Turkish J. Eng. Env. Sci.
29 (2005) , 361 – 368.

Elastoplastic Response of a Long Functionally Graded Tube Subjected to Internal Pressure

Ahmet N. ERASLAN
Middle East Technical University, Department of Engineering Sciences
Ankara-TURKEY
e-mail: aeraslan@metu.edu.tr
Tolga AKIS
Atılım University, Department of Civil Engineering, Ankara-TURKEY

Received 21.06.2005

Abstract

Plane strain analytical solutions to functionally graded elastic and elastic-plastic pressurized tube problems are obtained in the framework of small deformation theory. The modulus of elasticity of the tube material is assumed to vary radially according to a 2 parametric model in a general parabolic form. The analytical plastic model is based on Tresca's yield criterion, its associated flow rule and ideally plastic material behavior. Exact solutions of field equations for elastic and plastic deformations are obtained. It is shown that the elastoplastic response of the functionally graded pressurized tube is affected notably by the radial variation of modulus of elasticity. It is also shown mathematically that the nonhomogeneous solution presented here reduces to that of a homogeneous one by the appropriate choice of the material parameters.

Key words: Stress analysis, Elastoplasticity, Pressure chamber, Functionally graded material, Tresca's criterion, Ideal plastic.

Introduction

A sufficiently long tube subjected to pressure in the radial direction is usually referred to as a pressure chamber or pressure vessel and is one of the classical problems in engineering mechanics. A thick-walled tube under either internal or external pressure was treated comprehensively in purely elastic stress state by Timoshenko (1956), Timoshenko and Goodier (1970), Uğural and Fenster (1987), and Boresi et al. (1993), in the fully plastic stress state by Boresi et al. (1993), Mendelson (1968), and Nadai (1931), and in the elastic-plastic stress state by Parker (2001) and Perry and Aboudi (2003). Recent studies on the subject by Horgan and Chan (1999), Tutuncu and Ozturk (2001), Jabbari et al. (2002), and Ma et al. (2003) include tubes made of functionally graded materials (FGM) under pressure. However, it is evident from the list of existing literature that elastic-plastic treatment of the problem for FGM by analytical means has not been performed yet. It is therefore the objective of the present investigation to derive a consistent analytical solution in order to predict the elastic-plastic deformation behavior of a pressurized FGM tube.

A long tube of inner radius a with axially constrained ends is taken into account. The tube is subjected to internal pressure p_{in}, and is nonhomogeneous in composition so that its modulus of elasticity E varies radially according to a general parabolic form given by

$$E = E_0 \left[1 - n \left(\frac{r}{b}\right)^k\right]. \tag{1}$$

Here E_0 is the reference value of E, r the radial coordinate, b the radius and n and k are material parameters. With this form, a wide range of nonlinear and continuous profiles to describe reasonable variation of E in the material may be achieved. Concave, convex and linear E profiles may be selected by choosing suitable n and k values. In a related theoretical study by Horgan and Chan (1999) the variation of E is described by the nonlinear function $E = E_0(r/b)^n$. However, their model is not as flexible as the general parabolic model used in this work.

First a plane strain analytical solution to elastic deformation of the tube is obtained. Displacement formulation is performed and the resulting hypergeometric differential equation is solved by the introduction of appropriate transformation. Then a plastic deformation model based on Tresca's yield criterion, its associated flow rule and ideally plastic material behavior is developed and solved analytically. The reduction of these nonhomogeneous solutions to homogeneous counterparts is performed by setting $n = 0$. The results indicate that both elastic and elastoplastic responses of the tube subjected to internal pressure are affected notably by the material's nonhomogeneity.

