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Unsteady Viscous Flow Induced by Eccentric-Concentric Rotation of a Disk and the Fluid at Infinity

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Abstract

The unsteady viscous flow produced by a sudden coincidence of two axes while a disk and the fluid at infinity are initially rotating with the same angular velocity about non-coincident axes is examined. The velocity field and the shear stress components on the disk are found exactly with the use of the Laplace transform technique. In order to confirm the results obtained exactly, another solution that is valid at small times is also obtained. At the region near the disk, it is observed that the projections of the rotation centers of the fluid layers on the disk plane are in both the first and the second quadrant for the given flow geometry.

Key words: Eccentric and Concentric Rotation, Unsteady Flow, Newtonian Fluid.

Introduction

It is possible to determine the material moduli of non-Newtonian fluids if an apparatus consisting of two parallel disks rotating with the same angular velocity about two different axes perpendicular to the disks is used (Maxwell and Chartoff, 1965). For this flow, a symmetric condition is used because of the characteristic of the flow (Berker, 1982). However, when a disk and a fluid at infinity rotates with the same angular speed about non-coincident axes, the fluid at infinity is free of shear stress. Hence, we do not need an extra condition.

Coirier (1972) was the first to study the flow of a Newtonian fluid caused by non-coaxial rotation of a disk and of the fluid at infinity. He considered the rotation with both the same and different angular velocities. Erdoğan (1976a, 1977) investigated the same flow in the case of a porous disk, when they rotate with the same and slightly different angular velocities, respectively. Murthy and Ram (1978) studied the effect of heat transfer on the MHD flow for a Newtonian fluid when a porous disk and the fluid at infinity rotate eccentrically with the same angular velocity. Recently, Ersoy and Barış (2002a) reconsidered the flow given by Coirier (1972) and examined the velocity field according to the coordinates.

Erdoğan (1976b) considered the flow in the same geometry for a non-Newtonian fluid after Coirier's work. He investigated the flow of a second order fluid when the disk and the fluid at infinity rotate with slightly different angular velocities. Ersoy (2000) investigated the MHD flow of a conducting Oldroyd-B fluid due to non-coaxial rotation of a porous, insulated disk and the fluid at infinity with the same angular velocity. Ersoy and Barış (2002b) studied the flow induced by the rotation at the same angular velocity in the case of a porous disk for a second grade/order fluid.

Pop (1979) was the first to consider the unsteady flow produced by a disk and a fluid at infinity. He considered the problem for a Newtonian fluid and studied the unsteady flow induced when the disk and the fluid at infinity start impulsively to rotate with the same angular velocity about non-coincident axes. Kasiviswanathan and Rao (1987) presented an exact solution of the unsteady Navier-Stokes equations for the flow produced by an eccentrically rotating porous disk oscillating in its own plane and the fluid at infinity. Erdoğan (1997) studied the unsteady viscous flow resulting from rotation about non-coaxial axes while the disk and the fluid at infinity are initially rotating about a common axis. Hayat et al. (1999) obtained an exact analytic solution for the unsteady viscous flow induced by the oscillations of a porous disk in its own plane. They also discussed the unsteady flow due to the porous disk oscillating and the fluid at infinity rotating about an axis parallel to their first rotation axis. Erdoğan (2000) studied the flow due to non-coaxial rotation of a disk oscillating in its own plane and a viscous fluid at infinity. Hayat et al. (2001) examined Erdoğan's work (1997) for a porous disk in the presence of a magnetic field. Siddique et al. (2001) studied the same problem for a second grade fluid. However, they considered small values of the elastic parameter.

In this paper, we study the unsteady flow of a Newtonian fluid resulting from rotation about a common axis while a disk and the fluid at infinity are initially rotating with the same angular velocity about non-coincident axes. The disk and the fluid at infinity rotate with the same angular velocity throughout the flow. The initial condition is different from that in the papers mentioned above since it is the solution obtained by Coirier (1972). The velocity field and the shear stress components that are related to the force components in the x- and y-directions exerted by the fluid on the disk are found exactly. In order to verify the results obtained exactly, another solution that is valid at small time instants is also obtained.

