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Abstract

This paper studies robustness of a gradient-type CDMA uplink power control algorithm with respect to disturbances and time-delays. This problem is of practical importance because unmodeled secondary interference effects from neighboring cells play the role of disturbances, and propagation delays are ubiquitous in wireless data networks. We first show $L_{\mathbf{p}}$*-stability, for* $\mathbf{p} \in [1, \infty]$*, with respect to additive disturbances. We pursue* L_∞*-stability within the input-to-state stability (ISS) framework of Sontag [7], which makes explicit the vanishing effect of the initial conditions. Next, using the ISS property and a loop transformation, we prove that global asymptotic stability is preserved for sufficiently small time-delays in forward and return channels. For larger delays, we achieve global asymptotic stability by scaling down the step-size in the gradient algorithm.*

1. Introduction

In wireless communication networks, power must be regulated to maintain a satisfactory quality of service for users, and power control has been a significant research topic [1, 2, 3, 4, 5]. Increased power ensures longer transmission distance and higher data transfer rate, but it also consumes battery and produces greater amount of interference to neighboring users. In code division multiple access (CDMA) systems, this problem has been studied as an optimization problem, where the i^{th} user minimizes its power p_i, while maximizing the signal-to-interference ratio (SIR) at the base station,

$$\gamma_i(p) := \frac{Lh_i p_i}{\sum\limits_{k \neq i} h_i p_k + \sigma^2}, \tag{1}$$

where L is the spreading gain of the CDMA system, h_i is the channel gain between the i^{th} mobile and the base station, and σ^2 is the noise variance containing the contribution of the secondary background interference. To regulate the power of each user, Deb *et al.* [1], Zander [2], and Yates [3], pose the constrained optimization problem,

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$$\min p_i \qquad \text{subject to } \gamma_i(p) \geq \gamma_i^{tar}, \tag{2}$$

where γ_i^{tar} is a threshold chosen to ensure adequate quality of service. An alternative noncooperative game-theoretic formulation is given by Alpcan *et al.* [5], [4], in which each user tries to maximize

$$\max_i \; J_i = U_i(\gamma_i(p)) - P_i(p_i), \tag{3}$$

where U_i is a utility function for the i^{th} user, which represents the demand for bandwidth, and P_i represents the cost of power. The authors then propose the gradient-type power control law

$$\dot{p}_i = -\lambda_i \frac{\partial J_i}{\partial p_i} = \frac{dU_i}{d\gamma_i} \frac{L\lambda_i h_i}{\sum\limits_{k \neq i} h_k p_k + \sigma^2} - \lambda_i \frac{dP_i(p_i)}{dp_i}, \quad \lambda_i > 0, \tag{4}$$

and prove asymptotic stability of the Nash equilibrium under several assumptions on the functions $U_i(\cdot)$ and $P_i(\cdot)$, and on the number of users.

In this paper, we study the robustness of this control law against additive disturbances and time-delays. This study is important because of modelling errors, power noise, secondary interference effects, such as those from neighboring cells, and propagation delays. Our starting point is a passivity-based stability proof for the algorithm (4), presented in our recent paper [6]. Using the Lyapunov functions obtained from this passivity analysis, in this paper we first show that the controller (4) is robust to additive $L_\mathbf{p}$-disturbances. In particular, L_∞-disturbances are pursued here within the input-to state stability (ISS) framework of Sontag [7], which makes explicit the vanishing effect of initial conditions. We then proceed to the study of delays using this ISS property. We first represent the delayed system as a feedback interconnection of the nominal delay-free model, and a perturbation block, the ISS-gain of which depends on the amount of delay. Then we prove global asymptotic stability (GAS) for sufficiently small delays using the ISS Small-Gain Theorem of Teel *et al.* [8], [9]. For larger delays, we achieve GAS by scaling down the stepsize λ_i.

The paper is organized as follows. Section 2 reviews the first-order gradient power control algorithm and proves an $L_\mathbf{p}$-stability property with respect to additive disturbances. Section 3 derives bounds for time-delays that the system can tolerate without losing stability. For larger delays, it proposes a scaling of the step-size λ_i in (4). Conclusions are given in Section 4. Throughout the paper, we will use *projection functions* to ensure nonnegative values for physical quantities, such as power. Given a function $f(x)$, its positive projection is defined as

$$(f(x))_x^+ := \begin{cases} f(x) & \text{if } x > 0, \text{ or } x = 0 \text{ and } f(x) \geq 0 \\ 0 & \text{if } x = 0 \text{ and } f(x) < 0. \end{cases}$$