Elastic Deformation

Cylindrical polar coordinates (r, θ, z) are considered. A state of plane strain and infinitesimal deformations are presumed. Furthermore, the notation of Timoshenko and Goodier (1970) is used. Hence, in the formulation, σ_j and ϵ_j denote a normal stress and a normal strain component, respectively, and u is the radial component of the displacement vector. Strain-displacement relations for small strains, and the equations of generalized Hooke's law together with the equation of equilibrium in the radial direction (Timoshenko and Goodier, 1970)

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0, \tag{2}$$

form the basis for the entire analysis. In a state of plane strain, i.e. $\epsilon_z = 0$, stress-displacement relations take the forms for the radial, circumferential, and axial components, respectively, as

$$\sigma_r = \frac{E(r)}{r(1+\nu)(1-2\nu)}\left[\nu u + r(1-\nu)\frac{du}{dr}\right], \tag{3}$$

$$\sigma_\theta = \frac{E(r)}{r(1+\nu)(1-2\nu)}\left[(1-\nu)u + r\nu\frac{du}{dr}\right], \tag{4}$$

$$\sigma_z = \frac{E(r)}{r(1+\nu)(1-2\nu)}\left[\nu u + r\nu\frac{du}{dr}\right], \tag{5}$$

where ν is Poisson's ratio. Substitution of Eqs. (3) and (4) into Eq. (2), and the use of Eq. (1) lead to the governing differential equation for the radial displacement u

$$r^2\left[1 - n\left(\frac{r}{b}\right)^k\right]\frac{d^2u}{dr^2} + r\left[1 - n(1+k)\left(\frac{r}{b}\right)^k\right]\frac{du}{dr} - \frac{1-\nu-n\left[1-\nu(1+k)\right]\left(\frac{r}{b}\right)^k}{1-\nu}u = 0. \tag{6}$$

Equation (6) is a hypergeometric differential equation (Eraslan, 2003) and is reduced to standard form using a new variable $x = n\,(r/b)^k$ and applying the transformation $u(r) = ry(x)$. The result is

$$x(1-x)\frac{d^2y}{dx^2} + \frac{2+k-2(1+k)x}{k}\frac{dy}{dx} - \frac{1}{k(1-\nu)}y = 0. \tag{7}$$

The solution is found elsewhere (Abramowitz and Stegun, 1966) as

$$y(x) = C_1F(\alpha,\beta,\delta;x) + \hat{C}_2x^{-2/k}F(\alpha-\delta+1,\beta-\delta+1,2-\delta;x), \tag{8}$$

with C_i being an arbitrary constant and F the hypergeometric function defined by

$$F(\alpha,\beta,\delta;x) = 1 + \frac{\alpha\beta}{\delta\cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\delta(\delta+1)\cdot 2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\delta(\delta+1)(\delta+2)\cdot 3!}x^3 + \cdots. \tag{9}$$

The arguments α, β, and δ of F in Eq. (8) are determined as

$$\alpha = \frac{2}{(2+k)(1-\nu) + \sqrt{(1-\nu)[(2+k)^2(1-\nu)-4k]}}, \tag{10}$$

$$\beta = \frac{2}{(2+k)(1-\nu) - \sqrt{(1-\nu)[(2+k)^2(1-\nu)-4k]}}, \tag{11}$$

$$\delta = 1 + \frac{2}{k}. \tag{12}$$

From $u(r) = ry(n\,(r/b)^k)$, the solution to the elastic equation, Eq. (6), is obtained as

$$u(r) = C_1 P(r) + C_2 Q(r), \tag{13}$$

where

$$P(r) = rF\left(\alpha, \beta, \delta; n\,(r/b)^k\right), \tag{14}$$

$$Q(r) = \frac{1}{r} F\left(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n\,(r/b)^k\right). \tag{15}$$

Hence, the stresses become

$$\sigma_r = \frac{E(r)}{r(1+\nu)(1-2\nu)} \left\{ C_1 \left[\nu P + r(1-\nu)\frac{dP}{dr} \right] + C_2 \left[\nu Q + r(1-\nu)\frac{dQ}{dr} \right] \right\}, \tag{16}$$

$$\sigma_\theta = \frac{E(r)}{r(1+\nu)(1-2\nu)} \left\{ C_1 \left[(1-\nu)P + r\nu\frac{dP}{dr} \right] + C_2 \left[(1-\nu)Q + r\nu\frac{dQ}{dr} \right] \right\}, \tag{17}$$

$$\sigma_z = \frac{\nu E(r)}{r(1+\nu)(1-2\nu)} \left\{ C_1 \left[P + r\frac{dP}{dr} \right] + C_2 \left[Q + r\frac{dQ}{dr} \right] \right\}. \tag{18}$$

