Turkish Journal of Mathematics

Volume 48 | Number 6 Article 11

11-14-2024

Fibration category structures induced by enrichments

MEHMET AKİF ERDAL

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 1138 – 1155
© TÜBİTAK
doi:10.55730/1300-0098.3566

Fibration category structures induced by enrichments

Mehmet Akif ERDAL*

Department of Mathematics, Faculty of Arts and Sciences, Yeditepe University, İstanbul, Turkiye

Received: 09.04.2024 • **Accepted/Published Online:** 02.07.2024 • **Final Version:** 14.11.2024

Abstract: We study fibration category structures induced by enrichments over symmetric monoidal categories that are also fibration categories. Let $\mathcal{V}$ be a monoidal category that is also a fibration category. Assuming that $\mathcal{V}$ has an interval object, we demonstrate that the fibration category structure on $\mathcal{V}$ can be transferred to any $\mathcal{V}$-enriched category through corepresentable functors, provided certain power objects exist. Furthermore, we extend this result to its G-equivariant version for a group G, showing that, under mild conditions, the category of G-objects in a $\mathcal{V}$-enriched category admits a (nontrivial) fibration category structure. We also show that several categories of topological algebras and associative algebras, along with their G-equivariant analogues, can be structured as fibration categories obtained through this method. Finally, we present some applications of these results, including the recovery of existing findings related to (equivariant) K-theory and E-theory of operator algebras.

Key words: Fibration category, homotopy, enriched category, path object, equivariant homotopy

1. Introduction

In homotopy theory, fibration categories provide an efficient framework for many homotopy terminal constructions, such as homotopy fiber sequences, loop space objects, and homotopical cohomology theories. Various types of fibration (and their dual versions, cofibration categories) have been studied in the literature; see, e.g., [5], [4], and [26]. In this paper, we focus on Brown's version, namely the "category of fibrant objects." These fibration categories have been instrumental in studying the homotopy theories of various objects, such as C^*-algebras [30], spectral sequences [18], and cofibration categories [29].

A category of fibrant objects is defined as a category equipped with two classes of morphisms, called weak equivalences and fibrations, that satisfy certain conditions. In this paper, we specifically focus on the structures of categories of fibrant objects that are induced by enrichments over symmetric monoidal categories, which are also categories of fibrant objects. Examples of such appear frequently in the literature and have many important applications in noncommutative topology (see, e.g., [28, 30]). We denote $(\mathcal{V}, \otimes, [-,-], 1)$ as a closed symmetric monoidal category with the unit represented by the terminal object $1 = *$. Such monoidal categories are often called "semicartesian monoidal". Note that all cartesian monoidal categories are semicartesian, but not vice versa; e.g., **Cat** with the non-Cartesian tensor product in [12]. We consider $(\mathcal{V}, we_{\mathcal{V}}, fib_{\mathcal{V}})$ a category of fibrant objects satisfying the following "interval condition" (**IC**) that reads:

*Correspondence: mehmet.erdal@yeditepe.edu.tr

2010 *AMS Mathematics Subject Classification:* 18D20, 55P15, 55P91

(**IC**) *there is a factorization of the codiagonal* $* \amalg * \to \mathbb{I} \to *$ *as a composition of an acyclic fibration followed by a pseudo-cofibration.*

Here, an acyclic fibration is defined as a fibration that is also a weak equivalence, while a pseudo-cofibration is a morphism whose pullback power with an (acyclic) fibration is an (acyclic) fibration. For details regarding this condition, see Lemma 3.2 and the preceding paragraph. Note that in the case of a monoidal model category, this definition of pseudo-cofibration coincides with that given in [21].

We first prove the following theorem.

Theorem 1.1 *Suppose that* $(\mathcal{V}, \otimes, [-,-], 1)$ *is a closed symmetric monoidal category and that* $(\mathcal{V}, we_{\mathcal{V}}, fib_{\mathcal{V}})$ *is a category of fibrant objects satisfying* (**IC**)*. Let* $\mathcal{A}$ *be a subcategory of* $\mathcal{V}$ *that generates* $\mathcal{V}$ *and contains* $\mathbb{I}$*. If* $\mathcal{C}$ *is a finitely complete category that is enriched over* $\mathcal{V}$ *and powered over* $\mathcal{A}$*, then* $\mathcal{C}$ *is a category of fibrant objects in which* f *is a weak equivalence (resp. fibration) if, for every* $D \in \mathcal{C}$*, the morphism* $\mathcal{C}(D, f)$ *is a weak equivalence (resp. fibration) in* $\mathcal{V}$*.*

Let G be a group and let $\mathcal{V}$ and $\mathcal{C}$ be as defined in Theorem 1.1. Denote by $G\mathcal{V}$ and $G\mathcal{C}$ the categories of G-objects in $\mathcal{V}$ and $\mathcal{C}$, respectively. For A and B in $G\mathcal{C}$ (i.e. G-objects in $\mathcal{C}$), we let $\overline{G\mathcal{C}}(A, B)$ to denote the object $\mathcal{C}(A, B)$ in $\mathcal{V}$ together with the G-action given by conjugation (see 3.1 for the precise definition). Thus, $\overline{G\mathcal{C}}(A, B)$ is an object in $G\mathcal{V}$. For any $H \leq G$, the H-fixed point object functor $(-)^H : G\mathcal{V} \to \mathcal{V}$ is defined as the functor given by the limit over H; that is, for X in $G\mathcal{V}$, we have $X^H = \lim_{BH} X$, where BH is the delooping groupoid of H (a category with single object and elements of H as morphisms on this object), and X is considered a functor from BH to $\mathcal{V}$ with the restricted H-action. The equivariant version of Theorem 1.1 is presented as follows.

Theorem 1.2 *Let* G *be a group and* $\mathcal{F}$ *be a collection of subgroups of* G*. Let* $\mathcal{V}$ *and* $\mathcal{C}$ *be as defined in Theorem 1.1 such that limits defining* H*-fixed points exist in* $\mathcal{V}$ *for every* $H \in \mathcal{F}$*. Then,* $G\mathcal{C}$ *is a category of fibrant objects in which* f *is a weak equivalence (resp. fibration) if, for every* $D \in G\mathcal{C}$ *and* $H \in \mathcal{F}$*, the morphism* $\overline{G\mathcal{C}}(D, f)^H$ *is a weak equivalence (resp. fibration) in* $\mathcal{V}$*.*

The paper is organized as follows: In Section 2, we provide essential background on categories of fibrant objects. In Section 3, we state and prove the theorems presented above, along with other auxiliary results. We prove Theorem 1.1 by formally verifying each axiom of a category of fibrant objects separately. Subsequently, in Section 3.1, we outline the prerequisites for equivariant constructions and prove Theorem 1.2. In Section 4, we present several examples that follow from our results. We demonstrate that various categories of operator algebras, including Uuye and Schocket's homotopy theory for C^*-algebras, serve as examples of Theorem 1.1, while their equivariant versions correspond to Theorem 1.2. Additionally, we show that the categories of metric spaces and algebras (including G-algebras) over a unital ring R are categories of fibrant objects derived from our theorems. In Section 5, we apply our results to recover known findings, demonstrating that the equivariant KK-category, the equivariant E-category of G-C^*-algebras, and the kk-category of R-algebras are triangulated categories.

2. Preliminaries

In this section, we provide essential background on categories of fibrant objects for the readers' convenience. For further details, we refer the reader to [5].

Definition 2.1 *A category of fibrant objects is a category $\mathcal{C}$ with two distinguished subcategories: $we_{\mathcal{C}}$ of weak equivalences and $fib_{\mathcal{C}}$ of fibrations. Both subcategories contain the core of $\mathcal{C}$ (i.e. both include all isomorphisms) and satisfy the following axioms:*

FC1 *$\mathcal{C}$ has all finite products, and a terminal object $*$.*

FC2 *$we_{\mathcal{C}}$ has the 2-out-of-3 property; i.e. assuming $v \circ w$ is defined, if any two of v, w, and $v \circ w$ are in $we_{\mathcal{C}}$, then so is the third.*

FC3 *Pullbacks along (acyclic) fibrations exist and are again (acyclic) fibrations. Here, an acyclic fibration means a fibration that is also a weak equivalence; i.e. an arrow in the intersection $we_{\mathcal{C}} \cap fib_{\mathcal{C}}$.*

FC4 *For each object A of $\mathcal{C}$, there exists an object $\mathcal{P}A$ in $\mathcal{C}$ (called the path object) which satisfies the following: there is a weak equivalence $\omega_A : A \to \mathcal{P}A$ and a fibration $\delta_A = (\delta_0, \delta_1) : \mathcal{P}A \to A \times A$, and the composition $\delta_A \circ \omega_A$ is equal to the diagonal map $d : A \to A \times A$.*

FC5 *All objects are fibrant; i.e. for each object A, the unique map to the terminal object $A \to *$ is in $fib_{\mathcal{C}}$.*

If we do not require **FC5** and weaken **FC4** so that only fibrant objects have path objects, then such a $\mathcal{C}$ is called a *prefibration category*. The full-subcategory of fibrant objects $\mathcal{C}^{fib}$ in a prefibration category $\mathcal{C}$ is a category of fibrant object, see [26, Defn. 1.1.2 and Prop. 2.1.2].

