Turkish J. Eng. Env. Sci.
26 (2002) , 503 – 511.

Viscous Heating Effects in Viscoelastic Flow between Rotating Parallel-Disks

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Received 21.03.2002

Abstract

Viscoelastic flow between two rotating parallel-disks (plates) is studied. First the current state of the art is discussed and the limitations resulting from the simple flow assumptions are reported. In an attempt to consider the effects of finiteness of the plates and viscous dissipation, the flow problem is fully formulated; that is, no assumptions are made about the three-dimensional and nonisothermal nature of the flow. One of the major contributions to the formulation is to use a nonisothermal constitutive equation, the nonisothermal Oldroyd-B equation. In this case, the numerical solution scheme of resulting equations is too complex, even for the Oldroyd-B model, which is considered one of the simplest constitutive equations for polymeric flows. Instead, the problem is reduced to a one-dimensional nonisothermal infinite plate problem and the viscous dissipation effect is then investigated. It is determined here that the viscous dissipation effect results in a significant temperature rise in the flow field and may yield errors in the prediction of the material properties of highly viscous and elastic fluids. Some simple modifications in existing equations used for the prediction of fluid-material properties are proposed by considering viscous dissipation and free surface effects.

Key words: Polymeric (Viscoelastic) fluid, Parallel-Disks, Rheometry, Nonisothermal, Viscous dissipation effect.

Introduction

Viscoelastic flow between two rotational plates has received considerable attention in the literature since it is the basis for rheological measurements of the viscosity and normal stresses in non-Newtonian liquids (Bird *et al.,* 1987). Similar flow geometry can also be found in a particular extrusion processing of polymers (Tadmor and Gogos, 1979). The schematics of the geometry in both applications are shown in Figures 1(a) and (b). The simplest case of the flow geometry consists of two co-axial disks of radius R, and these disks are separated by a narrow gap of height H. In a rheological measurement, fluid samples are placed in the narrow gap, and the motion is generated by rotating the upper plate at a fixed rotation rate (Ω) while the bottom plate remains stationary. The dimensionless rotation rate is expressed by the Deborah number, $De = \lambda\Omega$, where λ is the relaxation time of the fluid. The torque required to achieve this rotation as well as the total force required to maintain the disks at a separation H and pressure distribution across one of the plates are measured to predict rheological properties in shear flow.

For prediction of the viscometric properties of a fluid, the flow is assumed to be steady, one-dimensional, and isothermal for all rotation rates. Early theoretical and experimental observations, however, have revealed that these simple flow assumptions are not always valid in practical situations, and prediction of rheological parameters under these assumptions may lead to significant errors (Bird *et al.,* 1987). The major sources of error in predicting using simple flow assumptions are elastic flow transitions, surface deflection at the edge, and temperature rise due to viscous dissipation, and they are discussed briefly below.

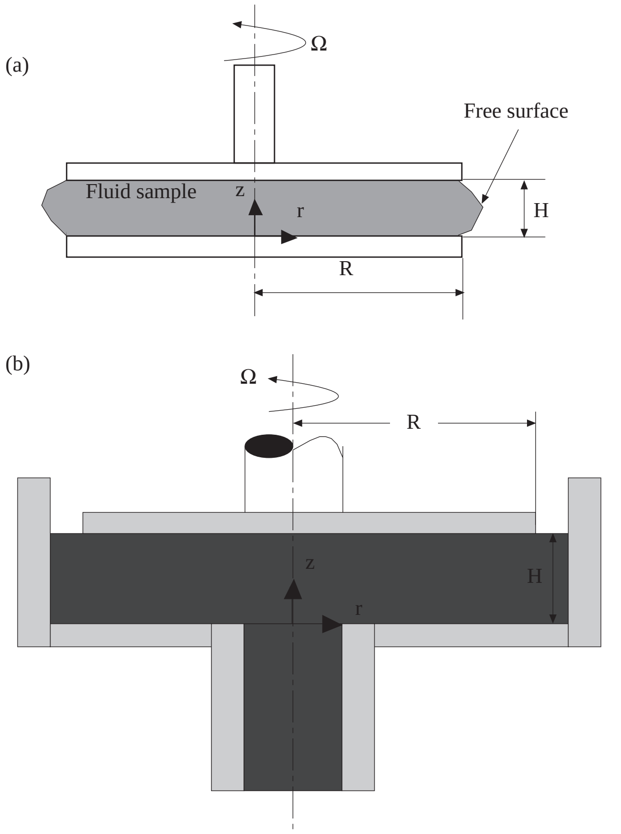

Figure 1. The schematics of (a) a typical parallel-plate rheometer, (b) a normal-stress extruder.