It should be noted that, for $n = 0$ from Eq. (1) $E = E_0$, from Eq. (9) $F(\alpha, \beta, \delta; 0) = 1$, from Eqs. (14) and (15) $P(r) = r$; $Q(r) = 1/r$, and therefore Eqs. (13) and (16)-(18) reduce to

$$u(r) = C_1 r + \frac{C_2}{r}, \tag{19}$$

$$\sigma_r(r) = \frac{E_0}{1+\nu} \left[\frac{C_1}{1-2\nu} - \frac{C_2}{r^2} \right], \tag{20}$$

$$\sigma_\theta(r) = \frac{E_0}{1+\nu} \left[\frac{C_1}{1-2\nu} + \frac{C_2}{r^2} \right], \tag{21}$$

$$\sigma_z(r) = \frac{2\nu E_0 C_1}{(1+\nu)(1-2\nu)}. \tag{22}$$

Equations (19)-(22) are nothing but the displacement and stress expressions for a homogeneous tube with axially constrained ends (Eraslan and Akis, 2005).

The boundary conditions to evaluate integration constants C_1 and C_2 are $\sigma_r(a) = -p_{in}$; $\sigma_r(b) = 0$, and upon application, one finds

$$C_1 = a(1+\nu)(1-2\nu)p_{in}[\nu Q(b) + b(1-\nu)Q'(b)]/D, \tag{23}$$

$$C_2 = -a(1+\nu)(1-2\nu)p_{in}[\nu P(b) + b(1-\nu)P'(b)]/D, \tag{24}$$

where

$$\begin{aligned} D \;=\; & E(a)\left\{[\nu Q(a) + a(1-\nu)Q'(a)][\nu P(b) + b(1-\nu)P'(b)] - [\nu P(a) + a(1-\nu)P'(a)]\right. \\ & \left.\times[\nu Q(b) + b(1-\nu)Q'(b)]\right\}. \end{aligned} \tag{25}$$

Homogeneous elastic solutions (Eraslan and Akis, 2005) indicate that the inner surface of the tube is critical where the stress state satisfies $\sigma_\theta > \sigma_z > \sigma_r$. Therefore, according to Tresca's yield criterion, the tube undergoes plastic deformation as soon as the pressure reaches the critical value p_e so that $\sigma_\theta - \sigma_r = \sigma_0$ at the inner surface. Here σ_0 stands for the uniaxial yield limit of the material. Carrying out the algebra, the nondimensional elastic limit pressure $\overline{p}_e$ is determined to be

$$\begin{aligned} \overline{p}_e \;=\; & \frac{p_e}{\sigma_0} = \left\{[\nu P(b) + b(1-\nu)P'(b)][\nu Q(a) + a(1-\nu)Q'(a)] - [\nu P(a) + a(1-\nu)P'(a)]\right. \\ & \left.\times[\nu Q(b) + b(1-\nu)Q'(b)]\right\} / \left\{(1-2\nu)[P(a) - aP'(a)][\nu Q(b) + b(1-\nu)Q'(b)]\right. \\ & \left.-(1-2\nu)[\nu P(b) + b(1-\nu)P'(b)][Q(a) - aQ'(a)]\right\}. \end{aligned} \tag{26}$$

It should be noted at this point that the hypergeometric function defined by the series, Eq. (9), converges slowly. Care must be exercised in calculating these functions. Several thousand terms may be required to be added in order to get a numerical value with sufficient accuracy. To be able to add such a large number of terms, each term should be factorized. For example, the fourth term T_4 in the series is obtained by the following calculation sequence:

$$t_1 = \frac{\alpha}{\delta};\; t_2 = \frac{\alpha+1}{\delta+1};\; t_3 = \frac{\alpha+2}{\delta+2};\; \hat{t}_1 = \frac{\beta}{1};\; \hat{t}_2 = \frac{\beta+1}{2};\; \hat{t}_3 = \frac{\beta+2}{3}. \tag{27}$$