For $f, g : A \to B$ in $\mathcal{C}$, f is said to be *right-homotopic* to g if there exist $H : A \to \mathcal{P}B$ such that $\delta_0 \circ H = h$ and $\delta_1 \circ H = g$ and f is *homotopic* to g if there exist a weak equivalence $w : A' \to A$ such that $f \circ w$ is right-homotopic to $g \circ w$. The homotopy relation is an equivalence relation on $\mathcal{C}(A, B)$, see [5]. The set of homotopy classes in $\mathcal{C}(A, B)$ is denoted by $\pi\mathcal{C}(A, B)$ and the resulting quotient category is denoted by $\pi\mathcal{C}$.

Theorem 2.2 ([5], Theorem 2.1) *The homotopy category $\mathrm{Ho}\,\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and has hom-sets given by*

$$\mathrm{Ho}\,\mathcal{C}(A, B) = \operatorname*{colim}_{A' \xrightarrow{\simeq} A} \pi\mathcal{C}(A', B).$$

The theorem implies that any morphism in the homotopy category of a category of fibrant objects can be written as a zig-zag of length 2.

A functor $F : \mathcal{C} \to \mathcal{D}$ between two categories of fibrant objects is called *exact* if it preserves fibrations, acyclic fibrations, terminal objects and pullbacks along fibrations.

Lemma 2.3 ([5], Ken Brown's Lemma) *If a functor between two categories of fibrant objects sends acyclic fibrations to weak equivalences, then it preserve all weak equivalences. In particular, exact functors preserve weak equivalences.*

Lemma 2.4 ([5], Factorization Lemma) *Let $(\mathcal{C}, we_{\mathcal{C}}, fib_{\mathcal{C}})$ be a category of fibrant objects. Then, every morphism f in C factors as $f = p \circ i$ with $f \in fib_{\mathcal{C}}$ and w is a section of an acyclic fibration.*

Factorization Lemma allows us to define homotopy fiber sequences in $\mathcal{C}$. Given $f : A \to B$ the homotopy fiber of f, denoted by $\mathrm{hofib}(f)$, is the pullback of the diagram $A \to B \leftarrow \mathcal{P}B$. The homotopy fiber of the path fibration $\mathcal{P}A \to A \times A$ is called the *loop space object of* A and denoted by ΩA. By definition, the loop space construction is functorial and defines an endofunctor on $\mathcal{C}$. The loop object ΩA is a group object and $\Omega\Omega A = \Omega^2 A$ is an abelian group object in the homotopy category $\mathrm{Ho}\,\mathcal{C}$. If $\mathcal{C}$ is a pointed category of fibrant objects (i.e. the terminal object is also initial), then the stable homotopy category $\mathrm{StHo}\,\mathcal{C}$ is given by the Spanier-Whitehead category; i.e. objects as pairs (A, n) where $A \in \mathcal{C}$ and $n \in \mathbb{N}$ and morphisms are given by

$$\mathrm{StHo}\,\mathcal{C}((A,n),(B,m)) = \operatorname*{colim}_k(\Omega^{n+k}A, \Omega^{m+k}B).$$

This is a triangulated category with shift $(-,n) \mapsto (-,n-1)$ and with distinguished triangles isomorphic to

$$(\Omega B, n) \xrightarrow{j} (F,n) \xrightarrow{i} (A,n) \xrightarrow{p} (B,n),$$

where p is a fibration with fiber F and j is the map induced by the action of the loop object on the fiber of p, see [5, Sec. 4] and [15].

3. Main results

Notation and convention

In this paper, we assume the categories are locally small and concrete, except possibly the homotopy categories of corresponding homotopical categories. For a $\mathcal{V}$-enriched category $\mathcal{C}$, we denote the $\mathcal{V}$-valued hom-functors by $\mathcal{C}(-,-)$ and $\mathcal{S}et$-valued hom-functors by $\mathcal{C}_0(-,-)$. We say a functor preserve (co)limits when it preserve (co)limits that exists in the codomain. See [27] for more details.

Let $\mathcal{V}$ be a category and $\mathcal{A}$ be a full subcategory of $\mathcal{V}$. We say $\mathcal{V}$ is generated by $\mathcal{A}$ if for every Y in $\mathcal{V}$ there exist a functor $F : J \to \mathcal{A}$ such that $Y = \mathrm{colim}_J\, F \circ \iota$ where $\iota : \mathcal{A} \to \mathcal{V}$ is the inclusion. We should also mention that similar notions exist in the literature, known as "dense" or "colimit dense" subcategories, see e.g. [1]. If $\mathcal{V}$ is cocomplete and $\mathcal{A} = \mathcal{V}$ then the condition above is trivially satisfied. Nontrivial examples of such categories also appear quite often. Here, we provide several well-known examples:

(i) If $\mathcal{V}$ is the category of simplicial sets, then one can choose $\mathcal{A}$ as its full-subcategory of simplicial n-simplices for every $n \in \mathbb{N}$, as every simplicial set is a colimit of simplicial simplices over its category of simplices.

(ii) More generally, if $\mathcal{V}$ is a category of preshaves over a small category C, then $\mathcal{A}$ can be chosen as all representable functors as every presheaf is a colimit of representables over its category of elements.

(iii) The category of Δ-generated spaces (see [11] originally based on J. Smith's unpublished work*) when Δ is chosen to be the subcategory of topological spaces of topological n-simplices.

(iv) More generally, if $\mathcal{V}$ is an accessible category in the sense of [2], then there is a small subcategory of "small objects" that generates $\mathcal{V}$ (see [2] for details).

(v) Yet another example is the category of compactly generated spaces (which is not an accessible category) where $\mathcal{A}$ is chosen to be the category of compact Hausdorff spaces.

*Dugger D (2003). Notes on Delta-generated spaces [online]. Website `https://pages.uoregon.edu/ddugger/delta.html` [accessed 22 April 2024].

Lemma 3.1 *Let $\mathcal{V}$ be a closed symmetric monoidal category that is generated by a subcategory $\mathcal{A}$. If $\mathcal{C}$ is a $\mathcal{V}$-enriched category that is powered over $\mathcal{A}$, then for every object D in $\mathcal{C}$ the corepresentable functor $\mathcal{C}(D,-) : \mathcal{C} \to \mathcal{V}$ preserves conical limits.*

Proof Let $F : I \to \mathcal{C}$ be a functor from a small category I. Then, $\mathcal{C}(D, \lim_I F)$ is a cone over $\mathcal{C}(D, F)$. Thus, there exist a morphism

$$q : \mathcal{C}(D, \lim_I F) \to \lim_J \mathcal{C}(D, F).$$

Since corepresentables into $\mathcal{S}et$ are continuous, $\mathcal{C}(D, \lim_J F)$ is a limiting cone for the underlying $\mathcal{S}et$-functor. In particular, q_0 is a bijection of the underlying sets. Now, let $X \in \mathcal{A}$ and consider the following diagram

$$\begin{array}{ccc}
\mathcal{V}_0(X, \mathcal{C}(D, \lim_J F)) & \xrightarrow{\mathcal{V}_0(X,q)} & \mathcal{V}_0(X, \lim_J \mathcal{C}(D, F)) \\
\downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} \\
\mathcal{C}_0(D, \pitchfork (X, \lim_J F)) & & \lim_J \mathcal{V}_0(X, \mathcal{C}(D, F)) \\
\downarrow{\scriptstyle \nu_0} & & \downarrow{\scriptstyle \nu_1} \\
\mathcal{C}_0(D, \lim_J \pitchfork (X, F)) & \xrightarrow{q_1} & \lim_J \mathcal{C}_0(D, \pitchfork (X, F))
\end{array}$$

The maps φ_0 and ν_1 are bijections as they are underlying maps of isomorphisms in $\mathcal{V}$ that comes from the powering over X. Since $\mathcal{V}$ is closed monoidal (so that $[X, -]$) preserves limits), φ_1 is a bijection. This also implies that $\mathcal{V}_0(X, -)$ is injective. Since powers preserve limits, we have $\pitchfork (X, \lim_J F) \cong \lim_J \pitchfork (X, F)$, so that ν_0 is a bijection. Since every other map in the diagram is a bijection and q_1 is an injection, then q_1 is also a bijection. For every Y in $\mathcal{V}$ there is a functor $F : J \to \mathcal{A}$ such that $Y \cong \operatorname{colim}_J F \circ \iota$. Then, we have

$$\mathcal{V}_0(Y, q) = \mathcal{V}_0(\operatorname{colim}_J F \circ \iota, q) \cong \lim_{j \in J} \mathcal{V}_0(F \circ \iota(j), q)$$

which is a bijection since $F \circ \iota(j)$ is in $\mathcal{A}$. Thus, by Yoneda lemma q is a isomorphism in $\mathcal{V}$. □

Here we mean $\mathcal{C}(D, -)$ preserve limits that exists in $\mathcal{C}$ and do not make any assumption about which limits exist. Note that if $Y = \operatorname{colim}_J F \circ \iota$ for some $F : J \to \mathcal{A}$ and A is an object in $\mathcal{C}$ such that $\lim_{j:J} \pitchfork (F \circ \iota(j), A)$ exists, then the power $\pitchfork (Y, A)$ is also defined via $\lim_{j:J} \pitchfork (F \circ \iota(j), A)$. The natural isomorphism $\mathcal{C}(D, \lim_{j:J} \pitchfork (F \circ \iota(j), A)) \cong \mathcal{V}(\operatorname{colim}_J F \circ \iota, \mathcal{C}(D, A))$ follows from formal properties of (co)limits and representables. If $\mathcal{C}$ has all such J-shaped limits for every Y in $\mathcal{V}$ with $Y \cong \operatorname{colim}_J F \circ \iota$, then $\mathcal{C}$ is powered over $\mathcal{V}$. In particular, if $\mathcal{C}$ is complete, then it is powered over $\mathcal{V}$, and thus, is $\mathcal{V}$-complete.