Governing Equations

Let us consider a Newtonian fluid filling the semi-infinite space $z \geq 0$ in a Cartesian coordinate system. The axis of rotation of the disk located at $z = 0$ and that of the fluid at infinity are in the plane $x = 0$. The disk and the fluid at infinity are initially rotating with the same angular velocity Ω about the z- and z'-axes, respectively, and the distance between the axes is denoted by ℓ (Figure 1). The disk suddenly starts to rotate with its initial angular velocity about the z'-axis. Therefore, the initial and boundary conditions are

$$\begin{aligned} &u = -\Omega\, y + \hat{f}(z), \quad v = \Omega x + \hat{g}(z), \\ &w = 0 \text{ at } t = 0 \qquad \text{for } z \geq 0, \end{aligned} \tag{1a}$$

$$\begin{aligned} &u = -\Omega\,(y - \ell), \quad v = \Omega x, \\ &w = 0 \text{ at } z = 0 \quad \text{for } t > 0, \end{aligned} \tag{1b}$$

$$\begin{aligned} &u = -\Omega\,(y - \ell), \quad v = \Omega\, x, \\ &w = 0 \text{ at } z \to \infty \quad \text{for } t \geq 0 \end{aligned} \tag{1c}$$

where $u,\ v,\ w$ denote the velocity components along the x, y, z-directions, respectively. The functions $\hat{f}(z)$ and $\hat{g}(z)$, obtained by Coirier (1972), are given by

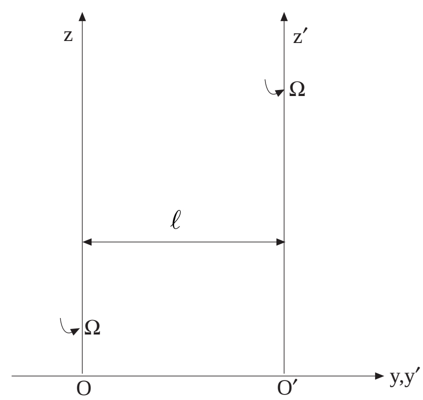

Figure 1. Flow geometry

$$\hat{f}(z) + i\,\hat{g}(z) = \Omega\,\ell\,\left(1 - e^{-\mathrm{k}\,z}\right) \tag{2}$$

where $i = \sqrt{-1}$, $\mathrm{k} = (1 + i)\,\sqrt{\Omega/(2\,\nu)}$, and ν is the kinematic viscosity of the fluid. The solution shown by Eq. (2) is summarized in the Appendix.

Thus, it seems reasonable to try a solution of the form

$$u = -\Omega\, y + f(z,t), \quad v = \Omega\, x + g(z,t), \quad w = 0. \tag{3}$$

This means that the flow is a result of superposition, in each z = constant plane, of a rigid body rotation with the angular velocity Ω about the z-axis and of a time-dependent rigid body translation that changes from plane to plane with the velocity $\{f(z,t);\ g(z,t);\ 0\}$ in a Cartesian coordinate system. Using Eqs. (1a-c) and (3), we have

$$f(z,0) = \hat{f}(z), \quad g(z,0) = \hat{g}(z) \quad \text{for } z \geq 0, \tag{4a}$$

$$f(0,t)=\Omega\,\ell,\quad g(0,t)=0\quad \text{for } t>0, \tag{4b}$$

$$f(\infty,t)=\Omega\,\ell,\quad g(\infty,t)=0\quad \text{for } t\geq 0. \tag{4c}$$

Substituting Eq. (3) into the Navier-Stokes equations, one obtains

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\Omega^2 x+\left(\nu\,\frac{\partial^2 f}{\partial z^2}-\frac{\partial f}{\partial t}+\Omega\,g\right), \tag{5a}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}=\Omega^2 y+\left(\nu\,\frac{\partial^2 g}{\partial z^2}-\frac{\partial g}{\partial t}-\Omega\,f\right), \tag{5b}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=0 \tag{5c}$$

where ρ is the density of the fluid and p is the modified pressure. Equations (5a-c) give

$$\nu\frac{\partial^2 f}{\partial z^2}-\frac{\partial f}{\partial t}+\Omega\,g=C_1(t), \tag{6a}$$

$$\nu\frac{\partial^2 g}{\partial z^2}-\frac{\partial g}{\partial t}-\Omega\,f=C_2(t). \tag{6b}$$

