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The 2-adic valuation of shifted Padovan and Perrin numbers and applications

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Abstract: We characterize the 2-adic valuation of $(P_n - 1)_{n\geq 0}$, where $(P_n)_{n\geq 0}$ denotes the Padovan sequence. In addition, we use this formula to find all the Cullen and Proth numbers that are Padovan numbers. We also fully describe the 2-adic order of $(R_n + 1)_{n\geq 0}$, where $(R_n)_{n\geq 0}$ denotes the Perrin sequence, and use it to find all Woodall and Proth numbers of the second kind which are Perrin numbers. As a consequence we find that 3, 5, 9, and 65 are the only Fermat numbers in the Padovan sequence; while 3 and 7 and 2 and 5 are the only numbers of Mersenne and Thâbit ibn Kurrah in the Perrin sequence respectively.

Key words: 2-adic valuation, Cullen number, Fermat number, mersenne number, Padovan number, Perrin number, Proth number, thâbit ibn Kurrah number, Woodall number

1. Introduction

Let p be a prime number. The p-adic valuation of an integer n is the exponent of the highest power of p that divides n. We denote it by $\upsilon_p(n)$. In recent years, the p-adic valuation of members of ternary linear recursive sequences has been studied. In 2014, Lengyel and Marques [19] completely determined $\upsilon_2(T_n)$ where $(T_n)_{n\geq 0}$ denotes the Tribonacci sequence. Two years later, Facó and Marques [13] did the same with $\upsilon_2(T_n - 1)$. In 2020, Young [22] characterized $\upsilon_2(T_n + 1)$ for $n \in \mathbb{Z}$. Bravo, Díaz and Ramírez [9, 10] gave exact formulas for $\upsilon_2(t_n)$, $\upsilon_3(t_n)$ and $\upsilon_3(t_n \pm 1)$ where $(t_n)_{n\geq 0}$ denotes the TriPell sequence. See also Irmak [16] for $\upsilon_2(S_n \pm 1)$ where $(S_n)_{n\geq 0}$ denotes the Tribonacci-Lucas sequence, and Anwar, Ismail and Rihane [4] for $\upsilon_3(N_n \pm 1)$ where $(N_n)_{n\geq 0}$ denotes the Narayana's cow sequence. Recently, Bilu et al. [7] described a formula for $\upsilon_3(T_n)$ for $n \geq 0$; while Alahmadi and Luca [1] proved that the formula for $\upsilon_2(T_n)$ of Marques and Lengyel [19] hold for all integers n, not only for positive ones.

The 2-adic valuation of Padovan and Perrin numbers was completely characterized by Irmak [17] in 2019. The Padovan sequence $(P_n)_{n\geq 0}$ is defined by the recurrence equation

$$P_{n+3} = P_{n+1} + P_n \quad \text{for} \quad n \geq 0,$$

with initial conditions $P_0 = P_1 = P_2 = 1$. Other sources such as OEIS* in its entry A000931 may start the Padovan sequence at a different location, in which case the results of this paper should be adjusted with

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appropriate offsets. The Perrin sequence $(R_n)_{n\geq 0}$ is defined by the same recurrence as the Padovan sequence but with different initial values. The first Padovan and Perrin numbers for $n \geq 0$ are respectively

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265, 351, 465, 616, 816, 1081, \ldots$$

and

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, 367, 486, 644, 853, 1130, \ldots.$$

These sequences can be extended to negative indices by $P_n = P_{n+3} - P_{n+1}$ for $n \leq -1$. The first Padovan and Perrin numbers for $n \leq -1$ are respectively

$$0, 1, 0, 0, 1, -1, 1, 0, -1, 2, -2, 1, 1, \ldots$$

and

$$-1, 1, 2, -3, 4, -2, -1, 5, -7, 6, -1, \ldots.$$

A Cullen number C_m is a number of the form

$$m2^m + 1 \quad \text{for} \quad m \geq 1.$$

These numbers were first introduced in 1905 by Cullen [11] and are also mentioned in Guy's book [15, Section B20]. In turn, a Woodall number W_m is of the form

$$m2^m - 1 \quad \text{for} \quad m \geq 1.$$

These numbers were first studied in 1917 by Cunningham and Woodall [12]. The first Cullen and Woodall numbers are respectively

$$3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, \ldots$$

and

$$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, \ldots.$$

The Cullen and Woodall numbers satisfy the recurrence relations:

$$C_0 = 3, \quad C_1 = 9, \quad C_n = 4C_{n-1} - 4C_{n-2} + 1 \quad \text{for } n \geq 2.$$

$$W_0 = 1, \quad W_1 = 7, \quad W_n = 4W_{n-1} - 4W_{n-2} - 1 \quad \text{for } n \geq 2.$$

The problem of finding Cullen and Woodall numbers belonging to other known sequences has been studied by several authors in the last decades, see for example [2, 5, 6, 8, 14, 20]. We point out that by the main result of Bilu, Marques and Togbé [8, Theorem 1] for a given linear recurrence $(G_n)_n$, under weak assumptions, and a given polynomial $T(x) \in \mathbb{Z}[x]$, if $G_n = mx^m + T(x)$, then $m \ll 1$ and $n \ll \log|x|$, where the implied constants depend only on $(G_n)_n$ and $T(x)$. Letting $(G_n)_{n\geq 0} \in \{(P_n)_{n\geq 0}, (R_n)_{n\geq 0}\}$, $x := 2$ and $T(x) := \pm 1$ we get $|P_n \cap C_m| < \infty$ and $|R_n \cap W_m| < \infty$. In the present paper, using a complete description of the 2-adic valuation of $(P_n - 1)_{n\geq 0}$ and $(R_n + 1)_{n\geq 0}$, and a different approach, we fully determine which Cullen and Woodall numbers are Padovan and Perrin numbers, respectively.

We also use the 2-adic valuation of $(P_n - 1)_{n\geq 0}$ to find the Padovan numbers which are Proth numbers. A Proth number is of the form

$$k \cdot 2^m + 1,$$

for odd k, m a positive integer, and $2^m > k$. The $2^m > k$ condition is needed since otherwise, every odd number > 1 would be a Proth number. The first few Proth numbers are

$$3, 5, 9, 13, 17, 25, 33, 41, 49, 57, 65, 81, 97, 113, 129, 145, 161, 177, 193, 209, 225, 241, 257, 289, \ldots.$$

The Cullen numbers are a special case of the Proth numbers with $k = m$ and the inequality restriction dropped. The Fermat numbers are a special case of the Proth numbers with $k = 1$.