Then

$$T_4 = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\delta(\delta+1)(\delta+2)\cdot 3!}x^3 = t_1 \times t_2 \times t_3 \times \hat{t}_1 \times \hat{t}_2 \times \hat{t}_3 \times x \times x \times x. \tag{28}$$

This calculation procedure avoids the evaluation of factorials of large numbers, which is practically not possible. The computer program HYPER, developed by one of the authors (Eraslan, 2002), implements this factorization technique. In this program the last term added, T_k, is determined such that the next term, T_{k+1}, satisfies $S_k + T_{k+1} = S_k$ in the 12-digit calculation, with S_k being the result of summing the first k terms. Further details may be found in the article by Orcan and Eraslan (2002). The code HYPER is used throughout this work for the calculation of hypergeometric functions.

To present some results pertaining to the elastic solution, the following formal nondimensional variables are used. Radial coordinate: $\overline{r} = r/b$, inner radius: $\overline{a} = a/b$, stress: $\overline{\sigma}_j = \sigma_j/\sigma_0$, and displacement: $\overline{u} = uE_0/(b\sigma_0)$. Furthermore, Poisson's ratio ν is taken as 0.3 throughout.

The elastic limit pressure for a homogeneous pressurized tube ($n = 0$) of inner radius $\overline{a} = 0.7$ is calculated from Eq. (26) as $\overline{p}_e = 0.255000$. From Eqs. (23) and (24) the corresponding integration constants in nondimensional forms are determined as $\overline{C}_1 = C_1 = 2.61170 \times 10^{-4}$ and $\overline{C}_2 = C_2/b^2 = 6.52925 \times 10^{-4}$. On the other hand, for an FGM tube of the same inner radius and under the same pressure having material parameters $n = -0.4$ and $k = 0.6$, integration constants are $\overline{C}_1 = 2.72204 \times 10^{-4}$ and $\overline{C}_2 = 5.19743 \times 10^{-4}$. Using these values, the stresses and displacement in the tube are calculated for both FGM and homogeneous tubes and are plotted in Figure 1. Solid lines show FGM and dashed lines show homogeneous results. A few points should be mentioned. The largest discrepancy between FGM and homogenous solutions is observed in the radial displacement. The axial stress in the homogeneous tube is constant throughout (see Eq. (22)), while it varies slowly in the FGM tube. The variation of elastic limit pressure $\overline{p}_e$ with material parameter n is calculated using k as a parameter and is plotted in Figure 2. The inner radius is $\overline{a} = 0.7$. Point $n = 0$ corresponds to the homogeneous tube and the limit for it is 0.255000. For $k = 0$, $E = E_0(1 - n) = constant$; hence, the tube is homogeneous, irrespective of the value of n.

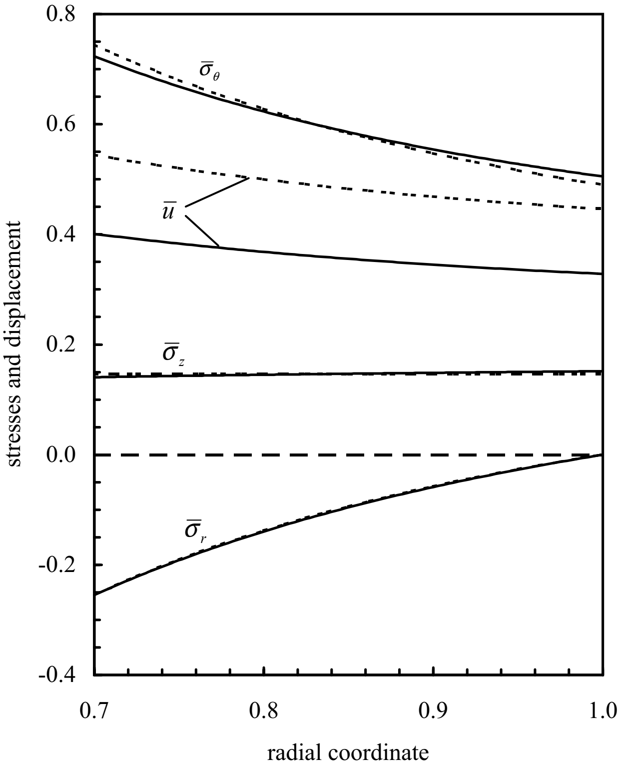

Figure 1. Comparison of stresses and displacement in an FGM tube ($n = -0.4$, $k = 1.4$; solid lines) to those in a homogeneous tube ($n = 0$; dashed lines) under internal pressure $\overline{p} = 0.255$.