Since the fluid at infinity has no shear stress, we find that $C_1(t)=0$ and $C_2(t)=-\Omega^2\ell$ with the help of Eq. (4c). Introducing $F(z,t)=f(z,t)+i\,g(z,t)$ and using Eqs. (6a-b), we get

$$\nu\frac{\partial^2 F}{\partial z^2}-\frac{\partial F}{\partial t}-i\,\Omega\,F=-i\,\Omega^2\ell. \tag{7}$$

If we introduce the following dimensionless variables

$$\mathrm{F}=\frac{f}{\Omega\,\ell}+i\,\frac{g}{\Omega\,\ell}-1,\quad \zeta=\sqrt{\frac{\Omega}{2\,\nu}}\,z,\quad \tau=\Omega\,t \tag{8}$$

then Eq. (7) and the conditions (4a-c) become

$$\frac{\partial^2 \mathrm{F}}{\partial \zeta^2}-2\,\frac{\partial \mathrm{F}}{\partial \tau}-2\,i\,\mathrm{F}=0, \tag{9}$$

and

$$\mathrm{F}(0,\tau)=0\quad(\tau>0), \tag{10a}$$

$$\mathrm{F}(\infty,\tau)=0\quad(\tau\geq 0), \tag{10b}$$

$$\mathrm{F}(\zeta,0)=-e^{-(1+i)\,\zeta}\quad(\zeta\geq 0). \tag{10c}$$

Solution of the problem

Setting

$$\mathrm{F}(\zeta,\tau)=H(\zeta,\tau)\,e^{-i\,\tau}, \tag{11}$$

we have

$$\frac{\partial^2 H}{\partial \zeta^2}-2\,\frac{\partial H}{\partial \tau}=0 \tag{12}$$

with the conditions as follows:

$$H(0,\tau)=0\quad(\tau>0), \tag{13a}$$

$$H(\infty,\tau)=0\quad(\tau\geq 0), \tag{13b}$$

$$H(\zeta,0)=-e^{-(1+i)\,\zeta}\quad(\zeta\geq 0). \tag{13c}$$

Let the Laplace transform of $H(\zeta,\tau)$ be $\bar{H}(\zeta,s)$, so that

$$\bar{H}(\zeta,s)=\int_0^\infty H(\zeta,\tau)\,e^{-s\,\tau}\,d\,\tau. \tag{14}$$

Taking the Laplace transforms of Eqs. (12)-(13a-b), we get

$$\bar{H}''-2\,s\,\bar{H}=2\,e^{-(1+i)\,\zeta}, \tag{15a}$$

$$\bar{H}(0,s)=0\quad(\tau>0), \tag{15b}$$

$$\bar{H}\,(\infty,s)=0\quad(\tau\geq 0) \tag{15c}$$

and we obtain the transformed solution as

$$\bar{H} = \frac{1}{s-i}\, e^{-\sqrt{2\,s}\,\zeta} - \frac{1}{s-i}\, e^{-(1+i)\,\zeta}. \tag{16}$$

In order to take the Laplace inversion of $\bar{H}$, we shall rewrite Eq. (16) as follows:

$$\bar{H} = \frac{1}{2\sqrt{s}} \left(\frac{1}{\sqrt{s}+\sqrt{i}} + \frac{1}{\sqrt{s}-\sqrt{i}} \right) e^{-\sqrt{2\,s}\,\zeta} - \frac{1}{s-i}\, e^{-(1+i)\,\zeta}. \tag{17}$$

With the help of Table 1 (Abramowitz and Stegun, 1965), we obtain

$$H = \frac{1}{2}\, e^{i\,\tau} \left[e^{\sqrt{2i}\,\zeta} \mathrm{erfc} \left(\sqrt{i\,\tau} + \frac{\zeta}{\sqrt{2\,\tau}} \right) + e^{-\sqrt{2\,i}\,\zeta} \mathrm{erfc} \left(-\sqrt{i\,\tau} + \frac{\zeta}{\sqrt{2\,\tau}} \right) \right] - e^{i\,\tau}\, e^{-(1+i)\,\zeta} \tag{18}$$

where erfc (.) denotes the complementary error function.