Finally, we also use the 2-adic valuation of $(R_n + 1)_{n\geq 0}$ to find the Perrin numbers which are Proth numbers of the second kind. A Proth number of the second kind is of the form

$$k \cdot 2^m - 1,$$

for odd k, m a positive integer, and $2^m > k$. The first of these numbers are

$$1, 3, 7, 11, 15, 23, 31, 39, 47, 55, 63, 79, 95, 111, 127, 143, 159, 175, 191, 207, 223, 239, 255, 287, \ldots.$$

Thâbit ibn Kurrah numbers and Woodall numbers are a special case of these numbers with $k = 3$ and $k = m$ respectively, and the inequality restriction dropped. The Mersenne numbers are a special case of these numbers with $k = 1$. Our results are as follows.

2. Results

Theorem 2.1 *For $n \geq 0$, we have*

$$\upsilon_2(P_n - 1) = \begin{cases} \infty, & \text{if } n = 0, 1, 2; \\ 0, & \text{if } n \equiv 3, 4, 6 \pmod 7; \\ \upsilon_2(n+2) + 1, & \text{if } n \equiv 5 \pmod 7; \\ \upsilon_2((n-1)(n+13)) + 1, & \text{if } n \equiv 1 \pmod 7; \\ \upsilon_2(n) + 1, & \text{if } n \equiv 0 \pmod{14}; \\ \upsilon_2(n+7) + 1, & \text{if } n \equiv 7 \pmod{14}; \\ \upsilon_2(n+5) + 2, & \text{if } n \equiv 9 \pmod{14}; \\ \upsilon_2((n-2)(n+26)) + 3, & \text{if } n \equiv 2 \pmod{28}; \\ \upsilon_2(n+12) + 4, & \text{if } n \equiv 16 \pmod{28}. \end{cases}$$

Theorem 2.2 *For $n \geq 0$, we have*

$$\upsilon_2(R_n + 1) = \begin{cases} 0, & \text{if } n \equiv 1, 2, 4 \pmod 7; \\ 1, & \text{if } n \equiv 5 \pmod 7; \\ \upsilon_2(n+7) + 2, & \text{if } n \equiv 0 \pmod 7; \\ \upsilon_2(n+11) + 1, & \text{if } n \equiv 3 \pmod 7; \\ \upsilon_2((n+1)(n+29)) + 1, & \text{if } n \equiv 6 \pmod 7. \end{cases}$$

Theorem 2.3 *The only Cullen numbers which are Padovan numbers are* 3*,* 9*, and* 65*.*

Theorem 2.4 *The only Proth numbers which are Padovan numbers are* 3*,* 5*,* 9*,* 49*,* 65*, and* 3329*.*

An immediate consequence is the following which is one of the main results of Adegbindin, Rihane, and Togbé [3, Theorem 4].

Corollary 2.5 *The only Fermat numbers in the Padovan sequence are* 3, 5, 9, *and* 65.

Theorem 2.6 *The only Woodall number that is a Perrin number is* 7.

Theorem 2.7 *The only Proth numbers of the second kind which are Perrin numbers are* 3, 7, *and* 39.

For $k = 1$ we deduce the following conclusion, which is one of the main results of Kafle, Rihane, and Togbé [18, Theorem 1.2].

Corollary 2.8 *The only Mersenne numbers in the Perrin sequence are* 3 *and* 7.

Except for 2 and 5 every Thâbit ibn Kurrah number is a Proth number of the second kind with $k = 3$, so we have the following result.

Corollary 2.9 *The only Thâbit ibn Kurrah numbers in the Perrin sequence are* 2 *and* 5.

3. Auxiliary results

The following result for the Padovan and Perrin numbers and their sums was found by Sokhuma [21, Proposition 2.2] with appropriate offsets.

Lemma 3.1 *For all positive integers* m, n *we have the relations:*

$$P_{n+m} = P_{n-1}P_{m-1} + P_nP_{m-2} + P_{n-2}P_{m-3},$$
$$R_{n+m} = P_{n-1}R_{m-1} + P_nR_{m-2} + P_{n-2}R_{m-3}.$$

The 2-adic valuation of Padovan and Perrin numbers was fully described by Irmak [17, Lemmas 2.5 and 2.6]. Their formulas are given below.

Lemma 3.2 *For* $n \geq 0$, *we get that*

$$\upsilon_2(R_n) = \begin{cases} 0, & \textit{if } n \equiv 0, 3, 5, 6, \pmod 7; \\ 1, & \textit{if } n \equiv 2 \pmod{14}; \\ 2, & \textit{if } n \equiv 9 \pmod{14}; \\ \upsilon_2(n-1)+1, & \textit{if } n \equiv 1 \pmod 7; \\ 1, & \textit{if } n \equiv 4 \pmod 7. \end{cases}$$

Lemma 3.3 *For* $n \geq 0$, *we obtain*

$$\upsilon_2(P_n) = \begin{cases} 0, & \textit{if } n \equiv 0, 1, 2, 5, \pmod 7; \\ \upsilon_2(n+4)+1, & \textit{if } n \equiv 3 \pmod 7; \\ \upsilon_2((n+3)(n+17))+1, & \textit{if } n \equiv 4 \pmod 7; \\ \upsilon_2((n+1)(n+8))+1, & \textit{if } n \equiv 6 \pmod 7. \end{cases}$$

The following three results generalize Lemmas 2.2, 2.3, and 2.4 in Irmak [17].