Let $(\mathcal{V}, \otimes, [-,-], 1)$ be a closed symmetric monoidal category and let $(\mathcal{V}, we_{\mathcal{V}}, fib_{\mathcal{V}})$ be a category of fibrant objects. Following [21], we call a morphism $f : X \to Y$ in $\mathcal{V}$ *pseudo-cofibration* if for every (acyclic) fibration $g : Z \to W$ the pull-back power

$$\mathcal{V}(Y, Z) \to \mathcal{V}(X, Z) \times_{\mathcal{V}(X,W)} \mathcal{V}(Y, W)$$

is an (acyclic) fibration. Note that every isomorphism is a pseudo-cofibration.

Lemma 3.2 *If $(\mathcal{V}, \otimes, [-,-], 1)$ is a closed symmetric monoidal category with 1 as the terminal object $*$ and $(\mathcal{V}, we_{\mathcal{V}}, fib_{\mathcal{V}})$ is a category of fibrant objects satisfying*

(**IC**) *the codiagonal* $* \amalg * \to *$ *exists and admits a factorization* $* \amalg * \xrightarrow{(j_0,j_1)} \mathbb{I} \xrightarrow{c_{\mathbb{I}}} *$ *such that* $c_{\mathbb{I}}$ *is an acyclic fibration, and* j_0 *and* j_1 *are pseudo-cofibrations.*

Then $\mathcal{V}(\mathbb{I}, X)$ *is a path object for every* X *in* $\mathcal{V}$*.*

Proof Since j_0, j_1 is also a pseudo-cofibration by [21, Lemma 6.4] . For every object X in $\mathcal{V}$ the map

$$\mathcal{V}((j_0, j_1), c_X) : \mathcal{V}(\mathbb{I}, X) \to \mathcal{V}(* \amalg *, X) \times_{\mathcal{V}(*\amalg *,*)} \mathcal{V}(X, *)$$

Similarly, since j_0 is an acyclic pseudo-cofibration,

$$\mathcal{V}(j_0, c_X) : \mathcal{V}(\mathbb{I}, X) \to \mathcal{V}(*, X) \times_{\mathcal{V}(*,*)} \mathcal{V}(X, *)$$

is an acyclic fibration. Moreover, we have $\mathcal{V}(*, X) \times_{\mathcal{V}(*,*)} \mathcal{V}(X, *) \cong X$. As a result, $\mathcal{V}(\mathbb{I}, X)$ is a path object for X. □

Proposition 3.3 *If* $\mathcal{V}$ *is a cartesian closed symmetric monoidal model category, then its full subcategory of fibrant objects satisfies* (**IC**)*.*

In this case, we can choose $\mathbb{I}$ as the object that appears in the factorization $*\amalg* \xrightarrow{(j_0,j_1)} \mathbb{I} \xrightarrow{c_{\mathbb{I}}} *$, of the codiagonal for the terminal object, as a cofibration (and thus, a pseudo-cofibration) followed by an acyclic fibration. Note that the inclusions $* \to * \amalg *$ are pseudo-cofibrations, and as pseudo-cofibrations are closed under compositions [21, Lemma 6.5], j_0 and j_1 are also pseudo-cofibrations.

Theorem 3.4 (Theorem 1.1) *Suppose that* $(\mathcal{V}, \otimes, [-,-], 1)$ *is a closed symmetric monoidal category and that* $(\mathcal{V}, we_{\mathcal{V}}, fib_{\mathcal{V}})$ *is a category of fibrant objects satisfying* (**IC**)*. Let* $\mathcal{A}$ *be a subcategory of* $\mathcal{V}$ *that generates* $\mathcal{V}$ *and contains* $\mathbb{I}$*. If* $\mathcal{C}$ *is a finitely complete category that is enriched over* $\mathcal{V}$ *and powered over* $\mathcal{A}$*, then* $\mathcal{C}$ *is a category of fibrant objects in which* f *is a weak equivalence (resp. fibration) if for every* $D \in \mathcal{C}$ *the morphism* $\mathcal{C}(D, f)$ *is a weak equivalence (resp. fibration) in* $\mathcal{V}$*.*

Proof We show that $\mathcal{C}$ satisfies each axiom given in the definition of the category of fibrant objects given in Definition 2.1. Axioms **FC1** and **FC2** are superfluous and **FC3** follows from Lemma 3.1 as the pullback of a fibration (resp. acyclic fibration) along any morphism in $\mathcal{V}$ is a fibration in $\mathcal{V}$ and $\mathcal{C}(D, -)$ preserves pullbacks.

For **FC4**, without loss of generality we assume $\mathcal{A}$ contains the factorization in (**IC**). Observe that for every B in $\mathcal{C}$

$$\mathcal{C}(D, \pitchfork(\mathbb{I}, B)) \cong \mathcal{V}(\mathbb{I}, \mathcal{C}(D, B)) \cong \mathcal{P}\mathcal{C}(D, B)$$

in $\mathcal{V}$. For any X in $\mathcal{V}$ we have $\mathcal{V}(*, X) \cong X$ and $\mathcal{V}(* \amalg *, X) \cong X \times X$, so that $\mathcal{V}(c_{\mathbb{I}}, X) \cong \omega_X$ and $\mathcal{V}((j_0, j_1), X) \cong \delta_X$ (in the notation of **FC4**, Definition 2.1). Let $\mathcal{P}_{\mathcal{C}} = \pitchfork(\mathbb{I}, -)$. For an object B in $\mathcal{C}$ consider the factorization

$$B \cong \pitchfork(*, B) \xrightarrow{\pitchfork(c_{\mathbb{I}}, B)} \mathcal{P}_{\mathcal{C}}(B) \xrightarrow{\pitchfork((j_0,j_1),B)} \pitchfork(* \amalg *, B) \cong B \times B.$$

Then, for every D in $\mathcal{C}$ we have

$$\mathcal{C}(D, B) \cong \mathcal{C}(D, \pitchfork(*, B)) \xrightarrow{\omega^*} \mathcal{C}(D, \mathcal{P}_{\mathcal{C}}(B)) \xrightarrow{\delta^*} \mathcal{C}(D, B \times B) \cong \mathcal{C}(D, B) \times \mathcal{C}(D, B),$$

where $\omega^* = \mathcal{C}(D, \pitchfork(c_{\mathbb{I}}, B))$ and $\delta^* = \mathcal{C}(D, \pitchfork((j_0, j_1), B))$. Note by 2 in 3.2 that $\omega^* \cong \omega_{\mathcal{C}(D,B)}$ and $\delta^* \cong \delta_{\mathcal{C}(D,B)}$ in the notation of **FC4**, Definition 2.1 and the composition is the diagonal of $\mathcal{C}(D, B)$ in $\mathcal{V}$. Since both $we_{\mathcal{V}}$ and $fib_{\mathcal{V}}$ contains the core of $\mathcal{V}$, $\pitchfork(c_{\mathbb{I}}, B)$ composed with the isomorphism $B \cong \pitchfork(*, B)$ is a weak equivalence and $\pitchfork((j_0, j_1), B)$ composed with the isomorphism $\pitchfork(* \amalg *, B) \cong B \times B$ is a fibration in $\mathcal{C}$. Therefore, the object $\mathcal{P}B := \pitchfork(\mathbb{I}, B)$ is a path object for every $B \in \mathcal{C}$.

Every object is clearly fibrant as $\mathcal{C}(D, A) \to \mathcal{C}(D, *) \cong *$ is a fibration in $\mathcal{V}$; thus, the axiom **FC5** holds as well. As a result we obtain that $\mathcal{C}$ is a category of fibrant objects. □

The theorem can be refined as follows:

Corollary 3.5 *Let $\mathcal{V}$ and $\mathcal{C}$ be as in Theorem 1.1 and T be a collection of objects in $\mathcal{C}$. Then $\mathcal{C}$ is a category of fibrant objects in which f is a weak equivalence (resp. fibration) if for every $D \in T$ the morphism $\mathcal{C}(D, f)$ is a weak equivalence (resp. fibration) in $\mathcal{V}$.*

Proposition 3.6 *Let $\mathcal{V}$ and $\mathcal{C}$ be as in Theorem 1.1. Then for every $D \in \mathcal{C}$, $\mathcal{C}(D, -) : \mathcal{C} \to \mathcal{V}$ is exact.*

The proposition is immediate from Lemma 3.1 and by definition of weak equivalences and fibrations in $\mathcal{C}$.

Observe that we do not need to assume the monoidal structure is coherent with the homotopical structure except for the objects in the factorization of the codiagonal. Of course, assuming such a coherence would have further implications.

Corollary 3.7 *Let $\mathcal{V}$ and $\mathcal{C}$ be as in Theorem 1.1 and assume for every X in $\mathcal{V}$, two maps from $*$ to X are homotopic in $\mathcal{V}$ if an only if they are right homotopic. Then* $\mathrm{Ho}\,\mathcal{C}$ *is equivalent to* $\pi\mathcal{C}$ *and for A and B in $\mathcal{C}$* $\mathrm{Ho}\,\mathcal{C}(A, B) \cong \mathrm{Ho}\,\mathcal{V}(*, \mathcal{C}(A, B))$*. In particular, if $\mathcal{V}$ is the full subcategory of fibrant objects in a cartesian closed monoidal model category with cofibrant unit, then* $\mathrm{Ho}\,\mathcal{C}$ *is equivalent to* $\pi\mathcal{C}$ *and* $\mathrm{Ho}\,\mathcal{C}(A, B) \cong \mathrm{Ho}\,\mathcal{V}(*, \mathcal{C}(A, B))$.