Using Eqs. (11) and (18), one finds

$$\frac{f}{\Omega\,\ell} + i\,\frac{g}{\Omega\,\ell} = 1 - e^{-(1+i)\,\zeta} + \frac{1}{2} \left[e^{\sqrt{2\,i}\,\zeta} \mathrm{erfc} \left(\sqrt{i\,\tau} + \frac{\zeta}{\sqrt{2\,\tau}} \right) + e^{-\sqrt{2\,i}\,\zeta} \mathrm{erfc} \left(-\sqrt{i\,\tau} + \frac{\zeta}{\sqrt{2\,\tau}} \right) \right]. \tag{19}$$

The dimensionless shear stress components in the fluid are obtained by

$$\frac{T_{xz} + i\,T_{yz}}{T_r} = \frac{\partial}{\partial\,\zeta} \left(\frac{f}{\Omega\,\ell} + i\,\frac{g}{\Omega\,\ell} \right) \tag{20}$$

where $T_r = \sqrt{\mu\,\rho\,\Omega^3/2}\,\ell$, and μ is the dynamic viscosity of the fluid. Using Eqs. (19)-(20), we obtain

$$\frac{T_{xz} + i\,T_{yz}}{T_r} = (1+i) \left[e^{-(1+i)\,\zeta} + \frac{1}{2}\, e^{\sqrt{2\,i}\zeta}\, \mathrm{erfc} \left(\sqrt{i\,\tau} + \frac{\zeta}{\sqrt{2\,\tau}} \right) - \frac{1}{2}\, e^{-\sqrt{2\,i}\,\zeta}\, \mathrm{erfc} \left(-\sqrt{i\,\tau} + \frac{\zeta}{\sqrt{2\,\tau}} \right) \right]$$

$$-\frac{\sqrt{2}}{2}\, \frac{1}{\sqrt{\pi\,\tau}} \left[e^{\sqrt{2\,i}\,\zeta - (\sqrt{i\,\tau} + \zeta/\sqrt{2\,\tau})^2} + e^{-\sqrt{2\,i}\,\zeta - (-\sqrt{i\,\tau} + \zeta/\sqrt{2\,\tau})^2} \right]. \tag{21}$$

Finally, we have the dimensionless shear stress components on the disk in the following form:

$$(\bar{\mathrm{T}}_{xz})_{\zeta=0} + i\,(\bar{\mathrm{T}}_{yz})_{\zeta=0} = (1+i) \left\{ 1 + \frac{1}{2} \left[\mathrm{erfc} \left(\sqrt{i\,\tau} \right) - \mathrm{erfc} \left(-\sqrt{i\,\tau} \right) \right] \right\} - \sqrt{\frac{2}{\pi\,\tau}}\, e^{-i\,\tau} \tag{22}$$

where $\bar{\mathrm{T}}_{xz} = T_{xz}/T_r$ and $\bar{\mathrm{T}}_{yz} = T_{yz}/T_r$.

Solution at small times

Although the solution given above is exact, we shall search for another solution that is valid for small values of time. Thus, our purpose is to compare the results obtained in two different forms. We shall write Eq. (16) as follows:

$$\bar{H} = \frac{1}{s}\, \frac{1}{(1 - i/s)} \left[e^{-\varphi\,\zeta} - e^{-(1+i)\,\zeta} \right] \tag{23}$$

where $\varphi = \sqrt{2\,s}$. It is well known that the series $\sum_{n=0}^{\infty} \mathrm{x}^n$ converges to $(1-\mathrm{x})^{-1}$ for $|\mathrm{x}| < 1$. Using this binomial series, it is possible to obtain the solution for small values of the time corresponding to large s, i.e.