Lemma 3.4 *Let w_n be either the nth Padovan or Perrin number. For $n, r, s \in \mathbb{Z}$ we have*

$$w_{rn+s} = R_r w_{r(n-1)+s} - R_{-r} w_{r(n-2)+s} + w_{r(n-3)+s}.$$

Proof Let $w_n = P_n$ and let $P(X) = X^3 - X - 1$ be the characteristic polynomial of $(P_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$. Denoting the zeros of P by α, β and γ, with

$$\alpha = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} = 1.32471\ldots$$

the only real zero, we have $P_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n$ and $R_n = \alpha^n + \beta^n + \gamma^n$ for $n \in \mathbb{Z}$, where $c_z = (7z^2 + z + 3)/23$. Using this and the fact that $\alpha\beta\gamma = 1$ we get

$$R_r P_{r(n-1)+s} - R_{-r} P_{r(n-2)+s} + P_{r(n-3)+s} = c_\alpha \alpha^{rn+s} + c_\beta \beta^{rn+s} + c_\gamma \gamma^{rn+s} = P_{rn+s}.$$

The proof that $R_{rn+s} = R_r R_{r(n-1)+s} - R_{-r} R_{r(n-2)+s} + R_{r(n-3)+s}$ is similar. We omit the details. □

Lemma 3.5 *For the integers j and $k, t \geq 1$, we get:*

$$P_{7\cdot 2^t k+j} \equiv \begin{cases} P_j \pmod{2^{t+2}}, & \textit{if } j \equiv 1, 2, 4 \pmod 7, \\ P_j + k \cdot 2^{t+1} \pmod{2^{t+2}}, & \textit{otherwise.} \end{cases}$$

Proof Suppose $j \equiv 1, 2, 4 \pmod 7$. We prove the first statement using induction on k. The base case $k = 1$ corresponds to Irmak [17, Lemma 2.3]. Assume that $P_{7\cdot 2^t k+j} \equiv P_j \pmod{2^{t+2}}$ holds for all positive integers $\leq k$. Using Lemma 3.4 with $w_n = P_n$, $n = k+1$, $r = 7\cdot 2^t$ and $s = j$ we have

$$P_{7\cdot 2^t(k+1)+j} = R_{7\cdot 2^t} P_{7\cdot 2^t k+j} - R_{-7\cdot 2^t} P_{7\cdot 2^t(k-1)+j} + P_{7\cdot 2^t(k-2)+j}. \tag{3.1}$$

From Lemma 2.4 in Irmak [17] with $j = 0$ we know that $R_{7\cdot 2^t} \equiv 3 \pmod{2^{t+2}}$. Moreover, considering that $2R_{-n} = R_n^2 - R_{2n}$ and using again Lemma 2.4 in Irmak [17] with $j = 7\cdot 2^t$ we arrive at $2R_{-7\cdot 2^t} = R_{7\cdot 2^t}^2 - R_{7\cdot 2^t + 7\cdot 2^t} \equiv 6 \pmod{2^{t+2}}$. Thus $R_{-7\cdot 2^t} \equiv 3 \pmod{2^{t+2}}$. From (3.1) it then follows that

$$\begin{aligned} P_{7\cdot 2^t(k+1)+j} &\equiv (3P_{7\cdot 2^t k+j} - 3P_{7\cdot 2^t(k-1)+j} + P_{7\cdot 2^t(k-2)+j}) \pmod{2^{t+2}} \\ &\equiv (3P_j - 3P_j + P_j) \pmod{2^{t+2}} \\ &\equiv P_j \pmod{2^{t+2}}. \end{aligned}$$

Assume $j \equiv 0, 3, 5, 6 \pmod 7$. Again we use strong induction on k to prove the second statement. The base case is again obtained by Irmak [17, Lemma 2.3]. From the induction hypothesis and the fact that $R_{\pm 7\cdot 2^t} \equiv 3 \pmod{2^{t+2}}$ it follows from (3.1) that

$$\begin{aligned} P_{7\cdot 2^t(k+1)+j} &\equiv (3P_{7\cdot 2^t k+j} - 3P_{7\cdot 2^t(k-1)+j} + P_{7\cdot 2^t(k-2)+j}) \pmod{2^{t+2}} \\ &\equiv (3P_j + 3k\cdot 2^{t+1} - 3P_j - 3(k-1)2^{t+1} + P_j + (k-2)2^{t+1}) \pmod{2^{t+2}} \\ &\equiv P_j + (k+1)2^{t+1} \pmod{2^{t+2}}. \end{aligned}$$

□

Lemma 3.6 *For the integers j and $k, t \geq 1$, we have*

$$R_{7\cdot 2^t k+j} \equiv \begin{cases} R_j \pmod{2^{t+2}}, & \text{if } j \equiv 0,2,6 \pmod 7, \\ R_j + k\cdot 2^{t+1} \pmod{2^{t+2}}, & \text{otherwise.} \end{cases}$$

In addition,

$$R_{7\cdot 2^t} \equiv 3 \pmod{2^{t+5}} \quad \text{for} \quad t \geq 3 \tag{3.2}$$

Proof Assume $j \equiv 0,2,6 \pmod 7$. We prove the first statement by induction on k. For $k=1$ the result is given by Irmak [17, Lemma 2.4]. Suppose that the equivalence holds for all positive integers do not exceed k. From Lemma 3.4 with $w_n = R_n$, $n = k+1$, $r = 7\cdot 2^t$ and $s = j$ we get

$$R_{7\cdot 2^t(k+1)+j} = R_{7\cdot 2^t} R_{7\cdot 2^t k+j} - R_{-7\cdot 2^t} R_{7\cdot 2^t(k-1)+j} + R_{7\cdot 2^t(k-2)+j}. \tag{3.3}$$

The above together with the fact that $R_{\pm 7\cdot 2^t} \equiv 3 \pmod{2^{t+2}}$ leads to $R_{7\cdot 2^t(k+1)+j} \equiv R_j \pmod{2^{t+2}}$.

Suppose $j \equiv 1,3,4,5 \pmod 7$. Again by strong induction on k we can prove the second statement. The base case follows again by Irmak [17, Lemma 2.4]. From (3.3) and the fact that $R_{\pm 7\cdot 2^t} \equiv 3 \pmod{2^{t+2}}$ we can conclude the inductive step.