Elastic flow instabilities

Although most of the previous work related to rotating parallel disks dealt with the prediction of secondary flow due to the elasticity of the fluids, few of these were able to report well enough the structure and origin of the instability. McKinley *et al.* (1991), Oztekin and Brown (1993), and Byars *et al.* (1994) have reported the following conclusions related to the flow transition.

Flow becomes time dependent and non-axisymmetric beyond a critical value of the dimensionless rotational rate, or Deborah number (i.e. $De_{crit} = 4.54 \pm 0.02$ for parallel-plate geometry). The temporal character of the secondary flow results in monotonic increase with time on the measurements of apparent viscosity and first normal stress. This transition is called shear-thickening or anti-thixotropic instability. The spatial character of the instability is a sensitive function of the fluid rheology and the aspect ratio of the finite disks. The origin of this instability is the large difference between the extra normal stresses in the streamwise direction and the direction of shear, which is inherent in viscoelastic fluids.

Edge (free surface) effect

The edge effect is usually important when the aspect ratio of a parallel-plate rheometer (R/H) is relatively small; that is, the instrument has finite dimensions. In this case, some distortion to the flow field near the edge may penetrate through to the flow far from the free surface and affect the result of the measurement. From earlier theoretical and experimental studies, it is known that there is reversal flow from the edge throughout the fluid and this formation may change flow characteristics even far away from the edge (see, for example, Griffiths *et al.*, 1969). In dealing with the theoretical prediction of the flow characteristics, all of the current work (with the exception of Olagunju 1994, Avagliano and Phan-Thien 1995) utilize semi-infinite geometry to obtain a solution. In the analysis by Olagunju (1994), the flow of a viscoelastic fluid is considered to be held by surface tension between two rotating finite plates. He obtained explicit formulas to account for contributions of edge effects on the torque and normal force. It was concluded that torque measurement was insensitive to the edge effect while normal stress was sensitive. These results were similar to those obtained by Shipman et al (1991). Avagligano and Phan-Thien (1995) also used finite plate geometry; however, their study concentrated on flow transitions instead of obtaining the correction factor due to edge effects.

Viscous dissipation effect

When a fluid is sheared, some of the work done is dissipated as heat. This shear-induced heating gives an inevitable increase in temperature within the fluid. Because of the high viscosities of polymeric fluids, especially polymer melts, a temperature rise due to viscous dissipation may considerably affect the isothermal flow field. Moreover, fluid parameters, such as viscosity and relaxation time, are very sensitive to temperature changes. In spite of this crucial importance of the temperature dependent flow phenomenon, relatively little attention has been paid to the non-isothermal flow of viscoelastic fluids until the last few years. Recent experimental observations on the dynamics and kinematics of nonisothermal viscoelastic flows have been reported by Yesilata *et al.* (2000) for contraction flows, and Rothstein and McKinley (2001), and Olagunju *et al.* (2002) for torsional flows. These studies have demonstrated remarkable modifications the onset conditions of purely elastic instabilities for both flows due to temperature gradients resulting from sequential heating or cooling imposed at the boundaries or the heat generated by viscous dissipation. One of the main conclusions drawn by Yesilata *et al.* (2000) is that modeling nonisothermal viscoelastic flows requires constitutive equations that include the effect of temperature gradient on the stress field. In other words, coupling between the momentum and energy conservation equations should be achieved through temperature-dependent constitutive equations along with temperature-dependent material properties. Therefore, the modeling of the viscous heating effect for viscoelastic fluids is quite different than that for inelastic non-Newtonian fluids, which has been reported previously by Turian and Bird (1963). A nonisothermal constitutive equation is necessary due to the relative difference between the observers' time scale and the material's internal time scale, according to the Morland-Lee hypothesis (see Luo and Tanner, 1987). Recently, significant progress has been made in the development of non isothermal constitutive equations and several proposed models presently exist in the literature (see Bird and Wiest (1995) for a review).