$$\bar{H} = \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{i}{s}\right)^n \left[e^{-\varphi\,\zeta} - e^{-(1+i)\,\zeta}\right] \tag{24}$$

or

$$\bar{H} = \sum_{n=0}^{\infty}(i)^n\,\frac{e^{-\varphi\,\zeta}}{s^{n+1}} - \sum_{n=0}^{\infty}(i)^n\,e^{-(1+i)\,\zeta}\,\frac{1}{s^{n+1}}. \tag{25}$$

From Table 1 (Abramowitz and Stegun, 1965), the Laplace inversion of $\bar{H}$ becomes

$$H = \sum_{n=0}^{\infty}(i)^n\,(4\,\tau)^n\,\mathrm{i}^{2n}\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - e^{-(1+i)\,\zeta}e^{i\,\tau} \tag{26}$$

where $\mathrm{i}^n\mathrm{erfc}\,(.)$ denotes the repeated integrals of the complementary error function and is given by

$$\mathrm{i}^{-1}\mathrm{erfc}\,\mathrm{x} = \frac{2}{\sqrt{\pi}}\,e^{-\mathrm{x}^2},\quad \mathrm{i}^{0}\mathrm{erfc}\,\mathrm{x} = \mathrm{erfc}\,\mathrm{x},\quad \mathrm{i}^{n}\mathrm{erfc}\,\mathrm{x} = \int_{\mathrm{x}}^{\infty}\mathrm{i}^{n-1}\mathrm{erfc}\,\phi\,d\phi\quad (n = 0,\,1,\,2,\,...). \tag{27}$$

From Eqs. (11) and (26), we have

$$\frac{f}{\Omega\,\ell} + i\,\frac{g}{\Omega\,\ell} = 1 - e^{-(1+i)\,\zeta} + e^{-i\,\tau}\sum_{n=0}^{\infty}(i)^n\,(4\,\tau)^n\,\mathrm{i}^{2n}\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) \tag{28}$$

or

$$\frac{f}{\Omega\,\ell} = \left[1 - e^{-\zeta}\,\cos\,\zeta\right] + (\cos\,\tau)\,[A(\zeta,\tau)] + (\sin\,\tau)\,[B(\zeta,\tau)], \tag{29a}$$

$$\frac{g}{\Omega\,\ell} = [e^{-\zeta}\,\sin\,\zeta] + (\cos\,\tau)\,[B(\zeta,\tau)] - (\sin\,\tau)\,[A(\zeta,\tau)] \tag{29b}$$

where

$$A(\zeta,\tau) = \mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - (4\,\tau)^2\,\mathrm{i}^4\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) + (4\,\tau)^4\,\mathrm{i}^8\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - ..., \tag{30a}$$

$$B(\zeta,\tau) = (4\,\tau)\,\mathrm{i}^2\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - (4\,\tau)^3\,\mathrm{i}^6\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) + (4\,\tau)^5\,\mathrm{i}^{10}\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - ...\quad. \tag{30b}$$

Bearing in mind the identity

$$\frac{d}{d\,\mathrm{z}}\,\mathrm{i}^n\mathrm{erfc}\,\mathrm{y} = -\left(\frac{d\,\mathrm{y}}{d\,\mathrm{z}}\right)\,\mathrm{i}^{n-1}\mathrm{erfc}\,\mathrm{y}, \tag{31}$$

one finds

$$\frac{T_{xz}}{T_r} = e^{-\zeta}\,(\cos\,\zeta + \sin\,\zeta) - \frac{\cos\,\tau}{\sqrt{2\,\tau}}\,[C(\zeta,\,\tau)] - \frac{\sin\,\tau}{\sqrt{2\,\tau}}\,[D(\zeta,\,\tau)], \tag{32a}$$

$$\frac{T_{yz}}{T_r} = e^{-\zeta}\,(\cos\,\zeta - \sin\,\zeta) - \frac{\cos\,\tau}{\sqrt{2\,\tau}}\,[D(\zeta,\,\tau)] + \frac{\sin\,\tau}{\sqrt{2\,\tau}}\,[C(\zeta,\,\tau)] \tag{32b}$$