We use induction on t to prove the last equivalence. Since $R_{56} = 6900995 \equiv 3 \pmod{256}$, it is true for $t=3$. Assume $R_{7\cdot 2^t} \equiv 3 \pmod{2^{t+5}}$ for fixed $t \geq 4$. Then $R_{7\cdot 2^t} = 3 + 2^{t+5}x_1$ for $x_1 \geq 0$. Let $T_n = \begin{pmatrix} R_n \\ R_{n+1} \\ R_{n+2} \end{pmatrix}$ and $B_0 = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix}$. By using Irmak [17, Eq. (2.5)] with $w_n = R_n$ and $n = m = 7\cdot 2^t$ we get

$$\begin{aligned} R_{7\cdot 2^{t+1}} &= T_{7\cdot 2^t}^T B_0^{-1} T_{7\cdot 2^t} \\ &= \frac{1}{5}\begin{pmatrix} R_{7\cdot 2^t} & R_{7\cdot 2^t+1} & R_{7\cdot 2^t+2} \end{pmatrix} \begin{pmatrix} -1 & -6 & 4 \\ -6 & -11 & 9 \\ 4 & 9 & -6 \end{pmatrix} \begin{pmatrix} R_{7\cdot 2^t} \\ R_{7\cdot 2^t+1} \\ R_{7\cdot 2^t+2} \end{pmatrix} \\ &= -\frac{1}{5}(R_{7\cdot 2^t}^2 + 11R_{7\cdot 2^t+1}^2 + 6R_{7\cdot 2^t+2}^2 + 12R_{7\cdot 2^t}R_{7\cdot 2^t+1} - 8R_{7\cdot 2^t}R_{7\cdot 2^t+2} - 18R_{7\cdot 2^t+1}R_{7\cdot 2^t+2}). \end{aligned} \tag{3.4}$$

Using Lemma 3.6 with $j = 1,2$ and $k = 1$ we can write $R_{7\cdot 2^t+1} = 2^{t+1} + 2^{t+2}x_2$ and $R_{7\cdot 2^t+2} = 2 + 2^{t+2}x_3$ for $x_2, x_3 \geq 0$. From this, the induction hypothesis and (3.4) we deduce that

$$\begin{aligned} R_{7\cdot 2^{t+1}} &= 3 - \frac{x_1}{5}\big(2^{t+4}(x_2 - x_3) + 2^{t+2}(6x_2+3) - 5\big)2^{t+6} - \frac{1}{5}(11 + 44x_2 + 44x_2^2 - 36x_3 + 24x_3^2 - 72x_2x_3)2^{2t+2} \\ &\equiv 3 \pmod{2^{t+6}} \end{aligned}$$

as claimed. □

We end this section of preliminaries with the following equivalences.

Lemma 3.7 *For positive integers k and $t \geq 3$, we have*

- $P_{7\cdot 2^t k+2} \equiv 1 \pmod{2^{t+5}}$,

- $P_{7\cdot 2^t k-26} \equiv 1 \pmod{2^{t+5}}$,
- $P_{7\cdot 2^t k-12} \equiv 1 \pmod{2^{t+4}}$ *for* $t \geq 2$,
- $R_{7\cdot 2^t k-1} \equiv -1 \pmod{2^{t+3}}$.

Proof To prove the first statement we use induction on k. Assume that $k = 1$. To show $P_{7\cdot 2^t+2} \equiv 1 \pmod{2^{t+5}}$ we use induction on t. For $t = 3$, $P_{58} = 8745217 \equiv 1 \pmod{256}$ is satisfied. Suppose that $P_{7\cdot 2^t+2} \equiv 1 \pmod{2^{t+5}}$ holds for fixed $t \geq 4$. Then $P_{7\cdot 2^t+2} = 1 + 2^{t+5}y_1$ for $y_1 \geq 0$. Let $T_n = \begin{pmatrix} P_n \\ P_{n+1} \\ P_{n+2} \end{pmatrix}$ and $B_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$. From Irmak [17, Eq. (2.5)] with $w_n = P_n$, $n = 7\cdot 2^t + 2$ and $m = 7\cdot 2^t$ we obtain

$$\begin{aligned} P_{7\cdot 2^{t+1}+2} &= T^T_{7\cdot 2^t+2} B_0^{-1} T_{7\cdot 2^t} \\ &= \begin{pmatrix} P_{7\cdot 2^t+2} & P_{7\cdot 2^t+3} & P_{7\cdot 2^t+4} \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{7\cdot 2^t} \\ P_{7\cdot 2^t+1} \\ P_{7\cdot 2^t+2} \end{pmatrix} \\ &= (2P_{7\cdot 2^t} - P_{7\cdot 2^t+2})P_{7\cdot 2^t+2} + (P_{7\cdot 2^t+2} - P_{7\cdot 2^t+1})P_{7\cdot 2^t+3} + (P_{7\cdot 2^t+1} - P_{7\cdot 2^t})P_{7\cdot 2^t+4}. \end{aligned} \tag{3.5}$$

Using Lemma 3.5 with $j = 0, 1, 3, 4$ and $k = 1$ we can write $P_{7\cdot 2^t} = 1 + 2^{t+1} + 2^{t+2}y_2$, $P_{7\cdot 2^t+1} = 1 + 2^{t+2}y_3$, $P_{7\cdot 2^t+3} = 2 + 2^{t+1} + 2^{t+2}y_4$ and $P_{7\cdot 2^t+4} = 2 + 2^{t+2}y_5$ for $y_i \geq 0$, $i = 2, 3, 4, 5$. From this, the induction hypothesis and (3.5) we obtain

$$\begin{aligned} P_{7\cdot 2^{t+1}+2} &= 1 + (3y_1 + 4y_1y_2 - 16y_1^2 + 2^t y_1 + 2^{t+1}y_1y_4)2^{t+6} + (2y_3y_5 - 2y_2y_5 - 2y_3y_4 - y_3 - y_5)2^{2t+3} \\ &\equiv 1 \pmod{2^{t+6}} \end{aligned}$$

as claimed. Now assume that

$$P_{7\cdot 2^t k+2} \equiv 1 \pmod{2^{t+5}} \tag{3.6}$$

for fixed $t \geq 3$ and for all positive integers k. From Lemma 3.4 with $w_n = P_n$, $n = k + 1$, $r = 7\cdot 2^t$ and $s = 2$ we get

$$P_{7\cdot 2^t(k+1)+2} = R_{7\cdot 2^t} P_{7\cdot 2^t k+2} - R_{-7\cdot 2^t} P_{7\cdot 2^t(k-1)+2} + P_{7\cdot 2^t(k-2)+2}. \tag{3.7}$$

Now from (3.2), the fact that $2R_{-n} = R_n^2 - R_{2n}$ and Lemma 3.6 with $j = 7\cdot 2^t$ and $k = 1$ we get

$$R_{-7\cdot 2^t} \equiv 3 \pmod{2^{t+5}} \quad \text{for} \quad t \geq 3. \tag{3.8}$$

Finally, from (3.6), (3.7), (3.2) and (3.8) we arrive at

$$\begin{aligned} P_{7\cdot 2^t(k+1)+2} &\equiv (3P_{7\cdot 2^t k+2} - 3P_{7\cdot 2^t(k-1)+2} + P_{7\cdot 2^t(k-2)+2}) \pmod{2^{t+5}} \\ &\equiv (3\cdot 1 - 3\cdot 1 + 1) \pmod{2^{t+5}} \\ &\equiv 1 \pmod{2^{t+5}}. \end{aligned}$$

All other statements are proved in a similar way. We omit the details of the second and third ones.