1. Elastic-plastic Deformation

In the plastic model, total strains are expressed as the superposition of elastic and plastic parts in the form $\epsilon_j = \epsilon_j^e + \epsilon_j^p$, where the superscripts e and p denote elastic and plastic, respectively. The total axial strain, for example, becomes

$$\epsilon_z = 0 = \frac{1}{E(r)}[\sigma_z - \nu(\sigma_r + \sigma_\theta)] + \epsilon_z^p. \tag{29}$$

Since $\sigma_\theta > \sigma_z > \sigma_r$ throughout the tube, Tresca's yield criterion reads

$$\sigma_\theta - \sigma_r = \sigma_0, \tag{30}$$

and the flow rule associated with this yielding is $\epsilon_\theta^p = -\epsilon_r^p$ and $\epsilon_z^p = 0$ (see for example Mendelson (1968) page 157). From Eq. (30) $\sigma_\theta = \sigma_r + \sigma_0$ and from Eq. (29) $\sigma_z = \nu(\sigma_0 + 2\sigma_r)$ as since $\epsilon_z^p = 0$. Using the equation of equilibrium, Eq. (2), and applying the boundary condition $\sigma_r(a) = -p_{in}$ we end up with the solution for the stresses in the plastic region. They are

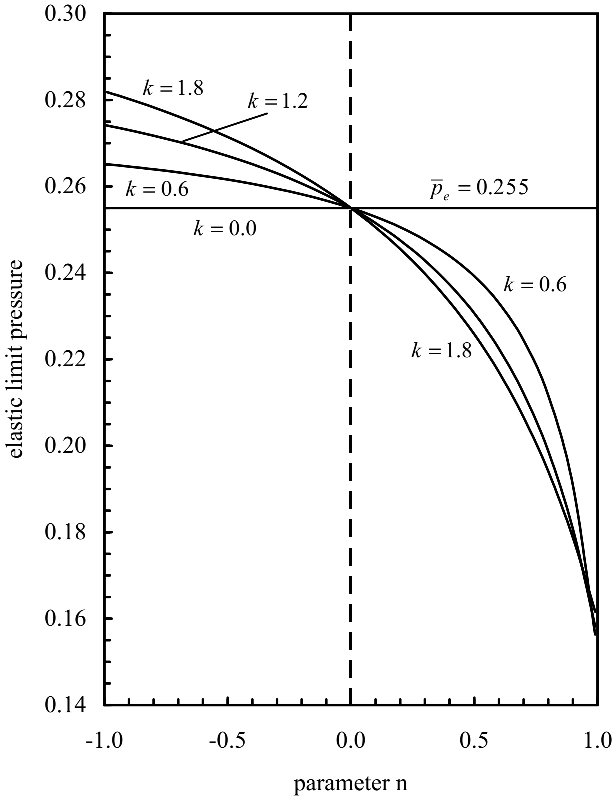

Figure 2. Variation of elastic limit pressure $\overline{p}_e$ with n using k as a parameter in FGM pressurized tubes of inner radius $\overline{a} = 0.7$.

$$\sigma_r = -p_{in} + (\ln r/a)\sigma_0, \tag{31}$$

$$\sigma_\theta = -p_{in} + (1 + \ln r/a)\sigma_0, \tag{32}$$

$$\sigma_z = -2\nu[p_{in} - (1/2 + \ln r/a)\sigma_0]. \tag{33}$$

Making use of Eqs. (31)-(33), strain-displacement relations: $\epsilon_r = u'$; $\epsilon_\theta = u/r$, and the associated flow rule $\epsilon_r^p + \epsilon_\theta^p = 0$, the sum $\epsilon_r + \epsilon_\theta$ is evaluated and simplified to give