This paper mainly deals with misprediction of rheological data due to viscous heating rather than its effect on rotational flow transitions. Three-dimensional formulation of the flow, which considers free surface effects at edges and viscous dissipation effect, is clearly outlined in Section 2. Inclusion of the time-dependency and nonaxisymmetry of the flow is the subject of a complex stability analysis and thus is outside the scope of this paper. One of the major contributions to the formulation is to use a nonisothermal constitutive equation, the nonisothermal Oldroyd-B equation, which is obtained from the non-isothermal Giesekus constitutive equation proposed by Wiest (1995). The Oldroyd-B constitutive equation predicts a constant viscosity and a constant first normal stress coefficient, and thus it is relatively a simple model for viscoelastic flows. The model is very suitable for Boger fluids that have nearly constant viscosities up to high shear rates. In order to explore whether a temperature gradient is present in the flow due to viscous dissipation, the solution found for one-dimensional and nonisothermal flow between infinite plates is applied to a typical viscoelastic fluid (PIB/PB Boger fluid) in Section 3.

Formulation

Dimensionless governing equations

Dimensionless governing equations for creeping flow ($Re = 0$) of a viscoelastic flow are given as

$$\underline{\nabla}.\underline{v} = 0 \tag{1}$$

$$-\underline{\nabla} p + \underline{\nabla}.\underline{\underline{S}} + \underline{\nabla}.(1-\beta_T)\underline{\underline{\dot{\gamma}}} = 0 \tag{2}$$

$$Pe\frac{DT}{Dt} - \nabla^2 T - Na[\underline{\underline{S}} : \nabla\ \underline{v} + (1-\beta_T)\underline{\underline{\dot{\gamma}}} : \nabla\ \underline{v}] = 0 \tag{3}$$

where $\underline{v}$ is the velocity vector, p is the pressure, $\underline{\underline{S}}$ is the polymeric contribution to the deviatoric stress tensor, and β_{T} is the ratio of temperature dependent polymer viscosity to total viscosity measured at reference temperature. Operators $\underline{\nabla}$, D/Dt, and (:) show gradient vector, substantial derivative and double-dot product of two tensors respectively. Clear definitions of these operators can be found in appendix section of the book by Bird *et al.* (1987). The total viscosity ($\eta = \eta_p + \eta_s$) is formed by contributions of polymer viscosity (η_p) and solvent viscosity (η_s). Definitions of these parameters along with the temperature shift factor (a_{T}), the rate-of-strain tensor ($\underline{\underline{\dot{\gamma}}}$), dimensionless temperature (T), Peclet number (Pe), and Nahme-Griffith number (Na) are given as

$$\beta_T = a_T[\eta_p/\eta] = a_T\beta, \quad a_T = \exp[C(\frac{1}{\tilde{T}} - \frac{1}{\tilde{T}_0})],$$

$$T = \frac{\tilde{T} - \tilde{T}_0}{\Delta\tilde{T}}, \quad \Delta\tilde{T} = \frac{1}{(da_T/d\tilde{T})_{\tilde{T}=\tilde{T}_0}}$$

$$\underline{\underline{\dot{\gamma}}} = \underline{\nabla v} + (\underline{\nabla v})^T, \quad Pe = \frac{\rho c_p H^2 \dot{\gamma}_R^2}{k}$$

$$Na = \frac{\eta H^2 \dot{\gamma}_R^2}{k\Delta\tilde{T}}, \quad \dot{\gamma}_R = \Omega R/H,$$

where ρ, c_p, and k, are respectively, density, heat capacity per unit mass, and thermal conductivity of the fluid. The superscript '$\sim$' is explicitly used here for some cases to emphasize that the parameter has a dimension, and thus $\tilde{T}$ denotes dimensional temperature. Since no reference temperature difference can be obtained from the boundary conditions, $\Delta\tilde{T}$ is selected as a rheological temperature difference (see Pearson, 1978). The polymeric contribution to the deviatoric stress tensor is given by nonisothermal form of Oldroyd-B constitutive equation

$$[1 - De_T \frac{D(\ln T)}{Dt}]\underline{\underline{S}} + De_T[\underline{v}.\nabla\underline{\underline{S}} - \underline{\underline{S}}.\underline{\underline{L}} - \underline{\underline{S}}.\underline{\underline{L}}^T] = \beta_T \underline{\underline{\dot{\gamma}}} \tag{4}$$

where the second term in the first parentheses is due to the non isothermal effect (see Wiest, 1995), $\underline{\underline{L}} = \underline{\nabla}\ \underline{v}$, and superscript T stands for transpose. $De_T = \lambda_T\Omega = a_T\lambda\Omega$ is the definition of the temperature dependent Deborah number with temperature dependent relaxation time (λ_T). The governing equations given above were scaled with $(H, \Omega^{-1}, H\Omega, \eta_0\Omega)$ for length, time, velocity, and stress respectively. The final forms of the continuity, momentum, and energy equations for steady and axisymmetric flow of a viscoelastic fluid are