Proof If $h : A \to \pitchfork(\mathbb{I}, B)$ is a right homotopy in $\mathcal{C}$ from f to g, then for any $D \in \mathcal{C}$ the induced map $\mathcal{C}(D, h) : \mathcal{C}(D, A) \to \mathcal{C}(D, \pitchfork(\mathbb{I}, B)) \cong \mathcal{V}(\mathbb{I}, \mathcal{C}(D, B))$ defines a right homotopy from $\mathcal{C}(D, f)$ to $\mathcal{C}(D, g)$. Conversely, if for every D in $\mathcal{C}$, $\mathcal{C}(D, f)$ is right homotopic to $\mathcal{C}(D, g)$ via $k_D : \mathcal{C}(D, A) \to \mathcal{V}(\mathbb{I}, \mathcal{C}(D, B))$, then define $h : A \to \pitchfork(\mathbb{I}, B)$ via the bijection

$$\mathcal{V}_0(\mathbb{I}, \mathcal{C}(A, B)) \cong \mathcal{C}_0(A, \pitchfork(\mathbb{I}, B))$$

by $h = (k_A)_0(id_A)$. Therefore, $f, g : A \to B$ in $\mathcal{C}$ are right homotopic if and only if for every $D \in \mathcal{C}$, $\mathcal{C}(D, f)$ and $\mathcal{C}(D, g)$ are right homotopic in $\mathcal{V}$.

We have $\mathcal{C}(A, B) \cong \mathcal{C}(A, \pitchfork(*, B)) \cong \mathcal{V}(*, \mathcal{C}(A, B))$. By assumption two maps $f_0, g_0 : * \to \mathcal{C}(A, B)$ are right homotopic if and only if they are homotopic, and if and only if $f_0 = g_0$ in the homotopy category of $\mathcal{V}$. Therefore, we have $\mathrm{Ho}\,\mathcal{C}$ is equivalent to $\pi\mathcal{C}$ and $\mathrm{Ho}\,\mathcal{C}(A, B) \cong \mathrm{Ho}\,\mathcal{V}(*, \mathcal{C}(A, B))$.

If $\mathcal{V}$ is the full subcategory of fibrant objects in a cartesian closed monoidal model category with cofibrant unit $*$, then the assumption in the statement above holds as $\mathcal{C}(A, B)$ is fibrant. □

Loop space objects in $\mathcal{C}$ are defined via powering, provided that the loop space objects in $\mathcal{V}$ are defined via internal hom. This happens, in particular, if the coequalizer of j_0, j_1 exists in $\mathcal{V}$.

Corollary 3.8 *Suppose that there exist S in $\mathcal{V}$ such that for any A in $\mathcal{V}$ the loop object ΩA is of the form $\mathcal{V}(S, A)$, then for every X in $\mathcal{C}$ the loop object ΩX is isomorphic to $\pitchfork(S, X)$ provided that $\mathcal{A}$ contains S.*

Proof Given the pull-back square

$$\begin{array}{ccc} \Omega A & \xrightarrow{\tilde{\iota}} & \pitchfork(\mathbb{I}, A) \\ {\scriptstyle \tilde{\delta_X}}\downarrow & & \downarrow{\scriptstyle \delta_X} \\ * & \xrightarrow{\iota} & A \times A \end{array}$$

applying $\mathcal{C}(D,-)$ we get

$$\begin{array}{ccc} \mathcal{C}(D, \Omega A) & \xrightarrow{\mathcal{C}(D,\tilde{\iota})} & \mathcal{C}(D, \pitchfork(\mathbb{I}, A)) \\ {\scriptstyle \mathcal{C}(D,\tilde{\delta_X})}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(D,\delta_X)} \\ \mathcal{C}(D,*) \cong * & \xrightarrow{\mathcal{C}(D,\iota)} & \mathcal{C}(D, A\times A) \cong \mathcal{C}(D,A) \times \mathcal{C}(D,A) \end{array}$$

which is a pullback in $\mathcal{V}$. Since $\mathcal{C}(D, \pitchfork(\mathbb{I}, A)) = \mathcal{V}(\mathbb{I}, \mathcal{C}(D,A)) \cong \mathcal{P}\mathcal{C}(D,A)$ we have $\mathcal{C}(D, \Omega A) \cong \Omega\mathcal{C}(D,A)$ for every D in $\mathcal{C}$. Moreover, we have

$$\mathcal{C}(D, \Omega A) \cong \Omega\mathcal{C}(D,A) \cong \mathcal{V}(S, \mathcal{C}(D,A)) \cong \mathcal{C}(D, \pitchfork(S,A)),$$

and thus, by Yoneda lemma $\Omega A \cong \pitchfork(S,A)$. □

If the enriching category is not a category of fibrant objects but just a prefibration category, then we have the following result.

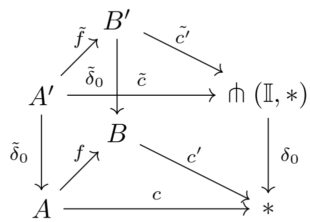
Corollary 3.9 *Suppose that* $(\mathcal{V}, \otimes, [-,-], 1)$ *is a closed symmetric monoidal category and that* $(\mathcal{V}, we_{\mathcal{V}}, fib_{\mathcal{V}})$ *is a prefibration category satisfying* (**IC**)*. Let* $\mathcal{A}$ *be a subcategory of* $\mathcal{V}$ *that generates* $\mathcal{V}$*. If* $\mathcal{C}$ *is a finitely complete category that is enriched over* $\mathcal{V}$ *and powered over* $\mathcal{A}$*, then* $\mathcal{C}$ *is a prefibration category in which* f *is a weak equivalence (resp. fibration) if for every* $D \in \mathcal{C}$ *the morphism* $\mathcal{C}(D,f)$ *is a weak equivalence (resp. fibration) in* $\mathcal{V}$*. Moreover, the homotopy category of* $\mathcal{C}$ *is equivalent to the homotopy category of* $\mathcal{C}^{fib}$*, its full subcategory of fibrant object.*

The proof of the first part is contained in the proof of Theorem 1.1. The last part follows from [26, Thm. 5.5.1 and 6.1.6]. In this particular case, since path objects are constructed functorially, for every object A in $\mathcal{C}$, $A \to *$ factorizes as $A \xrightarrow{w} A' \xrightarrow{f} *$ with $w \in we_{\mathcal{C}}$ and $f \in fib_{\mathcal{C}}$ where A' is obtained as the object fitting in the following pullback square

$$\begin{array}{ccc} A' & \xrightarrow{\tilde{c}} & \pitchfork(\mathbb{I}, *) \\ {\scriptstyle \tilde{\delta}_0}\downarrow & & \downarrow{\scriptstyle \delta_0} \\ A & \xrightarrow{c} & * \end{array}$$

with w being the section of $\tilde{\delta}_0$ (see the proof of Factorization Lemma in [5] and proof of [26, Prop. 2.1.2]). In particular, A' is uniquely determined by A. Moreover, for any map $f : A \to B$ there is an induced map

$f' : A' \to B'$ obtained in the obvious cubes of pullbacks.



In particular, we obtain a functor $R : \mathcal{C} \to \mathcal{C}^{fib}$ with $R(A) = A'$ for every A in $\mathcal{C}$, together with a natural weak equivalence (i.e. natural transformation that is componentwise weak equivalence) $w : \mathrm{id}_{\mathcal{C}} \Rightarrow R$ defined by the section of $\tilde{\delta}_0$, implying that $\mathcal{C}^{fib}$ is a right deformation retract of $\mathcal{C}$ in the sense of [9]. Consequently,

$$\mathrm{Ho}\,\mathcal{C}(A, B) \cong \mathrm{Ho}\,\mathcal{C}^{fib}(RA, RB) \cong \pi\mathcal{C}^{fib}(RA, RB).$$

3.1. Equivariant version

Let G be a group and $\mathcal{F}$ be a collection of subgroups of G. For any given category $\mathcal{D}$ denote by $G\mathcal{D}$ the functor category $[BG, \mathcal{D}]$ where BG denotes the delooping groupoid of G; i.e., $G\mathcal{D}$ is the category of G-objects and G-equivariant morphisms (i.e. natural transformations) in $\mathcal{D}$.

We first discuss a trivial (but uninteresting for our purposes) way of extending Theorem 1.1 to equivariant setting. Let $\mathcal{V}$ be as in Theorem 1.1. Assume the tensor unit in $\mathcal{V}$ is the terminal object. Suppose that for every $H \in \mathcal{F}$ and for every $X : BG \to \mathcal{C}$ in $G\mathcal{V}$, the limit $X^H := \lim_{BH} X \circ \iota_H$ exists in $G\mathcal{V}$, where $\iota_H : BH \to BG$ is the inclusion. Then, there is a immediate category of fibrant objects structure on $G\mathcal{V}$ in which a morphism f in $G\mathcal{V}$ is a weak equivalence (resp. fibration) if for every $H \in \mathcal{F}$ the induced morphism f^H is a weak equivalence (resp. fibration) in $\mathcal{V}$. This is immediate since the functors $-^H$ are defined via limits; and thus, preserve pullbacks, powers and terminal objects. This is also true for $G\mathcal{C}$ provided that relevant fixed points exist; that is, there is a category of fibrant objects structure on $G\mathcal{C}$ in which a G-morphism $f : A \to B$ is a weak equivalence (resp. fibration) if $f^H : A^H \to B^H$ is a weak equivalence (resp. a fibration).