If x and $f(x)$ are vectors, then $(f(x))_x^+$ is interpreted in the component-wise sense. When $(f(x))_x^+ = 0$, we say that the projection is *active*. When $(f(x))_x^+ = f(x)$, we say that the projection is *inactive*. We denote by $\|x\|$ the vector norm of x, and by $\|x\|_{L_\mathbf{p}}$ the $L_\mathbf{p}$-norm of $x(t)$, $p \in [1, \infty]$. For $d \in L_\infty$, we define $\|d\|_a = \lim\limits_{t \to \infty} \sup \|d(t)\|$. A system $\dot{x} = f(x, u)$ is said to be *input-to state stable* (ISS) if there exist class-K functions[1] $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any input $u(\cdot) \in L_\infty^m$ and $x_0 \in R^n$, the response $x(t)$ from the initial state $x(0) = x_0$ satisfies

$$\|x\|_{L_\infty} \leq \gamma_0(\|x_0\|) + \gamma\left(\|u\|_{L_\infty}\right), \quad \|x\|_a \leq \gamma(\|u\|_a).$$

2. Robustness to Disturbances

We first review the stability properties of the gradient-type power control law (4). As shown in Alpcan *et al.* [4], [5], the following assumption ensures that a unique Nash equilibrium p^* exists for the game (3).

Standing Assumption: *The function $P_i(\cdot)$ in (3) is twice continuously differentiable, nondecreasing, and strictly convex in p_i, i.e.,*

$$\frac{\partial P_i(p_i)}{\partial p_i} \geq 0, \quad \frac{\partial^2 P_i(p_i)}{\partial p_i^2} > 0, \quad \forall p_i, \tag{5}$$

and

$$U_i(\gamma_i) = u_i \log(\gamma_i + L), \tag{6}$$

where u_i is a constant, and γ_i and L are as in (1).

The choice of the logarithmic utility function in (6) is meaningful because it represents the maximum achievable bandwidth as in Shannon's Theorem [10]. Substituting this $U_i(\gamma_i)$ in (4) and adding projection $(\cdot)_{p_i}^+$ to ensure positivity of p_i, we obtain

$$n\dot{p}_i = \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + \frac{u_i \lambda_i h_i}{\sum_k h_k p_k + \sigma^2}\right)_{p_i}^+. \tag{7}$$

Note that in this derivation, the term $\sum_{k \neq i} h_k p_k + \sigma^2$ in (4) has been cancelled by the derivative of the logarithmic U_i, and replaced by $\sum_k h_k p_k + \sigma^2$. This means that we can represent (7) as in Figure 1, in which the diagonal entries Σ_i of the forward block are given by

$$\Sigma_i: \qquad \dot{p}_i = \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i \lambda_i w_i\right)_{p_i}^+, \quad i = 1, \cdots, M, \tag{8}$$

where

$$w := -h \cdot q, \tag{9}$$

[1] A function $\gamma(\cdot)$ is defined to be class-K if it is continuous, zero at zero, and strictly increasing.

$$h := \begin{bmatrix} h_1 & h_2 & \cdots & h_M \end{bmatrix}^T, \tag{10}$$

$$q := \varphi(y) = -\frac{1}{y+\sigma^2}, \tag{11}$$

$$y := h^T p. \tag{12}$$

In this representation the forward block corresponds to the mobiles and the feedback path corresponds to the base station. Stability of the equilibrium p^* is proved in [6], using passivity properties of both the feedforward and feedback paths:

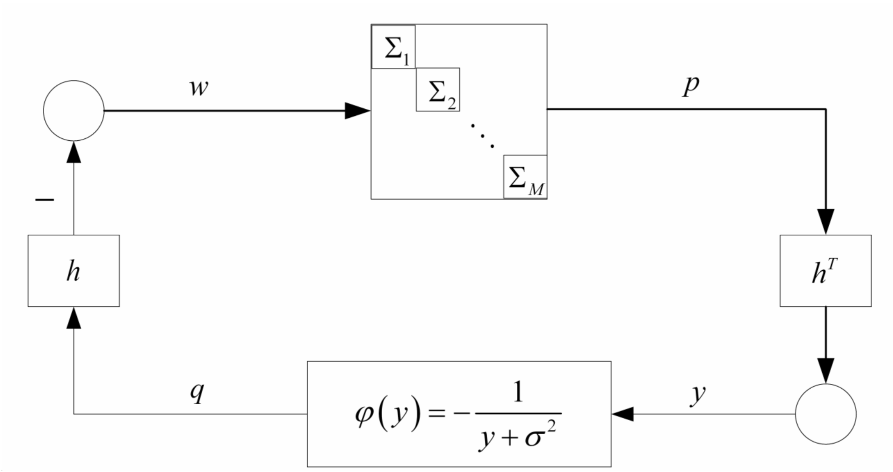

Figure 1. First-order gradient algorithm of CDMA power control.