$$\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} + \frac{\partial v_z}{\partial z} = 0 \tag{5}$$

$$-\frac{\partial p}{\partial r} + \{\frac{1}{r}\frac{\partial(rS_{rr})}{\partial r} + \frac{\partial(S_{rz})}{\partial z} - \frac{S_{\theta\theta}}{r}\} + \{\frac{1}{r}\frac{\partial[r(1-\beta_T)\dot{\gamma}_{rr}]}{\partial r} + \frac{\partial[(1-\beta_T)\dot{\gamma}_{rz}]}{\partial z} - \frac{(1-\beta_T)\dot{\gamma}_{\theta\theta}}{r}\} = 0 \tag{6}$$

$$\{\frac{1}{r^2}\frac{\partial(r^2S_{r\theta})}{\partial r} + \frac{\partial(S_{\theta z})}{\partial z}\} + \{\frac{1}{r^2}\frac{\partial[(1-\beta_T)r^2\dot{\gamma}_{r\theta}]}{\partial r} + \frac{\partial[(1-\beta_T)\dot{\gamma}_{\theta z}]}{\partial z}\} = 0 \tag{7}$$

$$-\frac{\partial p}{\partial z} + \{\frac{1}{r}\frac{\partial(rS_{rz})}{\partial r} + \frac{\partial(S_{zz})}{\partial z}\} + \{\frac{1}{r}\frac{\partial[(1-\beta_T)r\dot{\gamma}_{rz}]}{\partial r} + \frac{\partial[(1-\beta_T)\dot{\gamma}_{zz}]}{\partial z} = 0 \tag{8}$$

$$\begin{aligned} &Pe\{v_r\frac{\partial T}{\partial r} + v_z\frac{\partial T}{\partial z}\} - \{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{\partial^2 T}{\partial z^2}\} - Na\{[S_{rr} + (1-\beta_T)\dot{\gamma}_{rr}](\frac{\partial v_r}{\partial r}) \\ &+[S_{r\theta} + (1-\beta_T)\dot{\gamma}_{r\theta}](\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}) + [S_{rz} + (1-\beta_T)\dot{\gamma}_{rz}](\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}) \\ &+[S_{\theta\theta} + (1-\beta_T)\dot{\gamma}_{\theta\theta}](\frac{v_r}{r}) + [S_{\theta z} + (1-\beta_T)\dot{\gamma}_{\theta z}](\frac{\partial v_\theta}{\partial z}) \\ &+[S_{zz} + (1-\beta_T)\dot{\gamma}_{zz}](\frac{\partial v_z}{\partial z})\} = 0 \end{aligned} \tag{9}$$

where

$$\dot{\gamma}_{rr} = 2\frac{\partial v_r}{\partial r}, \dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} = r\frac{\partial}{\partial r}(\frac{v_\theta}{r}), \dot{\gamma}_{rz} = \dot{\gamma}_{zr} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}, \dot{\gamma}_{\theta\theta} = 2\frac{v_r}{r}, \dot{\gamma}_{\theta z} = \dot{\gamma}_{z\theta} = \frac{\partial v_\theta}{\partial z}, \dot{\gamma}_{zz} = 2\frac{\partial v_z}{\partial z}$$

The components of the constitutive equation given by equation (4) are expressed as

$$\Gamma S_{rr} + De_T\{-2\frac{v_\theta}{r}S_{r\theta} - 2[S_{rr}L_{rr} + S_{r\theta}L_{\theta r} + S_{rz}L_{zr}] = \beta_T\dot{\gamma}_{rr} \tag{10}$$

$$\Gamma S_{r\theta} + De_T\{\frac{v_\theta}{r}(S_{rr} - S_{\theta\theta}) - [S_{rr}L_{r\theta} + S_{r\theta}(L_{\theta\theta} + L_{rr}) + S_{rz}L_{z\theta} + S_{\theta\theta}L_{\theta r} + S_{\theta z}L_{zr}]\} = \beta_T\dot{\gamma}_{r\theta} \quad (11)$$

$$\Gamma S_{rz} + De_T\{-\frac{v_\theta}{r}S_{\theta z} - [S_{rr}L_{r\theta} + S_{rz}(L_{zz} + L_{rr}) + S_{\theta z}L_{\theta r} + S_{zz}L_{zr}]\} = \beta_T\dot{\gamma}_{rz} \quad (12)$$