On the other hand, there exists a more sophisticated category of fibrant object structure on $G\mathcal{C}$, which is obtained by using equivariant homotopical structure on hom-objects with conjugation actions. In terms of applications, the latter is more relevant to our interests.

Again, we let $\mathcal{V}$ be as defined in Theorem 1.1, which is also semicartesian monoidal (i.e. the tensor unit of $\mathcal{V}$ is the terminal object). We observe that for a given group G and a collection of subgroups $\mathcal{F}$, $G\mathcal{V}$ is a category of fibrant objects in which weak equivalences and fibrations are defined via fixed point objects. The category $G\mathcal{V}$ is also semicartesian symmetric monoidal where the action is defined diagonally. Since every object in the factorization given in (**IC**) has trivial G-actions, $G\mathcal{V}$ also satisfies (**IC**). Let $\mathcal{C}$ be a $\mathcal{V}$-enriched category. Since the inverse map gives an isomorphism $BG \cong BG^{\mathrm{op}}$, we have an isomorphism of categories $[BG, \mathcal{C}]^{\mathrm{op}} \cong [BG^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}]$ that sends a functor $A : BG \to \mathcal{C}$ to A^{op}. Then we can define a functor

$$\Pi^- : [BG, \mathcal{C}]^{\mathrm{op}} \times [BG, \mathcal{C}] \to [BG^{\mathrm{op}} \times BG, \mathcal{C}^{\mathrm{op}} \times \mathcal{C}]$$

which sends (A, B) to $(A^- \times B)$, where $A^- : BG^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is the object $\mathcal{A}$ with the action reversed. Let $d^- : BG \to BG^{\mathrm{op}} \times BG$ be the functor given by $g \mapsto (g^{-1}, g)$ for every morphism g in BG and let

$\mathbf{d}^-$: $[BG^{\mathrm{op}} \times BG, \mathcal{V}] \to [BG, \mathcal{V}]$ be the functor induced by d^- on functor categories. We define a functor $\overline{G\mathcal{C}} : G\mathcal{C}^{\mathrm{op}} \times G\mathcal{C} \to G\mathcal{V}$ by the following composition

$$[BG, \mathcal{C}]^{\mathrm{op}} \times [BG, \mathcal{C}] \xrightarrow{\Pi^-} [BG^{\mathrm{op}} \times BG, \mathcal{C}^{\mathrm{op}} \times \mathcal{C}] \xrightarrow{\mathcal{C}(-,-)} [BG^{\mathrm{op}} \times BG, \mathcal{V}] \xrightarrow{\mathbf{d}^-} [BG, \mathcal{V}].$$

In particular, for any two object A and B in $G\mathcal{C}$, $\overline{G\mathcal{C}}(A, B)$ is the composition

$$BG \xrightarrow{d^-} BG^{\mathrm{op}} \times BG \xrightarrow{A^{\mathrm{op}} \times B} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathcal{C}(-,-)} \mathcal{V} \quad .$$

More precisely, $\overline{G\mathcal{C}}(A, B)$ sends the unique object of BG to $\mathcal{C}(A, B)$ and sends a morphism g in BG to the composition

$$\mathcal{C}(A, B) \xrightarrow{\mathcal{C}(A(g^{-1}), B)} \mathcal{C}(A, B) \xrightarrow{\mathcal{C}(A, B(g))} \mathcal{C}(A, B) \quad .$$

In the underlying $\mathcal{S}et$-category a map $f : A \to B$ is sent to the composition $A \xrightarrow{g^{-1}} A \xrightarrow{f} B \xrightarrow{g} B$ which is in fact just the usual conjugation action, $(g \cdot f) : a \mapsto g \cdot f(g^{-1} \cdot a)$. Observe that the composition $\overline{G\mathcal{C}}(B, C) \times \overline{G\mathcal{C}}(A, B) \to \overline{G\mathcal{C}}(A, C)$ in $\mathcal{V}$ is G-equivariant where the action on the product is defined diagonally. Besides, the fixed point object $\overline{G\mathcal{C}}(A, B)^G$, which is given by the limit of over BG, is naturally isomorphic to the hom-object $G\mathcal{C}(A, B)$. We obtain that $G\mathcal{C}$ is $G\mathcal{V}$ enriched, see also [27, 3.4].

Now, for a given functor $F : I \to G\mathcal{C}$ we have a morphism

$$q : \overline{G\mathcal{C}}(D, \lim_I F) \to \lim_I \overline{G\mathcal{C}}(D, F)$$

in $G\mathcal{V}$; i.e. natural transformation between functors $BG \to \mathcal{V}$. This is because $\overline{G\mathcal{C}}(D, \lim_I F)$ is a cone over $\overline{G\mathcal{C}}(D, F)$. By Lemma 3.1, the underlying map is an isomorphism in $\mathcal{V}$. But this implies that q has an inverse in $G\mathcal{V}$, as in BG there is only one object and every morphism has an inverse. This means for every D in $\mathcal{C}$ the functor $\overline{G\mathcal{C}}(D, -)$ sends limits in $G\mathcal{C}$ to limits in $G\mathcal{V}$.

Let X be an object in $G\mathcal{A}$ with a trivial G-action (that is acts by identity morphisms in $\mathcal{V}$) and A be an object in $G\mathcal{C}$. Then define $\overline{G \pitchfork}(X, A) : BG \to \mathcal{C}$ as the functor that sends the unique object of BG to $\pitchfork(X, A)$ and a morphism $g : BG \to BG$ to the morphism $\pitchfork(X, A(g)) : \pitchfork(X, A) \to \pitchfork(X, A)$. Since powers preserve limits, we have natural isomorphisms in $\mathcal{V}$

$$\overline{G \pitchfork}(X, A^H) \cong \overline{G \pitchfork}(X, A)^H$$

for every $H \leq G$.

Theorem 3.10 (Theorem 1.2) *Let G be a group and $\mathcal{F}$ be a collection of subgroups of G. Let $\mathcal{V}$ and $\mathcal{C}$ be as defined in Theorem 1.1 such that for every $H \in \mathcal{F}$ limits defining H-fixed points exists in $\mathcal{V}$. Then $G\mathcal{C}$ is a category of fibrant objects in which f is a weak equivalence (resp. fibration) if for every $D \in G\mathcal{C}$ and $H \in \mathcal{F}$ the morphism $\overline{G\mathcal{C}}(D, f)^H$ is a weak equivalence (resp. fibration) in $\mathcal{V}$.*

Equivariant versions of Proposition 3.6, Corollary 3.7, Corollary 3.8, and Corollary 3.9 can be obtained in the same way.

The first mentioned category of fibrant objects structure on $G\mathcal{C}$, in which f is a weak equivalence (fibration) if f^H is so for every $H \in \mathcal{F}$, is usually not homotopically equivalent to the one given in Theorem 1.2. In fact, a weak equivalence in the latter is already a weak equivalence in the former, but not vice versa. Let $f : A \to B$ be a weak equivalence in the category of fibrant objects of Theorem 1.2; i.e.

$$\overline{G\mathcal{C}}(D, f)^H : \overline{G\mathcal{C}}(D, A)^H \to \overline{G\mathcal{C}}(D, B)^H$$

is a weak homotopy equivalence for every $H \leq G$ and for every $D \in G\mathcal{C}$. In particular, this is true for all such D on which G acts trivially. In this case, we have

$$\overline{G\mathcal{C}}(D, A)^H \cong \mathcal{C}(D, A)^H \cong \mathcal{C}(D, A^H).$$

Hence, f is a weak equivalence in the former category of fibrant objects. On the other hand, it is easy to find some A in $G\mathcal{C}$ with nontrivial action so that $\overline{G\mathcal{C}}(D, A)$ is not equivariantly weak equivalent to $\mathcal{C}(D, A)$ with the action on $\mathcal{C}(D, A)$ defined in the covariant side; i.e. for $f \in \mathcal{C}(D, A)$, $g \in G$ and for every $d \in D$ $(g \cdot f)(d) := g \cdot f(d)$. In fact, we can choose D with a nontrivial action so that $\overline{G\mathcal{C}}(D, A)$ is not weak equivalent to $\overline{G\mathcal{C}}(D, A)^H$ for some $H \leq G$ and A with the trivial action to obtain $\overline{G\mathcal{C}}(D, A)$ and $\mathcal{C}(D, A)$ are not equivariantly weak equivalent. Thus, the two fibration categories present different homotopy theories.

3.1.1. A remark on actions of group objects in $\mathcal{V}$

Let $\mathcal{V}$ and $\mathcal{C}$ be as in Theorem 1.1. We assume for convenience that $\mathcal{V}$ is cartesian monoidal. Let G be a Hopf group (i.e. a group internal to $\mathcal{V}$), see [14]. Then G is $\mathcal{V}$-enriched and following [14, 2.2, 2.3] we let $G\mathcal{C}$ be the category of G-objects in $\mathcal{C}$; that is, the category of $\mathcal{V}$-enriched functors $BG \to \mathcal{C}$. Let $\mathcal{F}$ be a set of closed subgroups of G, see [14, Sec. 6.1 and 6.2]. Again, being limits, fixed point functors commute with limits; in particular, pullbacks, powers and terminal objects. Therefore, $G\mathcal{V}$ is a category of fibrant objects as in Section 3.1, where weak equivalences and fibrations are created by fixed point functors. The functor $\overline{G\mathcal{C}}(D, -) : G\mathcal{C} \to G\mathcal{V}$ can be defined in the same way as above. Then, $\mathcal{V}$-group version of Theorem 1.2 follows.