where

$$C(\zeta,\tau) = \mathrm{i}^{-1}\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - (4\,\tau)^2\,\mathrm{i}^3\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) + (4\,\tau)^4\,\mathrm{i}^7\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - \ldots\,, \tag{33a}$$

$$D(\zeta,\,\tau) = (4\,\tau)\,\mathrm{i\;erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - (4\,\tau)^3\,\mathrm{i}^5\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) + (4\,\tau)^5\,\mathrm{i}^9\mathrm{erfc}\left(\frac{\zeta}{\sqrt{2\,\tau}}\right) - \ldots\,. \tag{33b}$$

Using the identity

$$\mathrm{i}^n\mathrm{erfc}\,0 = [2^n\,\Gamma(n/2+1)]^{-1} \quad (n = -1,\,0,\,1,\,2,\,\ldots), \tag{34}$$

the shear stress components on the disk are

$$(\mathrm{T}_{xz})_{\zeta=0} = 1 - \frac{\cos\,\tau}{\sqrt{2\,\tau}}\left[\frac{1}{2^{-1}\Gamma(0.5)} - \frac{(4\,\tau)^2}{2^3\,\Gamma(2.5)} + \frac{(4\,\tau)^4}{2^7\,\Gamma(4.5)} - \frac{(4\,\tau)^6}{2^{11}\,\Gamma(6.5)} + \ldots\right]$$

$$-\frac{\sin\,\tau}{\sqrt{2\,\tau}}\left[\frac{(4\,\tau)}{2\,\Gamma(1.5)} - \frac{(4\,\tau)^3}{2^5\,\Gamma(3.5)} + \frac{(4\,\tau)^5}{2^9\,\Gamma(5.5)} - \frac{(4\,\tau)^7}{2^{13}\,\Gamma(7.5)} + \ldots\right], \tag{35a}$$

$$(\mathrm{T}_{yz})_{\zeta=0} = 1 - \frac{\cos\,\tau}{\sqrt{2\,\tau}}\left[\frac{(4\,\tau)}{2\,\Gamma(1.5)} - \frac{(4\,\tau)^3}{2^5\,\Gamma(3.5)} + \frac{(4\,\tau)^5}{2^9\,\Gamma(5.5)} - \frac{(4\,\tau)^7}{2^{13}\,\Gamma(7.5)} + \ldots\right]$$

$$+\frac{\sin\,\tau}{\sqrt{2\,\tau}}\left[\frac{1}{2^{-1}\,\Gamma(0.5)} - \frac{(4\,\tau)^2}{2^3\,\Gamma(2.5)} + \frac{(4\,\tau)^4}{2^7\,\Gamma(4.5)} - \frac{(4\,\tau)^6}{2^{11}\,\Gamma(6.5)} + \ldots\right] \tag{35b}$$

where $\Gamma(.)$ is the gamma function defined by

$$\Gamma(n) = \int_0^\infty \mathrm{x}^{n-1}\,e^{-\mathrm{x}}\,d\mathrm{x}(n > 0). \tag{36}$$

Table 1. Table of Laplace transforms used in this paper, where $\mathrm{f}(s)$ is the Laplace transform of $\mathrm{F}(t)$ (Abramowitz and Stegun, 1965).

$\mathrm{f}(s)$	$\mathrm{F}(t)$
$\frac{e^{-k\,\sqrt{s}}}{\sqrt{s}\,(a+\sqrt{s})}\quad (k \geq 0)$	$e^{a\,k}\,e^{a^2\,t}\,\mathrm{erfc}\left(a\,\sqrt{t} + \frac{k}{2\,\sqrt{t}}\right)$
$\frac{1}{s+a}$	$e^{-a\,t}$
$\frac{e^{-k\,\sqrt{s}}}{s^{1+m/2}}\ (m = 0,\,1,\,2,\,\ldots;\,k \geq 0)$	$(4\,t)^{m/2}\,\mathrm{i}^m\mathrm{erfc}\left(\frac{k}{2\,\sqrt{t}}\right)$
$\frac{1}{s^m}\quad (m = 1,\,2,\,3,\,\ldots)$	$\frac{t^{m-1}}{(m-1)!}$

For small values of time, the solutions given by Eqs. (29a-b) and (35a-b) can be used instead of the exact solutions given by Eqs. (19) and (22), respectively. It is clear that Eqs. (29a-b) and (35a-b) are very convenient for $\tau \leq 0.25$.