$$\Gamma S_{\theta\theta} + De_T\{2\frac{v_\theta}{r}S_{r\theta} - 2[S_{r\theta}L_{r\theta} + S_{\theta\theta}L_{\theta\theta} + S_{\theta z}L_{z\theta}]\} = \beta_T\dot{\gamma}_{\theta\theta} \quad (13)$$

$$\Gamma S_{\theta z} + De_T\{\frac{v_\theta}{r}S_{rz} - [S_{\theta r}L_{rz} + S_{\theta z}(L_{zz} + L_{\theta\theta}) + S_{rz}L_{r\theta} + S_{zz}L_{z\theta}]\} = \beta_T\dot{\gamma}_{\theta z} \quad (14)$$

$$\Gamma S_{zz} + De_T\{-2[S_{rz}L_{rz} + S_{zz}L_{zz}]\} = \beta_T\dot{\gamma}_{zz} \quad (15)$$

where

$$\Gamma = \{1 - De_T(v_r\frac{\partial}{\partial r} + v_z\frac{\partial}{\partial z})\ln(T)\};$$

$$L_{rr} = \frac{\partial v_r}{\partial r}, L_{r\theta} = \frac{\partial v_\theta}{\partial r}, L_{rz} = \frac{\partial v_z}{\partial r}, L_{\theta r} = -\frac{v_\theta}{r}, L_{\theta\theta} = \frac{v_r}{r}, L_{\theta z} = 0, L_{zr} = \frac{\partial v_r}{\partial z}, L_{z\theta} = \frac{\partial v_\theta}{\partial z}, L_{zz} = \frac{\partial v_z}{\partial z}.$$

Boundary conditions

For a more general approach, we consider that the top plate rotates with constant angular velocity Ω and the bottom plate with angular velocity $s\Omega$ where $-1 \le s \le 1$. Corresponding boundary conditions are given below:

No-slip and no-penetration and isothermal conditions on the top ($z = 1$) and bottom ($z = 0$) plates,

$$v_r = v_z = 0, \quad v_\theta = r, T = 0 \quad (16)$$

$$v_r = v_z = 0, \quad v_\theta = rs, T = 0 \quad (17)$$

Symmetry conditions at the centerline ($r = 0$)

$$v_r = v_\theta = 0, \quad \frac{\partial v_z}{\partial r} = \frac{\partial T}{\partial r} = 0 \quad (18)$$

Boundary conditions at the free surface of $r = f(z)$, where $f(z)$ is dimensionless shape of the meniscus at the free surface: respectively, no flow through the boundary of the surface, stress balance at the boundary, and isothermal boundary

$$\underline{v}.\underline{\nabla} f = 0 \quad (19)$$

$$(p - p_a)\underline{n} + [\underline{\underline{S}} + (1 - \beta)\underline{\underline{\dot{\gamma}}}].\underline{n} = \sigma F\underline{n} \quad (20)$$

$$T = T_a \quad (21)$$

where p_a is atmospheric pressure, $\underline{n}$ is unit vector normal to the surface (outward), σ is surface tension, F is mean curvature of the surface, and T_a is atmospheric air temperature, which all are dimensionless. Since $f(z)$ is also a part of the solution, additional boundary conditions on the top and bottom plates are needed. One of the best approaches is to apply the pinning condition to the plates (see Olagunju, 1994), as given below

$$f(z = 0) = R \quad \text{and} \quad f(z = 1) = R. \quad (22)$$

Infinite Plate Solution (R >> H)

For the infinite plate case, by assuming that the top plate rotates while the bottom plate is stationary ($s = 0$), the velocity components in the flow field can be expressed as

$$v_r = v_z = 0 \quad v_\theta = v_\theta(r, z). \tag{23}$$

The corresponding boundary conditions for $z = 0$ and $z = 1$ are given as

$$v_\theta = 0, \quad T = 0 \tag{24}$$

$$v_\theta = r, \quad T = 0. \tag{25}$$

The process of the solution is summarized as follows:

- Continuity equation (1) is satisfied,
- The components of the rate-of-strain tensor all vanish except for the two components given below

$$\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} = r\frac{\partial}{\partial r}(\frac{v_\theta}{r}), \dot{\gamma}_{\theta z} = \dot{\gamma}_{z\theta} = \frac{\partial v_\theta}{\partial z}$$