4. Examples

4.1. Operator algebras

Our first example, which is the motivating example of the present paper, is the category of C^*-algebras, which was first given in [30]. These are particularly interesting as it is known that with the equivalences and fibrations induced by representable functors (as in our main theorems) do not come from a model structures for C^*-algebras, see [30, Appendix].

Notation 4.1 *The category of Banach spaces and contractions (i.e. bounded linear operators with norm at most* 1*) is denoted by* $\mathcal{B}an$*. This is a closed symmetric monoidal category with the projective tensor product. A Banach algebra is a monoid in this symmetric monoidal category. We denote by* $\mathcal{B}an\mathcal{A}lg$ *the category of monoids in* $\mathcal{B}an$*; that is, the subcategory of* $\mathcal{B}an$ *spanned by Banach algebras with morphisms as contractions that preserve the algebra structure. The category of* C^**-algebras and* $*$*-homomorphisms is denoted by* $C^*\mathcal{A}lg$*. Since every* $*$*-homomorphisms is already a contraction, see, e.g., the unpublished notes by Putnam*[†]*,* $C^*\mathcal{A}lg$ *can*

[†]Putnam IF (2019). Lecture notes on C^*-algebras [online]. Website https://web.uvic.ca/~ifputnam/ln/C*-algebras.pdf [accessed 22 April 2024].

be considered a subcategory of $\mathcal{B}an\mathcal{A}lg$. In particular, we have inclusions $C^*\mathcal{A}lg \to \mathcal{B}an\mathcal{A}lg \to \mathcal{B}an$. We denote the cartesian closed symmetric monoidal category of compactly generated and weakly Hausdorff topological spaces by $\mathcal{T}op$ and the category of compact Hausdorff spaces by $\mathcal{C}omp$. We denote by $s\mathcal{S}et$ the category of simplicial sets.

Let $\mathcal{C}$ be either of $\mathcal{B}an$, $\mathcal{B}an\mathcal{A}lg$, and $C^*\mathcal{A}lg$. Being normed, every object in $\mathcal{C}$ is a compactly generated weakly Hausdorff topological space with the norm topology. Hence, we can give the set $\mathcal{C}(A,B)$ with a topology by using the subspace topology of the compact open topology, which makes $\mathcal{C}(A,B)$ an object in $\mathcal{T}op$. It is straightforward that this topology is coherent with the composition, which is defined in the same way as in $\mathcal{T}op$, making $\mathcal{C}$ a $\mathcal{T}op$-enriched category.

For any given compact Hausdorff space X and a Banach space B, the space of continuous functions $\mathcal{T}op(X,B)$, by considering B with the norm topology, is a Banach space. In fact, the compact open topology on $\mathcal{T}op(X,B)$ coincides with the topology given by the sup-norm. If B is a Banach algebra, then there is a naturally induced Banach algebra structure on $\mathcal{T}op(X,B)$ and if further B is a C^*-algebra, then so does $\mathcal{T}op(X,B)$ (see also [30, 2.2]).

Since $\mathcal{T}op$ is cartesian closed we have a natural isomorphism

$$\mathcal{T}op(A, \mathcal{T}op(X,B)) \cong \mathcal{T}op(A \times X, B) \cong \mathcal{T}op(X, \mathcal{T}op(A,B)).$$

If A and B are Banach spaces, the above natural isomorphism induces a natural isomorphism

$$\mathcal{B}an(A, \mathcal{T}op(X,B)) \cong \mathcal{T}op(X, \mathcal{B}an(A,B)).$$

Similarly, we have natural isomorphisms

$$\mathcal{B}an\mathcal{A}lg(A, \mathcal{T}op(X,B)) \cong \mathcal{T}op(X, \mathcal{B}an\mathcal{A}lg(A,B))$$

and

$$C^*\mathcal{A}lg(A, \mathcal{T}op(X,B)) \cong \mathcal{T}op(X, C^*\mathcal{A}lg(A,B)).$$

The last one also appears in [30, Sec. 2.4]. This implies that $\mathcal{C}$ is powered over $\mathcal{C}omp$ where the powering is given by the set of continuous functions equipped with the relevant extra structure. In the category $\mathcal{T}op$ every object generated by compact Hausdorff spaces. Besides, $\mathcal{T}op$ is a monoidal model category with the Quillen model structure in which every object is fibrant. The path space object is given by the usual path space, which is given by the internal hom $\mathcal{T}op([0,1],-)$ in $\mathcal{T}op$. Note also that the interval $I=[0,1]$ fits into the obvious factorization $* \amalg * \to [0,1] \to *$ and each inclusion $* \to [0,1]$ are cofibrations (and hence, pseudo-cofibrations) and $[0,1] \to *$ is an acyclic fibration. Thus, by Theorem 1.1, we have the following:

Corollary 4.2 *Let $\mathcal{C}$ be either of $\mathcal{B}an$, $\mathcal{B}an\mathcal{A}lg$, $C^*\mathcal{A}lg$. Then $\mathcal{C}$ is a category of fibrant objects in which $f \in \mathcal{C}$ is a weak equivalence (fibration) if $\mathcal{C}(X,f)$ is a weak homotopy equivalence (resp. Serre fibration) for every $X \in \mathcal{C}$.*

The case $\mathcal{C} = C^*\mathcal{A}lg$ is already well-known, see [30], [28] and the cases $\mathcal{C} = \mathcal{B}an$ and $\mathcal{C} = \mathcal{B}an\mathcal{A}lg$ also follow from the unpublished preprint "A Baues fibration category structure on Banach and C^*-algebras" by Andersen and Grodal [‡].

[‡]Andersen KKS, Grodal J (1997). A Baues fibration category structure on Banach and C^*-algebras [online]. Website `https://web.math.ku.dk/~jg/papers/fibcat.pdf` [accessed 22 April 2024].

Recall that a functor between fibration categories is exact if it preserves fibrations, acyclic fibrations, terminal objects and pullbacks along fibrations. The following corollary is immediate from the definition:

Corollary 4.3 *The inclusions $C^*\mathcal{A}lg \hookrightarrow \mathcal{B}an\mathcal{A}lg \hookrightarrow \mathcal{B}an$ are exact.*

The corollary implies that we have inclusions

$$\operatorname{Ho} C^*\mathcal{A}lg \hookrightarrow \operatorname{Ho} \mathcal{B}an\mathcal{A}lg \hookrightarrow \operatorname{Ho} \mathcal{B}an.$$

4.2. Equivariant operator algebras

Let G be a group. Let again $\mathcal{C}$ be either of $\mathcal{B}an$, $\mathcal{B}an\mathcal{A}lg$ or $C^*\mathcal{A}lg$. Recall that $G\mathcal{C}$ denotes the category of G-objects in $\mathcal{C}$; i.e. the functor category $[BG, \mathcal{C}]$ where BG denotes the delooping groupoid of G.

Let $\mathcal{F}$ be a collection of subgroups of G. Then, from Theorem 1.2 combined with the nonequivariant case as discussed in the previous section, we obtain the the following corollary:

Corollary 4.4 *Let $\mathcal{C}$ be either of $\mathcal{B}an$, $\mathcal{B}an\mathcal{A}lg$, $C^*\mathcal{A}lg$. Then $G\mathcal{C}$ is a category of fibrant objects where f in $G\mathcal{C}$ is a weak equivalences (resp. fibration) if for every $D \in G\mathcal{C}$ and $H \leq G$ $\overline{G\mathcal{C}}(D, f)^H$ is a weak homotopy equivalence (resp. Serre fibration).*

As before, the following corollary is immediate.

Corollary 4.5 *The inclusions $GC^*\mathcal{A}lg \hookrightarrow G\mathcal{B}an\mathcal{A}lg \hookrightarrow G\mathcal{B}an$ are exact.*

Remark 4.6 *Using remarks of subsection 3.1.1, one can take G here to be a topological group and $\mathcal{F}$ be a collection of closed subgroups of G.*

Let $CC^*\mathcal{A}lg$ denote the full subcategory of commutative C^*-algebras. Note that the inclusion $CC^*\mathcal{A}lg \to \mathcal{B}an$ admits a left adjoint $\mathcal{B}an \to CC^*\mathcal{A}lg$ by the Adjoint functor theorem (see also [17, Thm. 11]). Moreover, by definition the loop object $\Omega : \mathcal{B}an \to \mathcal{B}an$ as powering by $\mathbb{S}^1$ restrict to $\Omega : CC^*\mathcal{A}lg \to CC^*\mathcal{A}lg$, which is also defined as powering by $\mathbb{S}^1$. By combining the exactness of the inclusion, if $\operatorname{Ho}\Omega : \operatorname{Ho}\mathcal{B}an \to \operatorname{Ho}\mathcal{B}an$ admits a left adjoint, then so does $\operatorname{Ho}\Omega : \operatorname{Ho} CC^*\mathcal{A}lg \to \operatorname{Ho} CC^*\mathcal{A}lg$ (see also [30, Corollary A.4]). But it is due to [30, Corollary A.5] that such a left adjoinf for the latter does not exist. Thus, the category of fibrant objects $\mathcal{B}an$ is not a full subcategory of fibrant objects in a model category. Same is true for the category of fibrant object $\mathcal{B}an\mathcal{A}lg$.