$$\frac{du}{dr} + \frac{u}{r} = \frac{D_1}{1 - n\left(\frac{r}{b}\right)^k} + \frac{D_2 \ln r}{1 - n\left(\frac{r}{b}\right)^k}, \tag{34}$$

where

$$D_1 = -(1+\nu)(1-2\nu)[2p_{in} - (1-2\ln a)\sigma_0]/E_0, \tag{35}$$

$$D_2 = 2(1+\nu)(1-2\nu)\sigma_0/E_0. \tag{36}$$

After some algebraic manipulations, the general solution of Eq. (34) is obtained and put into the form

$$u = \frac{C_3}{r} + \frac{D_1 r}{2} F\left(2/k, 1, 1+2/k; n\,(r/b)^k\right) + \frac{D_2 r}{2}\left\{\ln r F\left(1, 2/k, 1+2/k; n\,(r/b)^k\right) - \frac{2}{k^2}\left[\frac{k^2}{4} + \sum_{i=1}^{\infty}\frac{[n\,(r/b)^k]^i}{(i+2/k)^2}\right]\right\}, \tag{37}$$

where F is again the hypergeometric function defined by Eq. (9). Finally, plastic strains are obtained by subtracting elastic strains from total strains, i.e. $\epsilon_j^p = \epsilon_j - \epsilon_j^e$. The result is

$$\epsilon_\theta^p = -\epsilon_r^p = \frac{C_3}{r^2} + \frac{D_1}{2} F\left(2/k, 1, 1+2/k; n\,(r/b)^k\right) + \frac{D_2}{2}\left\{\ln r F\left(1, 2/k, 1+2/k; n\,(r/b)^k\right) - \frac{2}{k^2}\left[\frac{k^2}{4} + \sum_{i=1}^{\infty}\frac{[n\,(r/b)^k]^i}{(i+2/k)^2}\right]\right\} + \frac{(1+\nu)\left\{(1-2\nu)p_{in} - [1-\nu+(1-2\nu)\ln r/a]\sigma_0\right\}}{E_0\left[1-n\left(\frac{r}{b}\right)^k\right]}. \tag{38}$$

It should be noted that Eqs. (37) and (38) reduce by the substitution of $n = 0$ to

$$u = \frac{C_3}{r} + \frac{r[2D_1 + D_2(2\ln r - 1)]}{4}, \tag{39}$$

$$\epsilon_\theta^p = \frac{C_3}{r^2} + \frac{D_1}{2} + \frac{D_2[2\ln r - 1]}{4} + \frac{(1+\nu)\left\{(1-2\nu)p_{in} - [1-\nu+(1-2\nu)\ln r/a]\sigma_0\right\}}{E_0}. \tag{40}$$

This is the plastic solution for a plane strain homogeneous tube and may easily be verified by starting out with

$$\frac{du}{dr} + \frac{u}{r} = D_1 + D_2 \ln r. \tag{41}$$

The plastic region formed at the inner surface of the tube under the pressure $p_{in} = p_e$ propagates toward the outer surface for the values $p_{in} > p_e$. For $p_{in} > p_e$ the tube consists of an inner plastic region in $a < r \le r_{ep}$, and an outer elastic region in $r_{ep} \le r < b$, with r_{ep} being the plastic-elastic border. The solution requires the evaluation of 4 unknowns C_3, r_{ep}, C_1, and C_2. At the interface $r = r_{ep}$, the 2 stress components and radial displacements on both sides must be equal, i.e. $\sigma_r^p(r_{ep}) = \sigma_r^e(r_{ep})$; $\sigma_\theta^p(r_{ep}) = \sigma_\theta^e(r_{ep})$; $u^p(r_{ep}) = u^e(r_{ep})$. These continuity conditions are adjoined with the formal boundary condition $\sigma_r^e(b) = 0$ to get 4 nonredundant equations. Although these equations are linear in integration constants C_3, C_1, and C_2, the system as a whole is nonlinear. Newton iterations are used for the numerical solution.