- The operators defined in the equations (10) - (15) become

  $\Gamma = 1$, and $L_{r\theta} = \frac{\partial v_\theta}{\partial r}, L_{\theta r} = -\frac{v_\theta}{r}, L_{z\theta} = \frac{\partial v_\theta}{\partial z}$ (the other L's are "0")

- The stress components obtained from the constitutive equations are substituted into the momentum and energy equation that yield the following forms:

$$\frac{\partial p}{\partial r} = -\frac{2}{r} De_T \beta_T (\frac{\partial v_\theta}{\partial z})^2 \tag{26}$$

$$\frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\{(1-\beta_T)r^2[\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}]\} = 0 \tag{27}$$

$$\frac{\partial p}{\partial z} = 0 \tag{28}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{\partial^2 T}{\partial z^2} + Na\{(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r})^2 + (\frac{\partial v_\theta}{\partial z})^2\} = 0 \tag{29}$$

- The definition of velocity profile as $v_\theta = rw(z)$, where $w(z)$ is assumed to be nearly unaffected by temperature, and consideration of the order of magnitudes for the terms in energy equation gives the corresponding flow, stress and temperature field:

$$v_\theta = rz, \quad p = -De_T \beta_T r^2 \tag{30}$$

$$S_{rr} = S_{r\theta} = S_{rz} = S_{zz} = 0, \quad S_{\theta\theta} = 2De_T\beta_T r^2, \quad S_{\theta z} = -\beta_T r \tag{31}$$

$$T = \frac{1}{2}(Na)r^2 z(1-z) \tag{32}$$

Numerical example for a parallel plate rheometer

The solution obtained above is applied to a particular viscoelastic fluid (PIB/PB based Boger fluid with material properties of $\eta = 43\ Pas$, $\lambda = 1.8\ s$, $\beta = \eta_p/\eta_o = 0.35$ at $\tilde{T}_o = 295$ K). The radius and height of the rheometer are selected as $R = 20$ mm, and $H = 1$ mm, which are quite typical for a parallel-plate rheometer. For illustration purposes, "z" is kept constant ($z = 0.5$).

The radial profiles of temperature and normal stress in the 'θ' direction ($S_{\theta\theta}$) are given in Figures 2(a) and (b) as a function of rotation rate (Ω) or Deborah number, which is defined at $\tilde{T}_o = 295$K, $De = \lambda(\tilde{T}_o)\Omega$. The temperature increase due to viscous dissipation is negligible around the rotation center for all De values and also along the radial direction for small De values as illustrated in Figure 2(a). However, it increases significantly at higher De values, reaching almost 3K for $De = 5.4$ and 8K for $De = 9$ at the edge of the plates. The corresponding difference in fluid properties from the center to the edge is over 60% since a typical polymeric fluid such as the one used here is strongly temperature dependent. This feature of the fluid can be observed from the plot in Figure 2(b). Values of normal stresses with and without viscous heating effect greatly differ, especially for the edge and for increasing De values.