Discussion and Conclusions

When a disk and a fluid at infinity rotate with the same angular velocity about non-coincident axes, the fluid layer in each z = constant plane rotates as a rigid body with their angular velocity. The coordinates of the rotation centers of these fluid layers in this paper are obtained by $x/\ell = -g/\Omega\,\ell$ and $y/\ell = f/\Omega\,\ell$ for $0 \leq \zeta < \infty$. Thus, the velocity components in the planes parallel to the xy-plane are calculated according to the coordinates.

In our problem, the disk and the fluid at infinity are initially rotating with the same angular velocity about two parallel axes normal to the disk. Hence, the initial condition becomes the solution given by Coirier (1972). Because of the rotation about the common axis for $t > 0$, the fluid tends to a rigid body rotation and it rotates about the z'-axis in the steady flow, as expected (Figures 2-3). In the limit as $\tau \to \infty$, this result is readily acquired from Eq. (19) for every ζ.

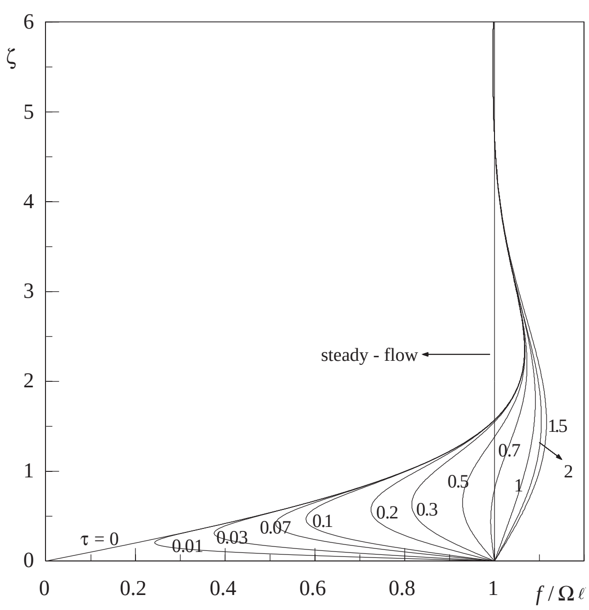

Figure 2. Variation of $f/\Omega\,\ell$ versus ζ for various values of τ.

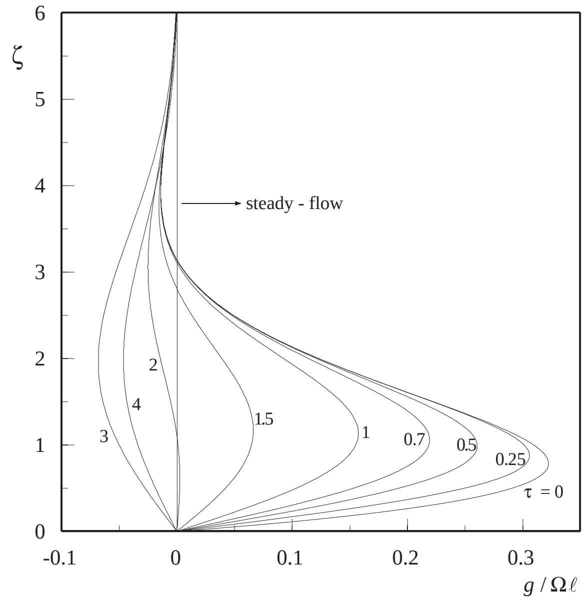

Figure 3. Variation of $g/\Omega\,\ell$ versus ζ for various values of τ.

At the region near the disk, the projections of the rotation centers of the fluid layers on the xy-plane are in the second quadrant until the computed value $\tau = 2.05158$. After this instant, the projections are in the first quadrant. The projections continue to be in both the first and second quadrants as time elapses, and they reach the point $\mathrm{O}'(0, \ell, 0)$ in the steady flow.