4.3. Metric spaces

Let $\mathcal{M}et$ denote the category of metric spaces and short maps (i.e. maps that do not increase distance). This category has finite limits. The terminal object is the point with the trivial metric. If $f : (A, d_A) \to (B, d_B)$ and $g : (C, d_C) \to (B, d_B)$ are short maps, then their pullback is the usual pullback in the category of sets with the sup-metric; that is,

$$A \times_B C = \{(a, c) \in A \times C \ : \ f(a) = g(c)\}$$

with the metric

$$d_{f,g}((a, c), (a', c')) = \max\{d_A(a, a'), d_A(c, c')\}.$$

Any metric space with the metric topology is first-countable. Thus, there is a forgetful functor $F : \mathcal{M}et \to \mathcal{T}op$ sending a metric space to the underlying topological space with the metric topology. For all metric spaces A and B, $\mathcal{M}et(A,B)$ is a closed subspace of $\mathcal{T}op(FA,FB)$. We can define a $\mathcal{T}op$-enrichment by inducing the subspace topology on the subset $\mathcal{M}et(A,B)$ from $\mathcal{T}op(FA,FB)$. The composition

$$\mathcal{M}et(B,C) \times \mathcal{M}et(A,B) \to \mathcal{M}et(A,C)$$

is the restriction of the composition in $\mathcal{T}op$ and hence is continuous. Besides, just as in the case of Banach spaces, $\mathcal{M}et$ is powered over compact Hausdorff spaces. If X is in $\mathcal{C}omp$ and A is in $\mathcal{M}et$, then define $\pitchfork(X,A) = C_0(X,FA)$, the set of continuous functions from X to A endowed with the sup-metric; $d(f,g) = \sup_{x\in X} d(f(x),g(x))$. Then for every X in $\mathcal{T}op$ and A,B in $\mathcal{M}et$, the isomorphism $\mathcal{T}op(X,\mathcal{T}op(FA,FB)) \cong \mathcal{T}op(FA,\mathcal{T}op(X,FB))$ in $\mathcal{T}op$ (note that $\mathcal{T}op$ is cartesian closed) gives rise to an isomorphism $\theta : \mathcal{T}op(X,\mathcal{M}et(A,B)) \leftrightarrow \mathcal{M}et(A,\pitchfork(X,B)) : \psi$. In fact, if $g : A \to \pitchfork(X,B)$ is a short map, then for all $x \in X$ and $a_1,a_2 \in A$

$$d(\theta(f)(x)(a_1),\theta(f)(x)(a_2)) = d(f(a_1)(x),f(a_2)(x)) \le \sup_{y\in X} d(f(a_1)(y),f(a_2)(y)) = d(f(a_1),f(a_2)) \le d(a_1,a_2).$$

Here we use d for all metrics but it is clear from the context that which metric space it is associated to. The last equality follows from shortness of f. Similarly, if $g : X \to \mathcal{M}et(A,B)$ is in $\mathcal{T}op(X,\mathcal{M}et(A,B))$, then

$$d(\psi(g)(a_1),\psi(g)(a_2)) = \sup_{x\in X} d(\psi(g)(a_1)(x),\psi(g)(a_2)(x)) = \sup_{x\in X} d(g(x)(a_1),g(x)(a_2)) \le d(a_1,a_2),$$

where the last equality follows due to shortness of $g(x)$ for all $x \in X$. Continuity of these maps follows due to the fact that $\mathcal{M}et(A,B)$ is closed in $\mathcal{T}op(FA,FB)$. Therefore, Theorem 1.1 applies to $\mathcal{M}et$ as well. Since $\mathcal{M}et$ lacks finite colimits (e.g., coproducts), it cannot be a model category.

For a group G, the G-equivariant version also holds for $G\mathcal{M}et$ (similar to the previous section) since H-fixed points exists in $\mathcal{M}et$ for all $H \le G$.

4.4. Algebras and G-algebras over a ring

Another example is the category of associative algebras over a ring R. An associative unital algebra is a monoid in the category $R\mathcal{M}od$ of R-modules. The category $\mathcal{A}lg_R$ of R-algebras is enriched over simplicial sets where the hom-object is given by the functor

$$\mathcal{A}lg_R^* : \mathcal{A}lg_R^{\mathrm{op}} \times \mathcal{A}lg_R \to s\mathcal{S}et$$

defined as $\mathcal{A}lg_R^*(A,B)_n = \hom_{\mathcal{A}lg_R}(A, B \otimes \mathbb{Z}^{\Delta^n})$ where

$$\mathbb{Z}^{\Delta^n} = \mathbb{Z}[x_0,\cdots,x_n]/\langle 1 - \sum_i x_i\rangle,$$

see [7, 3.1 (14)]. Besides, $\mathcal{A}lg_R$ is powered over simplexes as

$$\pitchfork(\Delta^n,A) = A \otimes \mathbb{Z}^{\Delta^n}$$

where A is an R-algebra. The category $s\mathcal{S}et$ is a cartesian closed monoidal model category with the Kan–Quillen model structure. However, not every object is fibrant. Therefore, by considering the underlying prefibration category structure, we apply Corollary 3.9 to $\mathcal{A}lg_R$.

Corollary 4.7 *The category $\mathcal{A}lg_R$ is a prefibration category of in which a homomorphism f is a weak equivalence (resp. fibration) if for every $X \in \mathcal{A}lg_R$, $\mathcal{A}lg_R(X, f)$ is a weak equivalence (resp. a Kan fibration) of simplicial sets in the Kan-Quillen model structure.*

An R-algebra that is fibrant in this prefibration category will be called a fibrant R-algebra, and the full-subcategory of fibrant R-algebras is denoted by $\mathcal{A}lg_R^{fib}$. Note that an R-algebra A is fibrant if $\mathcal{A}lg_R(D, A)$ is a Kan complex for every R-algerba R. Recall that $\mathcal{A}lg_R^{fib}$ is a category of fibrant objects [26, Prop. 2.1.2]. Moreover, $\mathcal{A}lg_R^{fib} \hookrightarrow \mathcal{A}lg_R$ induces an equivalence on homotopy categories.

For any group G and a collection $\mathcal{F}$ of subgroups of G, the G-equivariant version of Corollary 3.9 applies to $G\mathcal{A}lg_R$.

Corollary 4.8 *The category $G\mathcal{A}lg_R$ is a prefibration category in which a G-homomorphism f is a weak equivalence (resp. fibration) if for every $D \in G\mathcal{A}lg_R$ and $H \in \mathcal{F}$, $G\mathcal{A}lg_R(X, f)^H$ is a weak equivalence (resp. Kan fibration) of simplicial sets.*

A G-R-algebra is fibrant if its underlying R-algebra is in the prefibration category given in the previous corollary. Again, due to [26, Thm. 5.5.1 and 6.1.6] the inclusion of its full subcategory of fibrant objects in $G\mathcal{A}lg_R$ induces an equivalence on homotopy categories.

5. Some applications

The main applications we focus on are triangulated structures for the (equivariant) KK-theory and E-theory. We recover some known results bu using the category of fibrant objects structures that we have developed.

5.1. On equivariant KK-theory

For convenience, in this section, every C^*-algebra is assumed to be separable. Every G-Banach space with the norm topology is a G-anr, hence has the homotopy type of a G-CW-complex, see [3, Thm. 2.1 and 2.4, pp 10-11]. This, in particular, implies that a weak equivalence in $GC^*\mathcal{A}lg$ with the fibration category structure of Corollary 4.4 for $\mathcal{F}$ the collection of all closed subgroups of G is a homotopy equivalence.

Let $\mathcal{K} = \mathcal{K}(\ell^2(G \times \mathbb{N}))$. We follow equivariant version of Cuntz's picture, which is given by [25, pp. 68]. Let qA be the kernel of the codiagonal on A in $GC^*\mathcal{A}lg$; i.e. the limit of $A \star A \to A \leftarrow 0$ in $GC^*\mathcal{A}lg$, see [19, 5.2]. The G-equivariant KK-theory group $KK_G(A, B)$ is defined as the set of G-homotopy classes of G-$*$-homomorphisms from $q(A \otimes \mathcal{K})$ to $B \otimes \mathcal{K}$ [22, Thm. 4.3] (see also [8, 16, 19]). We have $- \otimes \mathcal{K}$ preserve pullbacks since this is true nonequivariantly and since limits in $GC^*\mathcal{A}lg$ are defined in the ambient category $C^*\mathcal{A}lg$. Recall that $GC^*\mathcal{A}lg$ is a category of fibrant objects in which $f \in GC^*\mathcal{A}lg$ is a weak equivalences (resp. fibration) if for every $D \in \mathcal{C}$ and $H \leq G$ the map $\overline{G\mathcal{C}}(D, f)^H$ is a weak homotopy equivalence (resp. Serre fibration) in $\mathcal{T}op$. Then, one defines a category of fibrant objects structure on $GC^*\mathcal{A}lg$ with the same fibrations and with weak equivalences being those maps that are sent to isomorphisms by the composition

$$GC^*\mathcal{A}lg \xrightarrow{Q} \operatorname{Ho} GC^*\mathcal{A}lg \xrightarrow{[qD, -\otimes\mathcal{K}]} \operatorname{Ho} GC^*\mathcal{A}lg,$$

where Q denotes the localization. Denote by $\operatorname{Ho}(G\underline{KK})$ the homotopy category of this category of fibrant objects. Then, $KK^G(A, B) := \operatorname{Ho}(G\underline{KK})(A, B)$. Moreover, due to stability of equivariant KK-theory with respect to $\mathcal{K}$, we obtain that the category $\mathrm{Ho}(G\underline{KK})$ is a triangulated category. This also recovers that equivariant KK-theory is triangulated (see [20, Appendix A]). See also Section 2 of [6].

Repeating similar arguments for Banach algebras and G-Banach algebras, one can also recover the analogues results of [23, Thm. 4.15] and [24, Thm. 2.1] for Banach Algebras. Note that the homotopy of maps of Banach algebras defined in *loc. cit.* is the same as the right homotopy for the category of fibrant objects of Corollaries 4.2 and 4.4, respectively.