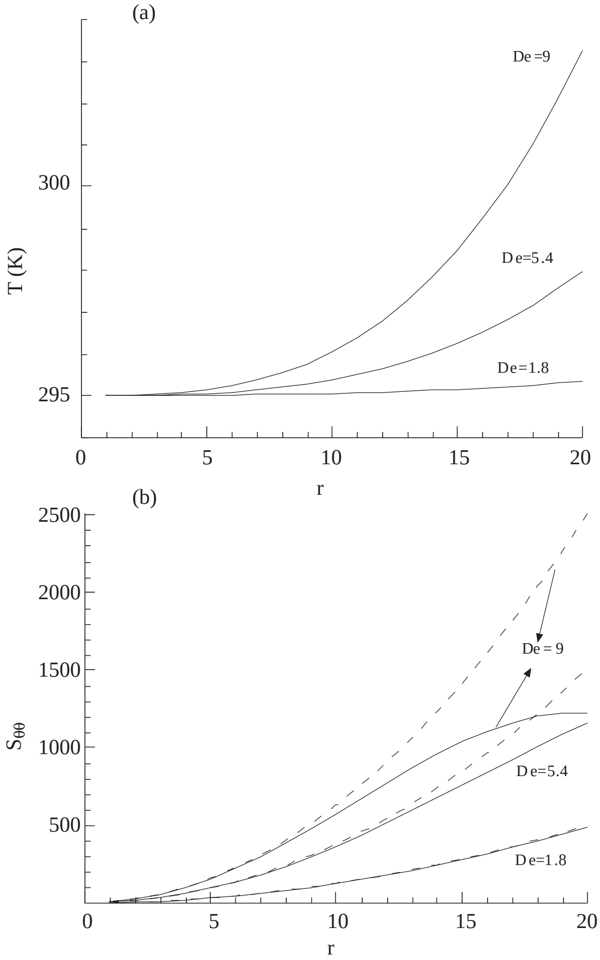

Figure 2. Radial profiles of (a) temperature and (b) normal stress in the 'θ' direction ($S_{\theta\theta}$). Plots are obtained for $\tilde{T}_o = 295$ K and $z = 0.5$. Solid and dashed lines are for stresses calculated with and without viscous dissipation effects respectively.

Proposed corrections in the current equations

The temperature gradient generated by viscous dissipation in the flow field seems significantly large for moderate and high rotation rates as shown in the previous section, and thus its effect on the prediction of material properties in parallel-plate rheometers should be considered. A practical way of accomplishing this is to modify some parameters in the current equations. The equations that are used to determine steady shear flow parameters of a viscoelastic fluid in a parallel-plate rheometer are given by Bird *et al.*, (1987) as follows:

$$\eta_o(\dot{\gamma}_R, \tilde{T}_o) = \frac{(M/2\pi R^3)}{\dot{\gamma}_R}[3 + \frac{d\ln(M/2\pi R^3)}{d\ln\dot{\gamma}_R}] \quad (33)$$

$$\Psi_1(\dot{\gamma}_R, \tilde{T}_o) - \Psi_2(\dot{\gamma}_R, \tilde{T}_o) = \frac{(\mathrm{N}/\pi R^2)}{\dot{\gamma}_R^2}[2 + \frac{d\ln(N/\pi R^2)}{d\ln\dot{\gamma}_R}] \quad (34)$$

$$\Psi_1(\dot{\gamma}_R, \tilde{T}_o) + \Psi_2(\dot{\gamma}_R, \tilde{T}_o) = \frac{1}{\dot{\gamma}_R^2}\frac{d(\Pi_{zz}(0))}{d\ln\dot{\gamma}_R} \quad (35)$$

$$\Psi_2(\dot{\gamma}_R, \tilde{T}_o) = \frac{P_a - d(\Pi_{zz}(R))}{\dot{\gamma}_R^2} \quad (36)$$

where M is torque, N is normal force required to keep separation constant, Ψ_1 and Ψ_2 are coefficients of first and second normal stresses for the fluid, $\dot{\gamma}_R$ is shear rate at the edge of the system, and Π_{zz} is normal pressure measured on the bottom plate. The shear rate at the edge is the reference parameter for measurements as can be seen from equations (33) through (36) and is defined as $\dot{\gamma}_R = \frac{\Omega R}{H}$. The reference temperature ($\tilde{T}_o$) in the equations is usually the temperature of the test environment or one of the plates, and the temperature rise in the flow field due to viscous dissipation is ignored. The dimensional form of equation (32) that describes local temperature in the rheometer is equal to

$$\tilde{T}(\tilde{r}, \tilde{z}) - \tilde{T}_o = \frac{\eta\Omega^2 R^2}{2kH^4}\tilde{r}^2(\tilde{z}H - \tilde{z}^2). \quad (37)$$

We define the arithmetic average of local temperatures in the bulk flow

$$\tilde{T}_0^* = \frac{\int_0^H\int_0^R \tilde{T}(\tilde{r}, \tilde{z})d\tilde{r}d\tilde{z}}{\int_0^H\int_0^R d\tilde{r}d\tilde{z}} \quad (38)$$

and propose a modified reference temperature

$$\tilde{T}_o^* = \tilde{T}_o\left\{1 + \frac{\eta\Omega^2 R^2\alpha^2}{36k\tilde{T}_0}\right\} \quad (39)$$

where $\alpha = R/H$. The value of any parameter on the left side of equations (33) through (36) indeed corresponds to the value at $\tilde{T}_o^*$, not at $\tilde{T}_o$.