To prove the last statement, we again use strong induction on k. Assume that $k=1$. By using the last equivalence in Lemma 3.6, we will show $R_{7\cdot 2^t-1}\equiv -1 \pmod{2^{t+3}}$. If $t=1$, it is obvious. Now assume that

$$R_{7\cdot 2^t-1}\equiv -1 \pmod{2^{t+3}}$$

holds for positive integer t. Our aim is to show $R_{7\cdot 2^{t+1}-1}\equiv -1 \pmod{2^{t+3}}$ for integer t. To do this, we take $w_n=R_n$, $n=7\cdot 2^t-1$, $m=7\cdot 2^t$ and $B_0=\begin{pmatrix}3&0&2\\0&2&3\\2&3&5\end{pmatrix}$ in Irmak [17, Eq. (2.5)]. Then

$$R_{7\cdot 2^{t+1}-1}=\begin{pmatrix}R_{7\cdot 2^t-1}\\R_{7\cdot 2^t}\\R_{7\cdot 2^t+1}\end{pmatrix}^T\begin{pmatrix}3&0&2\\0&2&3\\2&3&5\end{pmatrix}^{-1}\begin{pmatrix}R_{7\cdot 2^t}\\R_{7\cdot 2^t+1}\\R_{7\cdot 2^t+2}\end{pmatrix}.$$

Lemma 3.6 yields that $R_{7\cdot 2^t-1}=-1+y_1 2^{t+3}$, $R_{7\cdot 2^t}=3+y_2 2^{t+5}$, $R_{7\cdot 2^t+1}=2^{t+1}+y_3 2^{t+2}$ and $R_{7\cdot 2^t+2}=2+y_4 2^{t+2}$ for integers $y=i$, $i=1,2,3,4$. So

$$R_{7\cdot 2^{t+1}-1}\equiv -1 \pmod{2^{t+3}}$$

holds. Now, assume that $R_{7\cdot 2^t(k+l)-1}\equiv -1 \pmod{2^{t+3}}$ for $l\in\{-2,-1,0\}$. Our goal to prove that $R_{7\cdot 2^t(k+1)-1}\equiv -1 \pmod{2^{t+3}}$. Using Lemma 3.4 with $w_n=R_n$, $n=k+1$, $s=-1$ and $r=7\cdot 2^t$, we have

$$R_{7\cdot 2^t(k+1)-1}=R_{7\cdot 2^t}R_{7\cdot 2^tk-1}-R_{-7\cdot 2^t}R_{7\cdot 2^t(k-1)-1}+R_{7\cdot 2^t(k-2)-1}.$$

Since $R_{7\cdot 2^t}\equiv 3 \pmod{2^{t+5}}$, $2R_{-7\cdot 2^t}\equiv R^2_{7\cdot 2^t}-R_{7\cdot 2^{t+1}}\equiv 6 \pmod{2^{t+5}}$, $R_{7\cdot 2^t(k+l)-1}\equiv -1 \pmod{2^{t+3}}$ for $l\in\{-2,-1,0\}$, then we deduce

$$R_{7\cdot 2^t(k+1)-1}\equiv R_{7\cdot 2^t(k-2)-1}\equiv -1 \pmod{2^{t+3}}$$

as claimed. □

4. The proof of Theorem 2.1

Suppose $n\geq 0$.

In the proof of Theorem 2.1 we have used

Case 1: $n=0,1,2$

Recall that $\upsilon_p(0)=\infty$ is defined for every prime number p.

Case 2: $n\equiv 3,4,6 \pmod 7$

In this case, P_n is even by Lemma 3.3. Therefore $\upsilon_2(P_n-1)=0$.

Case 3: $n\equiv 5 \pmod 7$

We have $n\equiv -2 \pmod 7$ and then assuming $n\neq 5$ we can write $n=7\cdot 2^tk-2$ for some $t\geq 1$ and $k\equiv 1 \pmod 2$. By Lemma 3.5 with $j=-2$ we get $P_{7\cdot 2^tk-2}\equiv P_{-2}+k\cdot 2^{t+1} \pmod{2^{t+2}}$. Then $P_n-1\equiv k\cdot 2^{t+1} \pmod{2^{t+2}}$, from which it follows that $\upsilon_2(P_n-1)=t+1=\upsilon_2(7\cdot 2^tk)+1=\upsilon_2(n+2)+1$. The result also holds for $n=5$ since $\upsilon_2(P_5-1)=\upsilon_2(2)=\upsilon_2(5+2)+1$.

Case 4: $n \equiv 1 \pmod 7$

Here, we have $n \equiv 1, 8 \pmod{14}$. If $n \equiv 1 \pmod{14}$ then

$$\upsilon_2((n-1)(n+13)) = \begin{cases} \upsilon_2(n-1)+1, & \text{if} \quad n \equiv 1 \pmod 4, \\ \upsilon_2(n+13)+1, & \text{if} \quad n \equiv 3 \pmod 4. \end{cases}$$

When $n \equiv 1 \pmod 4$ we can write $n = 7 \cdot 2^t k + 1$ for some $t \geq 2$ and $k \equiv 1 \pmod 2$. Using Lemma 3.5 with $j = 1$ we get $P_n \equiv P_1 \equiv 1 \pmod{2^{t+2}}$. Therefore $P_n - 1 \equiv 0 \equiv 2^{t+2} \pmod{2^{t+2}}$. From the above we get $\upsilon_2(P_n - 1) = t + 2 = \upsilon_2(7 \cdot 2^t k) + 2 = \upsilon_2(n-1) + 2 = \upsilon_2((n-1)(n+13)) + 1$.