Proposition 1 *The equilibrium $p = p^*$ of the feedback system (8)-(12), represented as in Figure 1, is globally asymptotically stable.*

With this representation, we are now ready to show $L_\mathbf{p}$ and input-to-state stability of the power control algorithm (7) with respect to additive disturbances, such as secondary interference effects from neighboring cells. Denoting by d_{1i} and d_{2i} disturbances acting on the i^{th} mobile, we replace (7) with the perturbed model,

$$\dot{p}_i = \left(\frac{u_i \lambda_i h_i}{\sum_k h_k p_k + d_{2i} + \sigma^2} - \lambda_i \frac{dP_i(p_i)}{dp_i} + d_{1i} \right)^+_{p_i}, \tag{13}$$

and prove an $L_\mathbf{p}$-stability property ($\mathbf{p} \in [1, \infty]$):

Theorem 1 *Consider the power control system (13), where $P_i(p_i)$ satisfies, for all $p_i \geq 0$, $i = 1, 2, \cdots M$,*

$$\frac{\partial^2 P_i(p_i)}{\partial p_i^2} \geq \eta$$

where η is a positive constant. If $d_1 = [d_{11}, d_{12}, \cdots d_{1M}]$ and $d_2 = [d_{21}, d_{22}, \cdots d_{2M}]$ are $L_{\mathbf{p}}$-disturbances, $\mathbf{p} = [1, \infty)$, then (13) guarantees

$$\|p - p^*\|_{L_{\mathbf{p}}} \leq \bar{u}\bar{\lambda}(\alpha p)^{-\frac{1}{p}} \sqrt{\sum_i \frac{1}{u_i\lambda_i}(p_i(0) - p_i^*)^2} + \sqrt{2}\bar{u}\bar{\lambda}(\alpha_1 q)^{-\frac{1}{q}} \|\beta\|_{L_{\mathbf{p}}} \tag{14}$$

where

$$\alpha = \frac{\underline{u}\underline{\lambda}\eta}{\bar{u}}, \quad \beta = \frac{\bar{u}\bar{\lambda}}{\sqrt{2}\underline{u}\underline{\lambda}}\|d_1\| + \frac{\bar{u}\bar{\lambda}\bar{h}}{\sqrt{2}\sigma^4}\|d_2\| \tag{15}$$

$$\bar{u} = \max_i\{u_i\}, \ \underline{u} = \min_i\{u_i\}, \ \bar{\lambda} = \max_i\{\lambda_i\}, \ \underline{\lambda} = \min_i\{\lambda_i\}, \ \bar{h} = \max_i\{h_i\}, \ \underline{h} = \min_i\{h_i\} \tag{16}$$

and $\mathbf{q}$ and $\mathbf{p}$ are complementary indices, that is

$$\mathbf{p}^{-1} + \mathbf{q}^{-1} = 1. \tag{17}$$

When $\mathbf{p} = \infty$, the system satisfies the ISS estimate

$$\|p - p^*\| \leq \bar{u}\bar{\lambda}e^{-\alpha t}\sqrt{\sum_i \frac{1}{u_i\lambda_i}(p_i(0) - p_i^*)^2} + \frac{\sqrt{2}\bar{u}\bar{\lambda}}{\alpha}\|\beta_1\|_{L_\infty}. \tag{18}$$

Proof: The derivative of the storage function

$$V_1(p - p^*) = \frac{1}{2}\sum_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)^2 \tag{19}$$

along the solution of (13) is

$$\dot{V}_1 = \sum_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)\left(-\lambda_i\frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i + d_{1i}\right)^+_{p_i}. \tag{20}$$

We first note that

$$\frac{1}{u_i\lambda_i}(p_i - p_i^*)\left(-\lambda_i\frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i + d_{1i}\right)^+_{p_i} \leq \frac{1}{u_i\lambda_i}(p_i - p_i^*)\left(-\lambda_i\frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i + d_{1i}\right),$$

which follows because, if the projection is inactive then both sides of the inequality are equal, and if the projection is active, $p_i = 0$ and $-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i + d_{1i} < 0$, then the left hand side is zero, and the right hand side is non-negative. By adding and subtracting $u_i\lambda_i w_i^*$ and $\frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2}$, we obtain

$$\begin{aligned}
\dot{V} \leq& \sum_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)\left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + \underbrace{u_i\lambda_i w_i^*}_{\lambda_i \frac{dP_i(p_i^*)}{dp_i}} - \underbrace{u_i\lambda_i w_i^*}_{\frac{u_i\lambda_i h_i}{\sum_k h_k p_k^* + \sigma^2}} + \frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2} - \frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2} + u_i\lambda_i w_i + d_{1i}\right) \\
=& \sum_i \frac{(p_i - p_i^*)}{u_i}\left(-\frac{dP_i(p_i)}{dp_i} + \frac{dP_i(p_i^*)}{dp_i}\right) + \sum_i \frac{(p_i - p_i^*)}{u_i\lambda_i}\left(\frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2} - \frac{u_i\lambda_i h_i}{\sum_k h_k p_k^* + \sigma^2}\right) \\
&+ \sum_i \frac{(p_i - p_i^*)}{u_i\lambda_i}\left(\frac{u_i\lambda_i h_i}{\sum_k h_k p_k + d_{2i} + \sigma^2} - \frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2}\right) + \sum_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)\, d_{1i} \\
=& \sum_i \frac{(p_i - p_i^*)}{u_i}\left(-\frac{dP_i(p_i)}{dp_i} + \frac{dP_i(p_i^*)}{dp_i}\right) + \left(\frac{1}{y + \sigma^2} - \frac{1}{y^* + \sigma^2}\right)(y - y^*) \\
&+ \sum_i \left(\frac{1}{\sum_k h_k p_k + d_{2i} + \sigma^2} - \frac{1}{\sum_k h_k p_k + \sigma^2}\right) h_i (p_i - p_i^*) + \sum_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)\, d_{1i}.
\end{aligned} \tag{21}$$