Similar approach may be applied to modify reference (characteristic) shear rate, $\dot{\gamma}_R = \Omega\alpha$, on the left side of equations (33) through (36) for minimizing the free surface effects on measurements. The underlying assumption in these equations is no distortion at the free surface. In a real situation, however, there is surface deflection and it strongly affects the value of normal stress, N, in the equations, especially for small aspect ratios ($R/H < 10$). The explicit formula for calculating the axial surface deflection (see Shipman *et al.,* 1991)

$$\Delta R = \frac{\rho g H^3}{\sigma}\{\frac{1}{6}(z-\frac{1}{2})^3 - \frac{1}{24}(z)^2 + \frac{1}{48}\} \qquad (40)$$

can be averaged along the axial direction to redefine the reference shear rate in the equations above. The corrected reference shear rate $\dot{\gamma}_R^*$ given below may then be used instead of $\dot{\gamma}_R$ when deflections are significantly high

$$\dot{\gamma}_R^* = \frac{\Omega R^*}{H} = \dot{\gamma}_R\left\{1+\frac{7\rho g H^3}{44\sigma\Omega\alpha}\right\}. \qquad (41)$$

We are now able to replace these corrected (modified) reference values of temperature and shear rate with those on the left side of the equations (33) – (36). Namely, only ($\tilde{T}_o$,$\dot{\gamma}_R$) should be replaced with ($\tilde{T}_o^*$, $\dot{\gamma}_o^*$) and no change is necessary on the right side of the equations.

Summary and Concluding Remarks

The viscoelastic flow between two rotating parallel-plates has been studied in this paper. The motion generated in similar geometry is the basis for rheological measurements of the viscosity and normal stresses in non-Newtonian liquids. Similar flow geometry can also be found in polymer processing, such as a rotating normal stress extruder. In the most common case of flow, motion is generated by rotating one of the two coaxial disks as the other (usually the bottom plate) remains stationary. In rheological use of this geometry, the flow at all rotation rates is assumed to be steady, isothermal and one-dimensional. These approximations are valid with absence of any flow instability, free surface effect at the edge and temperature gradient due to viscous dissipation. Previous experimental and theoretical studies have, however, shown the presence of these three complicated phenomena and thus flow is proved to be time-dependent, three-dimensional, and nonisothermal at high rotation rates. In this paper, three-dimensional formulation of the flow, which considers free surface and viscous dissipation effects, is clearly outlined. The main contribution to the formulation is to use a nonisothermal constitutive equation, the nonisothermal Oldroyd-B equation. The numerical solution scheme of the resulting equations is found to be extremely complex. In order to explore whether temperature gradient is present in the flow due to viscous dissipation, the solution formulated for one-dimensional and nonisothermal flow between infinite plates is applied to a typical viscoelastic fluid (PIB/PB Boger fluid). From this illustrative example it is determined that the temperature gradient generated by viscous dissipation in the flow field is quite large for moderate and high rotation rates, and thus its effect on measured parameters should be considered. In order to minimize the errors in measurements due to viscous dissipation and free surface effects, modifications for reference temperature and shear rate are proposed here. These modifications allow better prediction and ensure that the effects of free surface and viscous dissipation taken into account. Further detailed experimental analysis of the flow is warranted to verify the validity of infinite plate solution obtained and the corrections proposed here.

Nomenclature

a_T	temperature shift factor
C	activation energy constant
c_P	heat capacity per unit mass
De	Deborah number
F	mean curvature of the surface
H	disk separation
k	thermal conductivity
M	torque
N	normal force
$\underline{n}$	unit vector normal to the surface (outward)
Na	Nahme-Griffith number
p	pressure
p_a	atmospheric pressure
Pe	Peclet number
R	disk-radius
r	radial coordinate
$\underline{\underline{S}}$	polymeric part of stress tensor
T	dimensionless temperature
$\tilde{T}_o$	reference temperature
$\tilde{T}_o^*$	modified reference temperature
T_a	atmospheric air temperature
$\underline{v}$	velocity vector
$\tilde{X}$	dimensional form of parameter X

X_T temperature dependent form of X
z axial coordinate
λ relaxation time of the fluid
θ azimuthal coordinate
Ω rotation rate
$\underline{\underline{\dot{\gamma}}}$ the rate-of-strain tensor
η total viscosity
η_p polymer viscosity
η_s solvent viscosity
$\Delta\tilde{T}$ rheological temperature difference
β the ratio of polymer viscosity to total viscosity
$\dot{\gamma}_R$ reference shear rate
$\dot{\gamma}_R^*$ modified reference shear rate
ρ density
σ surface tension
Ψ_1 coefficient of first normal stress
Ψ_2 coefficient of second normal stress
Π_{zz} normal pressure measured on the bottom plate

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