In case $n \equiv 3 \pmod 4$ we have $n \equiv -13 \pmod 4$ and we can write $n = 7 \cdot 2^t k - 13$ for some $t \geq 2$ and $k \equiv 1 \pmod 2$. Applying Lemma 3.5 with $j = -13$ we obtain $P_n \equiv P_{-13} \equiv 1 \pmod{2^{t+2}}$. Therefore $P_n - 1 \equiv 0 \equiv 2^{t+2} \pmod{2^{t+2}}$ and then $\upsilon_2(P_n - 1) = t+2 = \upsilon_2(7 \cdot 2^t k) + 2 = \upsilon_2(n+13) + 2 = \upsilon_2((n-1)(n+13)) + 1$.

On the other hand, when $n \equiv 8 \pmod{14}$ we have $\upsilon_2((n-1)(n+13)) = 0$. So it suffices to show that $\upsilon_2(P_n - 1) = 1$. For this we prove by induction on k that $P_{14k+8} \equiv 3 \pmod 4$, where $n = 14k + 8$ for $k \geq 0$. If $k = 0$ then $P_{22} = 351 \equiv 3 \pmod 4$. Assume $P_{14k+8} \equiv 3 \pmod 4$ for fixed $k \geq 1$. From the induction hypothesis and Lemma 3.1 with $n = 14k + 9$ and $m = 13$ we obtain

$$\begin{aligned} P_{14(k+1)+8} &= P_{14k+8}P_{12} + P_{14k+9}P_{11} + P_{14k+7}P_{10} \\ &= 21P_{14k+8} + 16P_{14k+9} + 12P_{14k+7} \equiv P_{14k+8} \equiv 3 \pmod 4. \end{aligned}$$

Case 5: $n \equiv 0 \pmod{14}$

Here we can write $n = 7 \cdot 2^t k$ for some $t \geq 1$ and $k \equiv 1 \pmod 2$. Then $P_n \equiv P_0 + k \cdot 2^{t+1} \pmod{2^{t+2}}$ by Lemma 3.5 with $j = 0$. It follows from the above that $\upsilon_2(P_n - 1) = t + 1 = \upsilon_2(7 \cdot 2^t k) + 1 = \upsilon_2(n) + 1$.

Case 6: $n \equiv 7 \pmod{14}$

We have $n \equiv -7 \pmod{14}$ and then we can write $n = 7 \cdot 2^t k - 7$ for some $t \geq 1$ and $k \equiv 1 \pmod 2$. From Lemma 3.5 with $j = -7$ we conclude that $P_n \equiv P_{-7} + k \cdot 2^{t+1} \pmod{2^{t+2}}$. Thus $\upsilon_2(P_n - 1) = t + 1 = \upsilon_2(7 \cdot 2^t k) + 1 = \upsilon_2(n+7) + 1$.

Case 7: $n \equiv 9 \pmod{14}$

Now we have $n \equiv -5 \pmod{14}$ and so we can write $n = 7 \cdot 2^t k - 5$ for some $t \geq 1$ and $k \equiv 1 \pmod 2$. Taking $j = -5$ in Lemma 3.5 we get $P_n \equiv P_{-5} \equiv 1 \pmod{2^{t+2}}$. Consequently $P_n - 1 \equiv 0 \equiv 2^{t+2} \pmod{2^{t+2}}$ and therefore $\upsilon_2(P_n - 1) = t + 2 = \upsilon_2(7 \cdot 2^t k) + 2 = \upsilon_2(n+5) + 2$.

Case 8: $n \equiv 2 \pmod{28}$

In this case $n \equiv 2, 30 \pmod{56}$. If $n \equiv 2 \pmod{56}$ then we write $n = 7 \cdot 2^t k + 2$ for some $t \geq 3$ and $k \equiv 1 \pmod 2$ or $n = 56x + 2$ for $x \geq 1$. From Lemma 3.7 it follows that $P_n - 1 \equiv 0 \equiv 2^{t+5} \pmod{2^{t+5}}$. Then $\upsilon_2(P_n - 1) = t + 5 = \upsilon_2(n-2) + 5 = \upsilon_2((n-2)(56x+28)) + 3 = \upsilon_2((n-2)(n+26)) + 3$ holds.

In case $n \equiv 30 \pmod{56}$ we have $n \equiv -26 \pmod{56}$ and so we can write $n = 7 \cdot 2^t k - 26$ for some $t \geq 3$ and $k \equiv 1 \pmod{2}$ or $n = 56x + 30$ for $x \geq 0$. From Lemma 3.7, $\upsilon_2(P_n - 1) = t + 5 = \upsilon_2(n + 26) + 5 = \upsilon_2((56x + 28)(n + 26)) + 3 = \upsilon_2((n - 2)(n + 26)) + 3$ holds.

Case 9: $n \equiv 16 \pmod{28}$

We have $n \equiv -12 \pmod{28}$ and then we can write $n = 7 \cdot 2^t k - 12$ for some $t \geq 2$ and $k \equiv 1 \pmod{2}$. Using Lemma 3.7 we have $P_n - 1 \equiv 0 \equiv 2^{t+4} \pmod{2^{t+4}}$. Then $\upsilon_2(P_n - 1) = t + 4 = \upsilon_2(7 \cdot 2^t) + 4 = \upsilon_2(n + 12) + 4$.

5. The proof of Theorem 2.2

Assume $n \geq 0$.

- **Case 1:** $n \equiv 1, 2, 4 \pmod{7}$.

  Here, R_n is even by Lemma 3.2. Hence $\upsilon_2(R_n + 1) = 0$.

- **Case 2:** $n \equiv 5 \pmod{7}$.

  We write $n = 7k + 5$ with $k \geq 0$. In order to prove that $\upsilon_2(R_n + 1) = 1$ it is sufficient to show that $R_n \equiv 1 \pmod{4}$. We do this by induction on k. If $k = 0$ then $R_n = R_5 = 5 \equiv 1 \pmod{4}$. Assume that $R_{7k+5} \equiv 1 \pmod{4}$ for fixed $k \geq 1$. Taking $n = 4$ and $m = 7k + 8$ in Lemma 3.1 we get

$$\begin{aligned} R_{7(k+1)+5} &= P_3 R_{7k+7} + P_4 R_{7k+6} + P_2 R_{7k+5} \\ &= 2(R_{7k+7} + R_{7k+6}) + R_{7k+5}. \end{aligned}$$

  From Lemma 3.2 we have that R_{7k+7} and R_{7k+6} are odd, so $R_{7k+7} + R_{7k+6} \equiv 0, 2 \pmod{4}$. In either case we obtain from the above equation and the induction hypothesis that $R_{7(k+1)+5} \equiv 1 \pmod{4}$.

