	7.36	7.49	1.71	4.21	4.28	1.64
	0	100	20.39	20.74	1.67	21.87	18.81	16.26	13.67	13.96	2.09
	100	100	20.81	20.99	0.88	22.28	19.11	16.56	14.34	14.41	0.48
SFSF 	0	0	0.91	0.95	4.43	3.01	2.59	15.93	0.91	0.93	2.38
	100	0	1.94	1.98	2.16	3.52	2.82	24.90	1.94	1.63	19.08
	0	100	10.89	14.07	22.58	15.27	15.28	0.09	10.54	11.06	4.75
	100	100	12.07	14.33	15.80	15.42	15.40	0.14	12.07	11.76	2.61
SSSC 	0	0	5.66	5.74	1.35	4.78	4.00	19.40	2.63	2.66	1.12
	100	0	6.69	6.77	1.20	5.55	4.85	14.51	3.10	3.13	0.97
	0	100	19.35	19.72	1.90	19.76	18.81	5.03	12.70	12.80	0.78
	100	100	19.76	19.98	1.12	19.83	19.11	3.76	13.23	13.26	0.21
SCSF 	0	0	1.45	1.65	11.99	5.10	4.00	27.58	1.24	1.14	7.98
	100	0	2.25	2.68	16.08	5.61	4.85	15.69	2.25	2.21	1.91
	0	100	13.11	15.37	14.70	19.58	18.81	4.11	10.95	12.80	14.45
	100	100	14.10	15.63	9.81	20.11	19.91	1.01	12.03	11.89	1.15
SSSF 	0	0	1.51	1.40	7.72	3.24	4.00	19.09	1.20	1.06	14.01
	100	0	2.54	2.43	4.44	3.56	4.85	26.57	2.01	1.75	15.35
	0	100	13.78	15.19	9.26	15.56	18.81	17.27	10.67	11.19	4.61
	100	100	15.21	15.45	1.58	15.74	19.11	17.64	12.14	11.88	2.21

Table 3. Comparison of fundamental frequencies ($\underline{\omega}$) of square Levy-plates on 2-parameter foundation ($\nu = 0.3$).

Boundary Conditions		SSSS			SCSC			SSSC			SFSF			SCSF			SSSF		
Foundation Coefficients		Fundamental frequency parameter ($\underline{\omega}$) (mesh:10x10, D=1)																	
$\underline{k}_1$	$\underline{k}_2$	This Study	Lam et al. (2000)	Relative Error %	This Study	Lam et al. (2000)	Relative Error %	This Study	Lam et al. (2000)	Relative Error %	This Study	Lam et al. (2000)	Relative Error %	This Study	Lam et al. (2000)	Relative Error %	This Study	Lam et al. (2000)	Relative Error %
0	100	47.49	48.62	2.32	53.19	54.68	2.72	32.64	32.90	0.79	50.04	51.32	2.49	37.55	37.98	1.13	36.74	37.15	1.10
0	1000	140.37	141.92	1.09	144.82	146.73	1.30	99.30	99.83	0.53	142.51	144.24	1.20	111.69	112.52	0.74	111.00	111.71	0.64
100	0	20.38	22.13	7.91	28.71	30.63	6.27	13.67	13.88	1.51	23.81	25.67	7.25	16.02	16.15	0.80	15.10	15.38	1.82
100	100	48.51	49.63	2.26	54.10	55.59	2.68	34.12	34.39	0.79	51.01	52.29	2.45	38.83	39.27	1.12	38.06	38.47	1.07
100	1000	140.72	142.20	1.04	145.15	147.13	1.35	99.79	100.33	0.54	142.85	144.61	1.22	112.13	113.00	0.77	111.44	112.23	0.70
1000	0	36.04	37.28	3.33	41.27	42.87	3.73	32.83	31.62	3.83	38.06	39.49	3.62	33.83	34.07	0.70	33.42	33.71	0.86
1000	100	56.89	58.00	1.91	61.70	63.17	2.33	45.34	45.64	0.66	59.02	60.28	2.09	48.96	49.42	0.93	48.35	48.79	0.90
1000	1000	143.82	145.43	1.11	148.16	150.12	1.31	104.17	104.72	0.53	145.91	147.62	1.16	116.03	116.92	0.76	115.37	116.12	0.65

References

Bowles, J.E., "Foundation Analysis and Design", McGraw-Hill, New York, 1988.

Çelik, A. and Saygun, A., "A Method for the Analysis of Plates on a Two Parameter Elastic Foundation", International Journal of Solids Structures, 36, 2891-2915, 1999.

Eisenberger, M. and Clastornik, J., "Beams on Variable Two Parameter Elastic Foundations", ASCE Journal of Engineering Mechanics, 113, 1454-1466, 1987.

El-Zafrany, A. and Fadhil, S., "A modified Kirchhoff Theory for Boundary Element Analysis of Thin Plates Resting on Two-Parameter Foundation", Engineering Structures, 18, 102-114, 1996.

Gülkan, P. and Alemdar, B.N., "An Exact Finite Element for a Beam on a Two-Parameter Elastic Foundation", Structural Engineering and Mechanics, 7, 259-276, 1999.

Hetényi, M., "Beams on Elastic Foundation: Theory with Applications in the Fields of Civil and Mechanical Engineering", Scientific Series, Volume XVI, University of Michigan Press, 1946.

Hrennikoff, A., "Framework Method and Its Technique for Solving Plane Stress Problems", Publ. International Association Bridge Structure Engineering, 9, 217-247, 1949.

Karaşin, A., "An Improved Finite Grid Solution for Plates on Generalized Foundations", PhD Dissertation, Middle East Technical University, Ankara, 2004.

Kerr, A.D., "Elastic and Viscoelastic Foundation Models", Journal of Applied Mechanics, 31, 491-498, 1964.

Lam, K.Y., Wang, C.M. and He, X.Q., "Exact Solutions for Levy-Plates on Two-Parameter Foundation using Green's Functions", Engineering Structures, 22, 364-378, 2000.

Razaqpur, A.G. and Shah, K.R., "Exact Analysis of Beams on Two-Parameter Elastic Foundations", International Journal of Solids Structures, 27, 435-454, 1991.

Sladek, J., Sladek, V. and Mang, H.A., "Meshless Local Boundary Integral Equation Method for Simply Supported and Clamped Plates Resting on Elastic Foundation", Computer Methods in Applied Mechanics and Engineering, 191, 5943-5959, 2002.

Vlasov, V.Z. and Leont'ev, U.N., "Beams, Plates and Shells on Elastic Foundation", Israel program for Scientific Translation, Jerusalem, 1966.

Wang, C.K., "Intermediate Structural Analysis", McGraw-Hill, New York, 1983.

Wang, C.M., Kitipornchai, S. and Xiang, Y., "Relationships Between Buckling Loads of Kirchhoff, Mindlin and Reddy Polygonal Plates on Pasternak Foundation", ASCE Journal of Engineering and Mechanics, 123, 1134-1137, 1997.

Wilson, E.L., "Three Dimensional Static and Dynamic Analysis of Structures", Computers and Structures, Inc., Berkeley, 2002.