Since $\left(\frac{1}{y+\sigma^2} - \frac{1}{y^*+\sigma^2}\right)(y - y^*) \leq 0$ and $P_i'' \geq \eta$, we obtain

$$\begin{aligned}
\dot{V} &\leq \sum_i -\frac{\eta}{u_i}(p_i - p_i^*)^2 + \sum_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)\, d_{1i} + \sum_i h_i \left|\frac{1}{y + d_{2i} + \sigma^2} - \frac{1}{y + \sigma^2}\right| |p_i - p_i^*| \\
&\leq -\frac{\eta}{\bar{u}}\|p - p^*\|^2 + \frac{1}{\underline{u\lambda}}\|p - p^*\|\,\|d_1\| + \sum_i \frac{h_i}{\sigma^4}|d_{2i}|\,|p_i - p_i^*| \\
&\leq -2\frac{\underline{u\lambda}\eta}{\bar{u}} V + \sqrt{2}\frac{\bar{u}\bar{\lambda}}{\underline{u\lambda}}\sqrt{V}\,\|d_1\| + \sqrt{2}\frac{\bar{u}\bar{\lambda}\bar{h}}{\sigma^4}\sqrt{V}\,\|d_2\| \\
&\leq -2\alpha V + 2\beta\sqrt{V}
\end{aligned}$$

which, from [11, Theorem 6.1], implies that

$$\left\|\sqrt{V}\right\|_{L_{\mathbf{p}}} \leq (\alpha \mathbf{p})^{-\frac{1}{\mathbf{p}}}\left\|\sqrt{V(0)}\right\| + (\alpha_1 \mathbf{q})^{-\frac{1}{\mathbf{q}}}\|\beta\|_{L_{\mathbf{p}}}, \tag{22}$$

and

$$\left\|\sqrt{V}\right\| \leq e^{-\alpha t}\left\|\sqrt{V(0)}\right\| + \frac{1}{\alpha}\|\beta\|_{L_\infty}. \tag{23}$$

Inequality (14) and (18) then follows from (22), (23), and

$$\|p - p^*\| \leq \sqrt{2}\bar{u}\bar{\lambda}\,\|W(t)\|.$$

□

3. Robustness to Time-Delays

We now prove that global asymptotic stability is preserved for sufficiently small time-delays between mobiles and the base station. This study is important because wireless data networks may exhibit significant propagation delays. Denoting by τ_i the round-trip delay for the i^{th} mobile, we represent the algorithm (8)-(12) as in Figure 2:

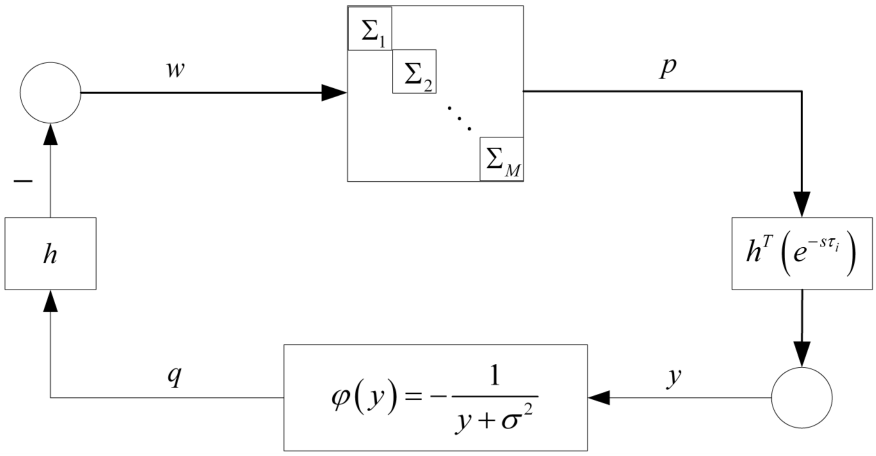

Figure 2. First-order gradient algorithm of CDMA power control in the case of time-delay.

where $h^T\left(e^{-s\tau_i}\right) := \left[\begin{array}{cccc} h_1 e^{-s\tau_1} & h_2 e^{-s\tau_2} & \cdots & h_M e^{-s\tau_M} \end{array}\right]$. To transform the delay robustness problem to the framework of Theorem 1, we add and subtract the term h^T from $h^T\left(e^{-s\tau_i}\right)$ in Figure 1, and represent it as in Figure 3, where the inner loop represents the nominal system without delay, and the outer loop is the perturbation due to delay.