- **Case 3:** $n \equiv 0 \pmod{7}$.

  We have $n \equiv -7 \pmod{7}$ and assuming that $n \neq 0$ we can write $n = 7 \cdot 2^t k - 7$ for some $t \geq 1$ and $k \equiv 1 \pmod{2}$. Using Lemma 3.6 with $j = -7$ we get $R_{7 \cdot 2^t k - 7} \equiv R_{-7} \equiv -1 \pmod{2^{t+2}}$. Then $R_n + 1 \equiv 0 \equiv 2^{t+2} \pmod{2^{t+2}}$ which implies that $\upsilon_2(R_n + 1) = t + 2 = \upsilon_2(7 \cdot 2^t k) + 2 = \upsilon_2(n + 7) + 2$. When $n = 0$ the result also follows since $\upsilon_2(R_0 + 1) = \upsilon_2(4) = 2 = \upsilon_2(0 + 7) + 2$.

- **Case 4:** $n \equiv 3 \pmod{7}$.

  Now $n \equiv -11 \pmod{7}$ and so we can write $n = 7 \cdot 2^t k - 11$ for some $t \geq 1$ and $k \equiv 1 \pmod{2}$. From Lemma 3.5 with $j = -11$ we obtain $R_{7 \cdot 2^t k - 11} \equiv R_{-11} + k \cdot 2^{t+1} \pmod{2^{t+2}}$. Then $R_n + 1 \equiv k \cdot 2^{t+1} \pmod{2^{t+2}}$ and thus $\upsilon_2(R_n + 1) = t + 1 = \upsilon_2(7 \cdot 2^t k) + 1 = \upsilon_2(n + 11) + 1$.

- **Case 5:** $n \equiv 6 \pmod{7}$.

  Now we have $n \equiv 6, 13 \pmod{14}$. When $n \equiv 6 \pmod{14}$ we have $\upsilon_2((n+1)(n+29)) = 0$. It is therefore sufficient to show that $\upsilon_2(R_n + 1) = 1$. Let $n = 14k + 6$ for $k \geq 0$. Next we proceed by induction on k to show that $R_{14k+6} \equiv 1 \pmod{4}$. The base case follows as $R_6 = 5 \equiv 1 \pmod{4}$. Suppose $R_{14k+6} \equiv 1$

(mod 4) for fixed $k \geq 1$. From the induction hypothesis and Lemma 3.1 with $n = 11$ and $m = 14k + 9$ we get

$$\begin{aligned} R_{14(k+1)+6} &= P_{10}R_{14k+8} + P_{11}R_{14k+7} + P_9R_{14k+6} \\ &= 12R_{14k+8} + 16R_{14k+7} + 9R_{14k+6} \equiv R_{14k+6} \equiv 1 \pmod 4. \end{aligned}$$

On the other hand, if $n \equiv 13 \pmod{14}$ then $n \equiv 13, 27 \pmod{28}$. In case $n \equiv 13 \pmod{28}$ we have $\upsilon_2((n+1)(n+29)) = 2$. Therefore, it is sufficient to show that $\upsilon_2(R_n + 1) = 3$. Put $n = 28k + 13$ for $k \geq 0$. Next we use induction to prove that $R_{28k+13} \equiv 7 \pmod{16}$. As $R_{13} = 39 \equiv 7 \pmod{16}$ the base case is followed. Assume $R_{28k+13} \equiv 7 \pmod{16}$ for fixed $k \geq 1$. From Lemma 3.1 with $n = 26$ and $m = 28k + 15$ we obtain

$$\begin{aligned} R_{28(k+1)+13} &= P_{25}R_{28k+14} + P_{26}R_{28k+13} + P_{24}R_{28k+12} \\ &= 816R_{28k+14} + 1081R_{28k+13} + 616R_{28k+12}. \end{aligned}$$

Note that R_{28k+12} is odd by Lemma 3.2, so $R_{28k+12} \equiv 1, 3, 5, 7, 9, 11, 13, 15 \pmod{16}$. In any case we obtain from the above equation and the induction hypothesis that $R_{28(k+1)+13} \equiv 7 \pmod{16}$.

Finally, in the case $n \equiv 27 \pmod{28}$ we have $n \equiv 27, 55 \pmod{56}$. When $n \equiv 27 \pmod{56}$ we have $n \equiv -29 \pmod{56}$ and then we can write $n = 7\cdot 2^t k - 29$ for some $t \geq 3$ and $k \equiv 1 \pmod 2$. From Lemma 3.7 it follows that $R_n + 1 \equiv 0 \equiv 2^{t+3} \pmod{2^{t+3}}$ giving $\upsilon_2(R_n+1) = t+3 = \upsilon_2(28)+\upsilon_2(2^{t-2}k-1)+\upsilon_2(n+29)+1 = \upsilon_2((7\cdot 2^t k-28)(n+29))+1 = \upsilon_2((n+1)(n+29))+1$. If $n \equiv 55 \pmod{56}$ then $n \equiv -1 \pmod{56}$ and so we can write $n = 7\cdot 2^t k - 1$ for some $t \geq 3$ and $k \equiv 1 \pmod 2$. Again from Lemma 3.7 we have that $\upsilon_2(R_n+1) = t+3 = \upsilon_2(n+1)+\upsilon_2(28)+\upsilon_2(2^{t-2}k+1)+1 = \upsilon_2((n+1)(7\cdot 2^t k+28))+1 = \upsilon_2((n+1)(n+29))+1$.