Since the shear stress components T_{xz} and T_{yz} do not depend on x and y, $(\mathrm{T}_{xz})_{\zeta=0}$ and $(\mathrm{T}_{yz})_{\zeta=0}$ are related to the x- and y-components of the force per unit area exerted by the fluid on the disk, respectively (Figure 4). At small times, the components become negative and positive in the x- and y-directions, respectively; however, the x-component is larger than the y-component. The components change their directions continuously and finally go to zero. As shown in Figure 4, these components become equal at some values of time (for example $(\mathrm{T}_{xz})_{\zeta=0} = (\mathrm{T}_{yz})_{\zeta=0} = 0.11928$ for $\tau = 1.37947$ and $(\mathrm{T}_{xz})_{\zeta=0} = (\mathrm{T}_{yz})_{\zeta=0} = -0.02951$ for $\tau = 4.22331$).

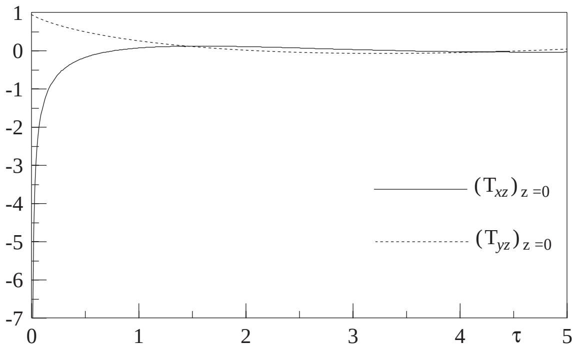

Figure 4. Variations of $(\mathrm{T}_{xz})_{\zeta=0}$ and $(\mathrm{T}_{yz})_{\zeta=0}$ versus τ.

Nomenclature

ℓ	eccentricity distance (L)
p	modified pressure (M L^{-1} T^{-2})
s	Laplace transform variable
t	time (T)
T_{xz}, T_{yz}	shear stress components (M L^{-1} T^{-2})
$(\mathrm{T}_{xz})_{\zeta=0}$, $(\mathrm{T}_{yz})_{\zeta=0}$	dimensionless shear stress components on the disk
u, v, w	velocity components in Cartesian coordinate system (L T^{-1})
x, y, z	Cartesian coordinates (L)
ζ	non-dimensional vertical distance
μ	dynamic viscosity (M L^{-1} T^{-1})
ν	kinematic viscosity (L^2 T^{-1})
ρ	density (M L^{-3})
τ	non-dimensional time
Ω	common angular velocity of the disk and the fluid at infinity (T^{-1})

Appendix

The initial condition in this paper is the solution given by Coirier (1972). He considered the Newtonian fluid case and studied the flow produced when a disk and the fluid at infinity rotate about the z- and z'-axes, respectively (for the notation used in this paper, see Figure 1). In the case of rotation with the same angular velocity, he assumed the velocity field to be

$$u = -\Omega\, y + \hat{f}(z), \quad v = \Omega\, x + \hat{g}(z), \quad w = 0. \quad \text{(A1)}$$

Substitution of Eq. (A1) into the Navier-Stokes equations leads to

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \Omega\,(\Omega\, x + \hat{g}) + \nu\, \hat{f}'', \quad \text{(A2a)}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -\Omega\,(-\Omega\, y + \hat{f}) + \nu\, \hat{g}'', \quad \text{(A2b)}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = 0. \quad \text{(A2c)}$$

Introducing $\hat{F}(z) = \hat{f}(z) + i\,\hat{g}(z)$ and using Eqs. (A2a-c), we have

$$\hat{F}'' - \frac{i\,\Omega}{\nu}\,\hat{F} = -\frac{i\,\Omega^2 \ell}{\nu} \quad \text{(A3)}$$

With the conditions $\hat{F}(0) = 0$ and $\hat{F}(\infty) = \Omega\,\ell$, the solution to Eq. (A3) is given in Eq. (2).

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