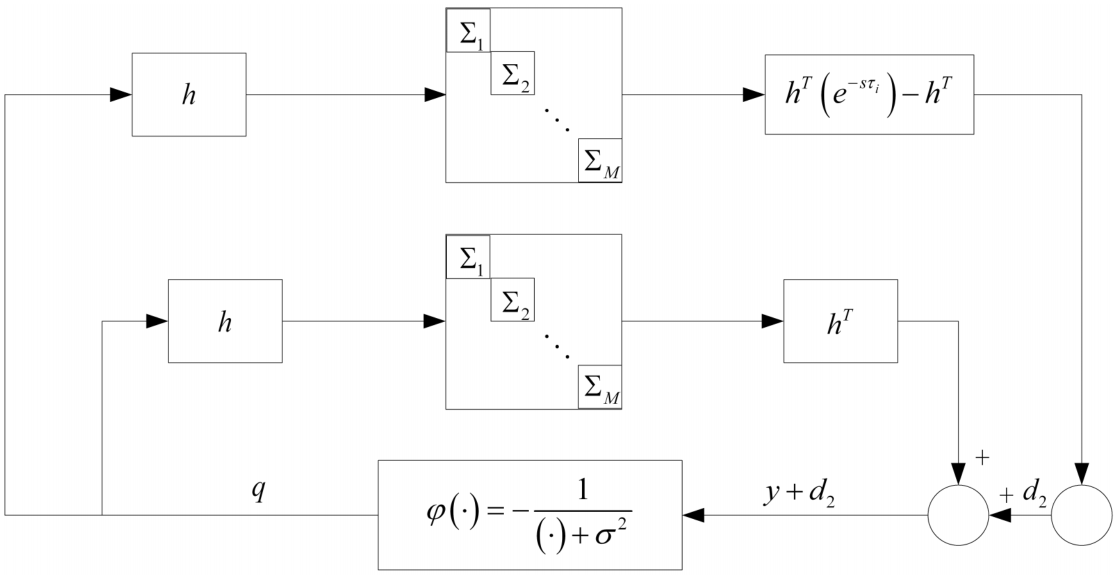

Figure 3. Equivalent system of gradient algorithm of CDMA power control after loop-transformation.

With this representation we prove stability using a small-gain argument. From Theorem 1, it is not difficult to show that the ISS gain of the feedback path from d_2 to $q - q^*$ is

$$g_1 = \frac{\|h\|}{\sigma^4}\left(\frac{\sqrt{2}\bar{u}\bar{\lambda}}{\alpha}\frac{\bar{u}\bar{\lambda}\bar{h}}{\sqrt{2}\sigma^4} + \frac{1}{\sigma^4}\right). \tag{24}$$

In Theorem 2 below, we also show that the feedforward path from $q-q^*$ to d_2 has gain

$$g_2 = \sqrt{2M\bar{h}}\bar{\tau}\left(\frac{\eta_1\bar{u}^2\bar{\lambda}^2\|R\|}{\underline{u\lambda}\eta_2} + \bar{u}\bar{\lambda}\bar{h}\right) \tag{25}$$

where η_1 and η_2 are positive constants defined in (28) and,

$$\bar{\tau} := \max_i\{\tau_i\}. \tag{26}$$

This means that for sufficiently small $\bar{\tau}$, the small-gain condition

$$g_1 g_2 < 1 \tag{27}$$

holds and GAS is preserved. If $\bar{\tau}$ is not sufficiently small, then we can scale down the stepsize λ_i in the power control (10) to recover GAS:

Theorem 2 *Consider the feedback interconnection in Figure 3, and suppose that* $P_i(p_i)$, $i = 1, 2, \cdots M$, *are such that for all* $p_i \geq 0$,

$$\eta_1 \geq P_i''(p_i) \geq \eta_2 \tag{28}$$

with $\eta_1 > \eta_2 > 0$. *If either the delay* $\bar{\tau}$ *or the stepsize* λ_i *is small enough that (27) is satisfied, then the power control scheme (8)-(12) guarantees global asymptotic stability.*

Proof: We prove the theorem in three steps. In the first step we give the gain from $q-q^*$ to $\dot{p}$ in the feedforward path, and in the second step the gain from $\dot{p}$ to d_2 in the forward path. By these two steps we show that the feedforward path in Figure 3 has gain g_2 as in (25). Then in the third step, we show that the feedback path has a complementary gain g_1 as in (24), and using the Small-Gain theorem, we get the conclusion.