5.2. On equivariant E-theory

The equivariant E-theory for separable G-C^*-algebras is well studied. For the definition and details we refer to [13, Ch. 6]. By using the last example of the previous section, we can give a definition of equivariant E-theory groups as hom-spaces in the stable homotopy category of a category of fibrant objects.

Then, $GC^*\mathcal{A}lg$ admits a structure of a category of fibrant objects where f is a weak equivalence (resp. a fibration) if $f \otimes \mathcal{K}(\mathcal{H})$ is a weak equivalence (resp. a fibration) in $GC^*\mathcal{A}lg$ with the structure of the category of fibrant objects from Corollary 4.8. This is the equivariant analogue of [30, Proposition 2.24]. Denote this new category of fibrant objects by $GC^*\mathcal{A}lg^{\mathcal{K}(\mathcal{H})}$. Following the definition of equivariant E-theory in [13, Ch. 6], we see that $E_G(A,B) = \mathrm{Ho}(GC^*\mathcal{A}lg^{\mathcal{K}(\mathcal{H})})(\Omega A \otimes \mathcal{K}(\mathcal{H}), \Omega B)$; i.e. the equivariant E-theory can be obtained as a hom-set in the stable homotopy category of a category of fibrant objects $GC^*\mathcal{A}lg^{\mathcal{K}(\mathcal{H})}$.

5.3. On (equivariant) bivariant algebraic k-theory

We can also recover that the kk-category as defined in [7, Sec. 4] and its equivariant version in [10] are triangulated categories obtained as stabilizations of categories of fibrant objects given in subsection 4.4. Most of the material given here are present in [7], except the fibration category structures.

First observe that the notion of elementary homotopy for R-algebras as given in [7, Def. 3.1.1] coincides with the right homotopy for fibrant objects in the prefibration category given in Corollary 4.7, and same is true for the homotopy classes of homomorphisms. Moreover, due to the prefibration category structure on $\mathcal{A}lg_R$ as given in Corollary 4.7, we can put a bound on the length of the zigzags that appear in the definition of the homotopy in [7, Def. 3.1.1]. In fact, between fibrant R-algebras, notions of elementary homotopy and homotopy coincides due to Corollary 3.7 (note also that $s\mathcal{S}et$ is a cartesian closed monoidal model category with Kan–Quillen model structure).

In [7] bivariant k-theory kk for R-algebras is defined and it is shown (in Cor. 6.3.4 *loc. cit.*) that for every D in $\mathcal{A}lg_R$, the functor $kk(D,-)$ is homological. Then, the prefibration category structure on $\mathcal{A}lg_R$ induces a prefibration category structure on the quotient category $\underline{kk} := \mathcal{A}lg_R / \underset{kk}{\sim}$ where $f \underset{kk}{\sim} g$ in $\mathcal{A}lg_R$ if $kk(D,f) = kk(D,g)$ for every D in $\mathcal{A}lg_R$. Note that the loop objects in $\underline{kk}$ and $\mathcal{A}lg_R$ coincides and it is also shown therein that $\underline{kk}(A,B) \cong \underline{kk}(\Omega A, \Omega B)$. Therefore, $\mathrm{StHo}\,\underline{kk} \cong \mathrm{Ho}\,\underline{kk}$. Being the stable homotopy category of a category of fibrant objects, $\mathrm{Ho}\,\underline{kk}$ is triangulated. This gives [7, 6.2.4].

The equivariant version is given in [10], and it can be shown in the same way that equivariant kk-theory is also obtained as hom-space in the stable homotopy category of a category of fibrant objects.

References

[1] Adámek J, Brooke-Taylor AD, Campion T, Positselski L, Rosický J. Colimit-dense subcategories. Commentationes Mathematicae Universitatis Carolinae 2019; 60 (4): 447-462. http://dx.doi.org/10.14712/1213-7243.2019.021

[2] Adámek J, Rosický J. Locally Presentable and Accessible Categories. Cambridge, UK: Cambridge University Press, 1994. https://doi.org/10.1017/CBO9780511600579

[3] Bartsch T. Topological Methods for Variational Problems with Symmetries. Berlin, Germany: Springer-Verlag, 2006. https://doi.org/10.1007/BFb0073859

[4] Baues HJ. Algebraic Homotopy. Cambridge, UK: Cambridge University Press, 1989. https://doi.org/10.1017/CBO9780511662522

[5] Brown KS. Abstract homotopy theory and generalized sheaf cohomology. Transactions of the American Mathematical Society 1973; 186: 419-458. https://doi.org/10.2307/1996573

[6] Bunke U, Engel A, Land M. A stable ∞-category for equivariant KK-theory. arXiv preprint 2021; arXiv:2102.13372. https://doi.org/10.48550/arXiv.2102.13372

[7] Cortiñas G, Thom A. Bivariant algebraic K-theory. Journal für die reine und angewandte Mathematik 2007; 2007 (610): 71-123. https://doi.org/10.1515/CRELLE.2007.068

[8] Cuntz J. A new look at KK-theory. K-Theory 1987; 1 (1): 31-51. http://dx.doi.org/10.1007/BF00533986

[9] Dwyer WG, Hirschhorn PS, Kan DM, Smith JH. Homotopy limit functors on model categories and homotopical categories. Providence, RI, USA: American Mathematical Society, 2004. https://doi.org/10.1090/surv/113

[10] Ellis E. Equivariant algebraic kk-theory and adjointness theorems. Journal of Algebra 2014; 398: 200-226. https://doi.org/10.1016/j.jalgebra.2013.09.023

[11] Fajstrup L, Rosicky J. A convenient category for directed homotopy. Theory and Applications of Categories 2008; 21 (1): 7-20.

[12] Foltz F, Lair C, Kelly GM. Algebraic categories with few monoidal biclosed structures or none. Journal of Pure and Applied Algebra 2008; 17 (2): 171-177. https://doi.org/10.1016/0022-4049(80)90082-1

[13] Guentner E, Higson N, Trout J. Equivariant E-Theory for C^*-Algebras. Memoirs of the American Mathematical Society 2000; 148 (703). https://doi.org/10.1090/memo/0703

[14] Guillou B, May JP, Rubin J. Enriched model categories in equivariant contexts. Homology, Homotopy and Applications 2010; 12 (2): 1-35. https://dx.doi.org/10.4310/HHA.2019.v21.n1.a10

[15] Heller A. Stable homotopy categories. Bulletin of the American Mathematical Society 1968; 74 (1): 28-63. https://doi.org/10.1090/S0002-9904-1968-11871-3

[16] Higson N. Bivariant K-theory and the Novikov conjecture. Geometric and Functional Analysis 2000; 10 (3): 563-581. https://doi.org/10.1007/PL00001630

[17] Lieberman M, Rosický J, Vasey S. Hilbert spaces and C^*-algebras are not finitely concrete. Journal of Pure and Applied Algebra 2023; 227 (4). https://doi.org/10.1016/j.jpaa.2022.107245

[18] Livernet M, Whitehouse S. Homotopy theory of spectral sequences. Homology, Homotopy and Applications 2024; 26 (1): 69-86. https://dx.doi.org/10.4310/HHA.2024.v26.n1.a5

[19] Meyer R. Equivariant Kasparov Theory and Generalized Homomorphisms. K-Theory 2000; 21 (3): 201-228. http://dx.doi.org/10.1023/A:1026536332122

[20] Meyer M, Nest R. The Baum–Connes conjecture via localisation of categories. Topology 2006; 45 (2): 209-259. https://doi.org/10.1016/j.top.2005.07.001

[21] Muro F. Dwyer–Kan homotopy theory of enriched categories. Journal of Topology 2015; 8 (2): 377-413. https://doi.org/10.1112/jtopol/jtu029

[22] Nest R, Voigt C. Equivariant Poincaré duality for quantum group actions. Journal of Functional Analysis 2010; 258 (5): 1466-1503. https://doi.org/10.1016/j.jfa.2009.10.015

[23] Paravicini W. kk-Theory for Banach algebras I: the non-equivariant case. Journal of Functional Analysis 2015, 268 (10): 3108-3161. https://doi.org/10.1016/j.jfa.2015.03.009

[24] Paravicini W. kk-Theory for Banach algebras II: equivariance and Green–Julg type theorems. Journal of Functional Analysis 2015; 268 (10): 3162-3210. https://doi.org/10.1016/j.jfa.2015.03.010

[25] Phillips NC. Equivariant K-Theory and Freeness of Group Actions on C^*-Algebras. Lecture Notes in Mathematics (LNM, volume 1274). Berlin, Germany: Springer-Verlag, 1987. https://doi.org/10.1007/BFb0078657

[26] Radulescu-Banu A. Cofibrations in homotopy theory. arXiv Preprint 2009; arXiv:math/0610009. https://doi.org/10.48550/arXiv.math/0610009

[27] Riehl E. Categorical homotopy theory. Cambridge, UK: Cambridge University Press, 2014. https://doi.org/10.1017/CBO9781107261457

[28] Schochet C. Topological methods for C^*-algebras. III. Axiomatic homology. Pacific Journal of Mathematics 1984; 96 (1): 193-211.

[29] Szumilo K. Two models for the homotopy theory of cocomplete homotopy theories. PhD, University of Bonn, Bonn, Germany, 2014.

[30] Uuye O. Homotopical algebra for C^*-algebras. Journal of Noncommutative Geometry 2013; 7 (4): 981-1006. https://doi.org/10.4171/JNCG/141