6. Padovan Cullen numbers

Let us start by proving by induction that

$$\alpha^{n-4} \leq P_n - 1 \quad \text{for} \quad n \geq 3. \tag{6.1}$$

If $n = 3, 4, 5$, then $\alpha^{-1} < 0.8 < 1 = P_3 - 1$, $\alpha^0 = 1 = P_4 - 1$, and $\alpha < 1.4 < 2 = P_5 - 1$. Suppose that $\alpha^{k-4} \leq P_k - 1$ for all $3 \leq k \leq n-1$. Taking into account that α^{-1} is a zero of $X^3P(1/X)$ and using the recurrence equation of the Padovan sequence we obtain

$$\alpha^{n-4} = \alpha^{n-4}(\alpha^{-2} + \alpha^{-3}) = \alpha^{n-6} + \alpha^{n-7} \leq P_{n-2} + P_{n-3} - 2 < P_n - 1.$$

Assume $m, n \geq 4$. Using (6.1) in equation $P_n = C_m$ we obtain $\alpha^{n-4} \leq P_n - 1 = m2^m \leq 2^{3m/2}$, and thus

$$n < 4m + 4. \tag{6.2}$$

Now from Theorem 2.1 and the fact that $\upsilon_2(n) \leq \log n/\log 2$ for all $n \in \mathbb{Z}^+$ we deduce that

$$\upsilon_2(P_n - 1) \leq \frac{\log(n-2) + \log(n+26)}{\log 2} + 3 \quad \text{for all} \quad n \geq 4. \tag{6.3}$$

Using (6.3) in equation $P_n = C_m$ we obtain $m \le \upsilon_2(m2^m) = \upsilon_2(P_n - 1) \le ((\log(n-2) + \log(n+26))/\log 2) + 3$. Thus by (6.2) we get

$$m < \frac{\log(2m+1) + \log(2m+15)}{\log 2} + 5.$$

This inequality yields $m \le 15$. Therefore $n \le 47$ since $P_{48} = 525{,}456 > 491{,}521 = C_{15}$. A computational search reveals that $(n,m) \in \{(5,1),(9,2),(16,4)\}$ are the only solutions of equation $P_n = C_m$ for $n \in [0,47]$ and $m \in [1,15]$. This completes the proof of Theorem 2.3.

7. Padovan Proth numbers

Suppose $n \ge 4$. Using (6.1) in equation $P_n = k \cdot 2^m + 1$ we get $\alpha^{n-4} \le P_n - 1 = k \cdot 2^m < 2^m \cdot 2^m = 2^{2m}$. Therefore,

$$n < 5m + 4. \tag{7.1}$$

Now from (6.3) in equation $P_n = k \cdot 2^m + 1$ we obtain $m = \upsilon_2(k \cdot 2^m) = \upsilon_2(P_n - 1) \le ((\log(n-2) + \log(n+26))/\log 2) + 3$. So from (7.1) we get

$$m < \frac{\log(5m+2) + \log(5m+30)}{\log 2} + 3.$$

This inequality is only true if $m \le 16$. So $n \le 83$ by (7.1) and $k < 2^{16} = 65536$. Through a computational search we find that $(k,n,m) \in \{(1,5,1),(1,7,2),(1,9,3),(3,15,4),(1,16,6),(13,30,8)\}$ are the only solutions of the equation $P_n = k \cdot 2^m + 1$ for odd k with $2^m > k \in [1,65535]$, $m \in [1,16]$ and $n \in [0,83]$. This ends the proof of Theorem 2.4.

8. Perrin Woodall numbers

Assume $n \ge 2$ and $m \ge 4$. From Theorem 2.2 and the fact that $\upsilon_2(n) \le \log n/\log 2$ for all $n \ge 1$ we get

$$\upsilon_2(R_n + 1) \le \frac{\log(n+1) + \log(n+29)}{\log 2} + 1. \tag{8.1}$$

Using (8.1) in equation $R_n = W_m$ we get

$$m < \frac{\log(n+1) + \log(n+29)}{\log 2} + 1. \tag{8.2}$$

On the other hand, let us prove by induction that

$$\alpha^{n-2} < R_n \quad \text{for} \quad n \ge 2. \tag{8.3}$$

If $n = 2,3,4$, then $\alpha^0 = 1 < 2 = R_2$, $\alpha < 1.4 < 2 = R_3$, and $\alpha^2 < 1.8 < 2 = R_4$. Suppose that $\alpha^{k-2} < R_k$ for all $2 \le k \le n-1$. From the fact that $-\alpha^{-3} - \alpha^{-2} + 1 = 0$ and from the recurrence equation of the Perrin sequence it follows that

$$\alpha^{n-2} = \alpha^{n-2}(\alpha^{-2} + \alpha^{-3}) = \alpha^{n-4} + \alpha^{n-5} < R_{n-2} + R_{n-3} = R_n.$$

Using (8.3) in equation $R_n = W_m$ we get $\alpha^{n-2} < R_n < R_n + 1 = m2^m \leq 2^{3m/2}$, and therefore

$$n < 3.7m + 2. \tag{8.4}$$

From (8.2) and (8.4) we then get

$$m < \frac{\log(3.7m+3) + \log(3.7m+31)}{\log 2} + 1.$$

This inequality implies that $m \leq 12$. Thus $n \leq 38$ since $R_{39} = 57,918 > 49,151 = W_{12}$. The proof of Theorem 2.6 ends with a computational search which confirms that $(n, m) = (7, 2)$ is the only solution of equation $R_n = W_m$ for $n \in [0, 38]$ and $m \in [1, 12]$.

9. Perrin Proth numbers of the second kind

Assume $n \geq 2$. Using (8.3) in equation $R_n = k \cdot 2^m - 1$ we obtain $\alpha^{n-2} < R_n + 1 = k \cdot 2^m < 2^{2m}$, and then

$$n < 5m + 2. \tag{9.1}$$

From (8.1) in equation $R_n = k \cdot 2^m - 1$ we get $m < (\log(n+1) + \log(n+29))/\log 2) + 1$. Thus, by (9.1) we get

$$m < \frac{\log(5m+3) + \log(5m+31)}{\log 2} + 1.$$

From this inequality, it follows that $m \leq 13$. So $n \leq 66$ by (9.1) and $k < 2^{13} = 8192$. The proof of Theorem 2.7 ends with a computational search that proves that $(k, n, m) \in \{(1, 0, 2), (1, 3, 2), (1, 7, 3), (5, 13, 3)\}$ are the only solutions of the equation $R_n = k \cdot 2^m - 1$ for odd k with $2^m > k \in [1, 8191]$, $m \in [1, 13]$ and $n \in [0, 66]$.

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