Step 1: We let

$$V_1(p-p^*) = \frac{1}{2}\sum_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)^2$$

as in (19). Following the same steps as (21), we obtain

$$\begin{aligned}\dot{V} &\leq \sum_i \tfrac{1}{u_i}(p_i - p_i^*)\left(-\tfrac{dP_i(p_i)}{dp_i} + \tfrac{dP_i(p_i^*)}{dp_i}\right) + (q-q^*)(y-y^*)\\ &\leq -\tfrac{\eta_2}{\bar{u}}\|p-p^*\|^2 + \|q-q^*\|\,\|R\|\,\|p-p^*\|\\ &\leq -2\tfrac{\underline{u\lambda}\eta_2}{\bar{u}}V + \sqrt{2\bar{u}\bar{\lambda}}\,\|q-q^*\|\,\|R\|\sqrt{V}.\end{aligned}$$

From [11, Theorem 6.1], we have

$$\left\|\sqrt{V(t)}\right\| \le e^{-\frac{\underline{u}\lambda\eta_2}{\bar{u}}t}\left\|\sqrt{V(0)}\right\| + \frac{\bar{u}\sqrt{\bar{u}\bar{\lambda}}\,\|h\|}{\sqrt{2}\underline{u}\underline{\lambda}\eta_2}\|q-q^*\|_{L_\infty}$$

which, with $\|p(t)-p^*\| \le \sqrt{2\bar{u}\bar{\lambda}}\left\|\sqrt{V(t)}\right\|$, yields

$$\|p(t)-p^*\|_{L_\infty} \le \frac{\sqrt{\bar{u}\bar{\lambda}}}{\sqrt{\underline{u}\underline{\lambda}}}\|p(0)-p^*\| + \frac{\bar{u}^2\bar{\lambda}\,\|R\|}{\underline{u}\underline{\lambda}\eta_2}\|q-q^*\|_{L_\infty} \tag{29}$$

$$\|p(t)-p^*\|_a \le \frac{\bar{u}^2\bar{\lambda}\,\|R\|}{\underline{u}\underline{\lambda}\eta_2}\|q-q^*\|_a. \tag{30}$$

Next, because $\left|\left(-\lambda_i\frac{dP_i(p_i)}{dp_i}+u_i\lambda_i w_i\right)^+_{x_i}\right| \le \left|-\lambda_i\frac{dP_i(p_i)}{dp_i}+u_i\lambda_i w_i\right|$,

$$\|\dot{p}_i\| \le \left\|-\lambda_i\frac{dP_i(p_i)}{dp_i}+\lambda_i\frac{dP_i(p_i^*)}{dp_i}+u_i\lambda_i w_i^*-u_i\lambda_i w_i\right\| \le \left\|-\lambda_i\frac{dP_i(p_i)}{dp_i}+\lambda_i\frac{dP_i(p_i^*)}{dp_i}\right\|+\|u_i\lambda_i h_i q^*-u_i\lambda_i h_i q\|.$$

Thus, from (28), we obtain

$$\|\dot{p}\| \le \bar{\lambda}\eta_1\|p-p^*\| + \bar{u}\bar{\lambda}\bar{h}\|q-q^*\|,$$

which implies, from (29) and (30)

$$\|\dot{p}\|_{L_\infty} \le \frac{\bar{\lambda}\eta_1\sqrt{\bar{u}\bar{\lambda}}}{\sqrt{\underline{u}\underline{\lambda}}}\|p(0)-p^*\| + \left(\frac{\eta_1\bar{u}^2\bar{\lambda}^2\,\|R\|}{\underline{u}\underline{\lambda}\eta_2}+\bar{u}\bar{\lambda}\bar{h}\right)\|q-q^*\|_{L_\infty}, \tag{31}$$

$$\|\dot{p}(t)\|_a \le \left(\frac{\eta_1\bar{u}^2\bar{\lambda}^2\,\|R\|}{\underline{u}\underline{\lambda}\eta_2}+\bar{u}\bar{\lambda}\bar{h}\right)\|q-q^*\|_a. \tag{32}$$

Step 2: Next, we claim that the subsystem from $\dot{p}$ to d_2 satisfies

$$\|d_2\|_a \le \sqrt{2M\bar{h}}\bar{\tau}\,\|\dot{p}(t)\|_a, \tag{33}$$

$$\|d_2\|_{L_\infty} \le \sqrt{2M\bar{h}}\bar{\tau}\left(\|\dot{p}\|_{L_\infty} + \sup_{-\bar{\tau}<t\le 0}\left\|-\lambda\frac{dP(p(t))}{dp} - diag\left\{\begin{array}{ccc} u_1\lambda_1 & \cdots & u_M\lambda_M\end{array}\right\}w(t)\right\|\right). \tag{34}$$