An FGM tube of inner radius $\overline{a} = 0.7$ with material parameters $n = -0.4$ and $k = 0.6$ undergoes plastic deformation when internal pressure reaches $\overline{p}_e = 0.260594$. The tube behaves elastic-plastic for $p_{in} > p_e$. Assigning $\overline{p}_{in} = 0.33 > \overline{p}_e$, Newton iterations are carried out to compute the unknowns. They are obtained as $\overline{C}_3 = C_3/b^2 = 9.65200 \times 10^{-4}$, $\overline{r}_{ep} = r_{ep}/b = 0.835838$, $\overline{C}_1 = 3.87282 \times 10^{-4}$, $\overline{C}_2 = 7.39470 \times 10^{-4}$. The corresponding stresses and displacement are plotted in Figure 3. In this figure, $\overline{\epsilon}_j^p = \epsilon_j^p E_0/\sigma_0$ is the normalized plastic strain and ϕ is the stress variable calculated from $\phi = \overline{\sigma}_\theta - \overline{\sigma}_r$. As seen, $\phi = 1$ in the plastically deformed region, verifying the ideally plastic behavior of the tube.

The propagation of elastic-plastic border radius r_{ep} with increasing pressures from elastic to fully plastic stress states is also investigated. The parameters $\overline{a} = 0.7$ and $k = 0.6$ are used in these calculations. Results for 3 different n values are shown in Figure 4. At $\overline{r}_{ep} = 1.0$, 3 curves intersect at the same point, corresponding to $\overline{p}_{in} = 0.356675$. This is the fully plastic limit of the tubes and turns out to be independent of the variation of E in the material.

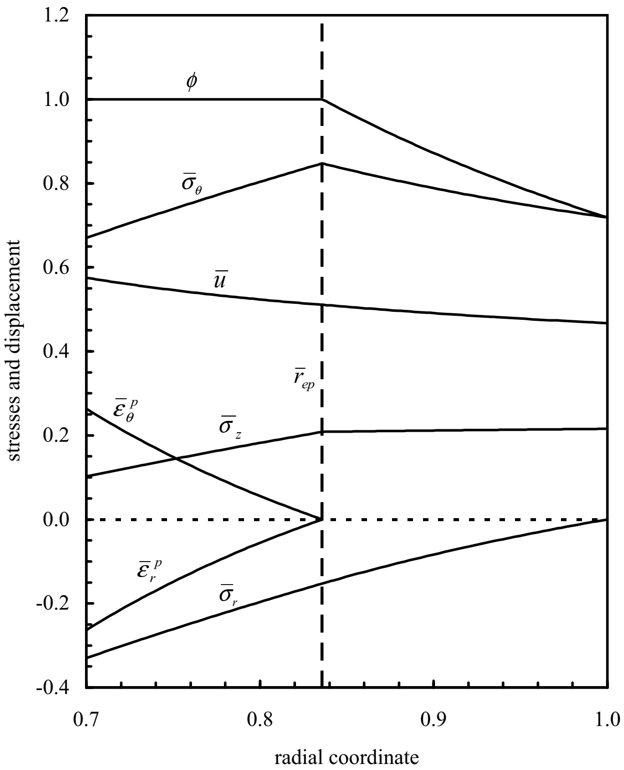

Figure 3. Stresses, plastic strains and displacement in a partially plastic FGM tube ($n = -0.4$, $k = 0.6$) of inner radius $\overline{a} = 0.7$ under internal pressure $\overline{p} = 0.33$.