To prove this, we first note that

$$\begin{aligned}|d_2(t)| &= \left|\sum_{i=1}^M h_i p_i(t-\tau_i) - \sum_{i=1}^M h_i p_i(t)\right| \le \sum_{i=1}^M h_i\int_{t-\tau_i}^t |\dot{p}_i(\sigma)|\,d\sigma \\ &\le \sum_{i=1}^M h_i\int_{\max\{0,t-\tau_i\}}^t |\dot{p}_i(\sigma)|\,d\sigma + \sum_{i=1}^M h_i\int_{\min\{0,t-\tau_i\}}^0 |\dot{p}_i(\sigma)|\,d\sigma\end{aligned}$$

which implies by Young's Inequality

$$\begin{aligned}
|d_2(t)|^2 &\le 2\left(\sum_{i=1}^{M} h_i \int_{\max\{0,t-\tau_i\}}^{t} |\dot{p}_i(\sigma)|\, d\sigma\right)^2 + 2\left(\sum_{i=1}^{M} h_i \int_{\min\{0,t-\tau_i\}}^{0} |\dot{p}_i(\sigma)|\, d\sigma\right)^2 \\
&\le 2M\sum_{i=1}^{M}\left(h_i \int_{\max\{0,t-\tau_i\}}^{t} |\dot{p}_i(\sigma)|\, d\sigma\right)^2 + 2M\sum_{i=1}^{M}\left(h_i \int_{\min\{0,t-\tau_i\}}^{0} |\dot{p}_i(\sigma)|\, d\sigma\right)^2 \\
&\le 2M\bar{h}\sum_{i=1}^{M}\left(\int_{\max\{0,t-\tau_i\}}^{t} |\dot{p}_i(\sigma)|\, d\sigma\right)^2 + 2M\bar{h}\sum_{i=1}^{M}\left(\int_{\min\{0,t-\tau_i\}}^{0} |\dot{p}_i(\sigma)|\, d\sigma\right)^2.
\end{aligned}$$

Applying Cauchy-Schwarz inequality to each term, we get

$$|d_2(t)|^2 \le 2M\bar{h}\bar{\tau}\sum_{i=1}^{M}\int_{\max\{0,t-\tau_i\}}^{t} |\dot{p}_i(\sigma)|^2\, d\sigma + 2M\bar{h}\bar{\tau}\sum_{i=1}^{M}\int_{\min\{0,t-\tau_i\}}^{0} |\dot{p}_i(\sigma)|^2\, d\sigma$$

which implies that the vector norm of d_2 satisfies

$$\begin{aligned}
\|d_2\| &\le \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^{M}\int_{\max\{0,t-\tau_i\}}^{t} |\dot{p}_i(\sigma)|^2\, d\sigma + 2M\bar{h}\bar{\tau}\sum_{i=1}^{M}\int_{\min\{0,t-\tau_i\}}^{0} |\dot{p}_i(\sigma)|^2\, d\sigma} \\
&\le \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^{M}\int_{\max\{0,t-\tau_i\}}^{t} |\dot{p}_i(\sigma)|^2\, d\sigma} + \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^{M}\int_{\min\{0,t-\tau_i\}}^{0} |\dot{p}_i(\sigma)|^2\, d\sigma}.
\end{aligned}$$

Because $\max\{0, t-\tau_i\} \ge \max\{0, t-\bar{\tau}\}$ and $\min\{0, t-\tau_i\} \ge \min\{0, t-\bar{\tau}\}$, we get

$$\|d_2\| \le \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^{M}\int_{\max\{0,t-\bar{\tau}\}}^{t} |\dot{p}_i(\sigma)|^2\, d\sigma} + \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^{M}\int_{\min\{0,t-\bar{\tau}\}}^{0} |\dot{p}_i(\sigma)|^2\, d\sigma}.$$

By changing the sequence of the sum and integral, we obtain

$$\begin{aligned}
\|d_2\| &\le \sqrt{2M\bar{h}\bar{\tau}\int_{\max\{0,t-\bar{\tau}\}}^{t}\sum_{i=1}^{M} |\dot{p}_i(\sigma)|^2 d\sigma} + \sqrt{2M\bar{h}\bar{\tau}\int_{\min\{0,t-\bar{\tau}\}}^{0}\sum_{i=1}^{M} |\dot{p}_i(\sigma)|^2 d\sigma} \\
&\le \sqrt{2M\bar{h}\bar{\tau}^2 \|\dot{p}(\sigma)\|_{L_\infty}^2} + \sqrt{2M\bar{h}\bar{\tau}\int_{\min\{0,t-\bar{\tau}\}}^{0} \|\dot{p}(\sigma)\|^2\, d\sigma},
\end{aligned}$$

from which (33) and (34) follows.

Combining (31)- (32) and (33)-(34) from Steps 1 and 2, we conclude that the L_∞-gain and asymptotic gain of the feedforward path are:

$$\|d_2\|_a \le g_2 \|q - q^*\|_a\,, \tag{35}$$

$$\begin{aligned}
\|d_2\|_{L_\infty} &\le g_2 \|q - q^*\|_{L_\infty} + \sqrt{2M\bar{h}}\bar{\tau}\tfrac{\bar{\lambda}\eta_1\sqrt{\bar{u}\bar{\lambda}}}{\sqrt{\underline{u}\underline{\lambda}}}\|\tilde{p}(0)\| \\
&\quad + \sup_{-\bar{\tau}<t\le 0}\left\| -\lambda\tfrac{dP(p(t))}{dp} - diag\left\{\begin{array}{ccc} u_1\lambda_1 & \cdots & u_M\lambda_M \end{array}\right\} w(t)\right\|.
\end{aligned} \tag{36}$$

where g_2 is as in (25).

Step 3: Finally, we show that the feedback path has a complementary gain g_1 as in (24). For the inner loop in Figure 3, it follows from Theorem 1 that

$$\|q-q^*\| = \left\| \frac{1}{y+d_2+\sigma^2} - \frac{1}{y^*+\sigma^2} \right\| \le \frac{1}{\sigma^4}\|y-y^*+d_2\| \le \frac{\|h\|}{\sigma^4}\|p-p^*\| + \frac{1}{\sigma^4}\|d_2\|$$

and, thus

$$\|q-q^*\|_a \le g_1 \|d_2\|_a \,, \tag{37}$$

$$\|q-q^*\|_{L_\infty} \le \frac{\|h\|}{\sigma^4}\bar{u}\bar{\lambda}e^{-\alpha t}\sqrt{\sum_i \frac{1}{u_i\lambda_i}\left(p\left(0\right)-p^*\right)^2} + g_1\|d_2\|_{L_\infty} . \tag{38}$$

Substituting (37) and (38) into (35) and(36), and using the small-gain condition (27), we conclude

$$\|d_2\|_a \le 0, \tag{39}$$

$$\|d_2\|_{L_\infty} \le \frac{\frac{\|h\|}{\sigma^4}g_2\|p\left(0\right)-p^*\| + \frac{\sqrt{2M\bar{h}\bar{u}\bar{\lambda}}\bar{\lambda}\eta_1}{\sqrt{\underline{u\lambda}}}\bar{\tau}\|p\left(0\right)-p^*\| + \sup\limits_{-\bar{\tau}<t\le 0}\left\|-\lambda\frac{dP(p(t))}{dp} - q\left(t\right)\right\|}{1-g_1g_2}. \tag{40}$$

Finally, from Theorem 1, we get

$$\|p-p^*\|_a \le 0 \tag{41}$$

$$\begin{aligned}\|p-p^*\|_{L_\infty} \le & \frac{(1-g_1g_2)\|p(0)-p^*\| + \frac{\sqrt{2}\bar{u}^2\bar{\lambda}^2\bar{h}\|h\|}{\sqrt{2}\alpha\sigma^8}g_2\|p(0)-p^*\|}{1-g_1g_2} \\ & + \frac{\frac{\sqrt{2}\bar{u}^2\bar{\lambda}^3\bar{h}}{\sqrt{2}\sigma^4\alpha}\frac{\sqrt{2M\bar{h}\bar{u}\bar{\lambda}}\eta_1}{\sqrt{\underline{u\lambda}}}\bar{\tau}\|p(0)-p^*\| + \frac{\sqrt{2}\bar{u}^2\bar{\lambda}^2\bar{h}}{\sqrt{2}\sigma^4\alpha}\sup\limits_{-\bar{\tau}<t\le 0}\left\|-\lambda\frac{dP(p(t))}{dp} - q(t)\right\|}{1-g_1g_2},\end{aligned} \tag{42}$$

which proves global asymptotic stability as defined in [12].

If the small-gain condition violates (27), then we can scale down the user-dependent stepsize λ_i by $\kappa > 0$, and rewrite (27) as

$$\frac{\|h\|}{\sigma^4}\left(\frac{\kappa^2\sqrt{2}\bar{u}^2\bar{\lambda}^2\bar{h}}{\sqrt{2}\sigma^4\alpha} + \frac{1}{\sigma^4}\right)\sqrt{2M\bar{h}}\bar{\tau}\kappa\left(\frac{\eta_1\bar{u}^2\bar{\lambda}\|h\|}{\underline{u}\eta_2} + \bar{u}\bar{h}\right) < 1 \tag{43}$$

which is satisfied for sufficiently small κ. Thus, for any delay $\bar{\tau}$, the scaled controller

$$\dot{p}_i = \frac{u_i\kappa\lambda_i h_i}{\sum\limits_k h_k p_k + \sigma^2} - \kappa\lambda_i\frac{dP_i\left(p_i\right)}{dp_i}, \tag{44}$$

where κ is as (43), achieves GAS. □

4. Conclusion

We have addressed robustness of the first-order gradient power control algorithm in [5] against disturbances and time-delay. Using an ISS property of the nominal, delay-free, system, and a small-gain argument, we showed that global asymptotic stability is preserved in the presence of small time-delays. For larger delays, we achieved GAS by scaling down the step-size of the gradient algorithm. One shortcoming of reducing the gains, however, is that it may cause degradation in performance. To avoid this degradation, we are currently studying dynamic redesigns which employ lead filters to counteract delays. The design of such filters can be pursued within the passivity framework of [6].

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