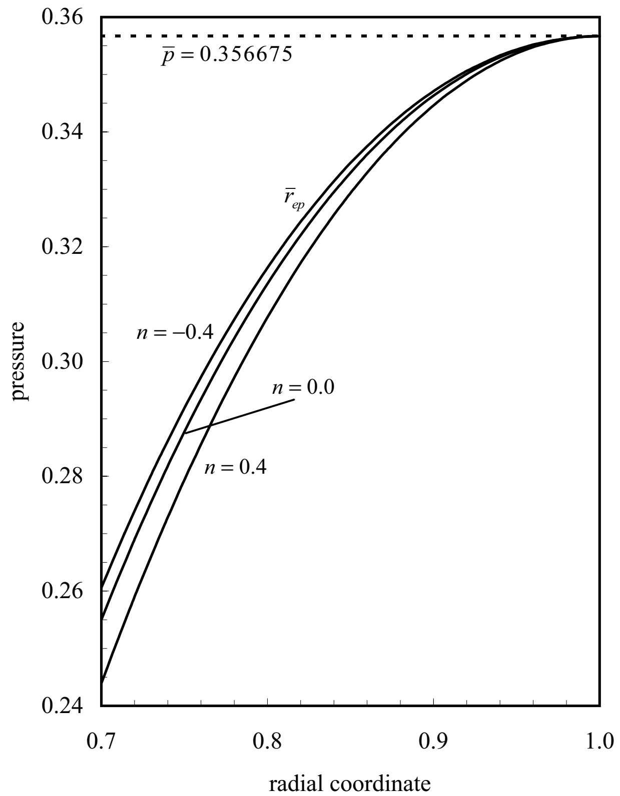

Figure 4. Propagation of elastic plastic border radius $\overline{r}_{ep}$ in an FGM tube with increasing pressures for $\overline{a} = 0.7$ and $k = 0.6$ using n as a parameter.

References

Abramowitz, M. and Stegun, A.I. (Eds.), Handbook of Mathematical Functions, 5th. Printing, US Government Printing Office, 1966.

Boresi, A.P., Schmidt, R.J. and Sidebottom, O.M., Advanced Mechanics of Materials, 5th ed., Wiley, New York, 1993.

Eraslan, A.N., "Inelastic Deformations of Rotating Variable Thickness Solid Disks by Tresca and von Mises Criteria". Int. J. of Computational Engineering Science 3, 89-101, 2002.

Eraslan, A.N., "Elastic-Plastic Deformations of Rotating Variable Thickness Annular Disks with Free, Pressurized and Radially Constrained Boundary Conditions". Int. J. of Mechanical Sciences 45, 643-667, 2003.

Eraslan, A.N. and Akis, T., "Yielding of Two-Layer Shrink-Fitted Composite Tubes Subject to Radial Pressure". Forschung im Ingenieurwesen/Engineering Research (in press) 2005.

Horgan, C.O. and Chan, A.M., "The Pressurized Hollow Cylinder or Disk Problem for Functionally Graded Isotropic Linearly Elastic Materials". Journal of Elasticity 55, 43-59, 1999.

Jabbari, M., Sohrabpour, S. and Eslami, M.R., "Mechanical and Thermal Stresses in a Functionally Graded Hollow Cylinder due to Radially Symmetric Loads". Int. J. of Pressure Vessels and Piping 79, 493-497, 2002.

Ma, L., Feng, X.Q., Gau, K.W. and Yu, S.W., "Elastic and Plastic Analyses of Functionally Graded Elements". Functionally Graded Materials VII Materials Science Forum 423-4, 731-736, 2003.

Mendelson A., Plasticity: Theory and Application, Macmillan, New York, 1968.

Nadai, A., Plasticity; a Mechanics of the Plastic State of Matter, McGraw-Hill, New York, London, 1931.

Orcan, Y. and Eraslan, A.N., "Elastic-Plastic Stresses in Linearly Hardening Rotating Solid Disks of Variable Thickness". Mechanics Research Communications 29, 269-281, 2002.

Parker, A.P., "Autofrettage of Open-End Tubes - Pressures, Stresses, Strains, and Code Comparisons". Journal of Pressure Vessel Technology - Transactions of the ASME 123, 271-281, 2001.

Perry, J. and Aboudi, J., "Elasto-Plastic Stresses in Thick Walled Cylinders". Journal of Pressure Vessel Technology - Transactions of the ASME 125, 248-252, 2003.

Timoshenko, S.P., Strength of Materials: Part II Advanced Theory and Problems, 3rd. ed., D. van Nostrand Company, New York, 1956.

Timoshenko, S.P. and Goodier, J.N., Theory of Elasticity, 3rd ed., McGraw-Hill, New York, 1970.

Tutuncu, N. and Ozturk, M., "Exact Solutions for Stresses in Functionally Graded Pressure Vessels". Composites: Part B 32, 683-686, 2001.

Uğural, A.C. and Fenster, S.K., Advanced Strength and Applied Elasticity, 2nd. SI ed., Prentice Hall, New York, 1987.