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Coefficient problems for a certain subclass of analytic and univalent functions

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Abstract: In the present work, some new subclasses of analytic and univalent functions are introduced and some geometric properties such as coefficient estimates problem are studied for them. Furthermore, we show that our results are generalization for some earlier work in the literature and we show this by comparing ours with those related.

Key words: $\delta-$close-to-starlike, $\delta-$close-to-convex, $\delta-$quasistarlike, $\delta-$quasiconvex, symmetric point, complex order

1. Introduction and preliminaries

Let A be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ given by the following series expansions:

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots + a_n z^n + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \; a_n \in \mathbb{C}. \tag{1.1}$$

The class A is known as the class of normalized functions with normalization conditions $f(0) = 0$ and $f'(0) = 1$ in the literature. The subclass of all univalent functions of A is denoted by S (see [16]). Many mathematicians were interested in coefficient estimates for this class. Within a short period, in 1916, Bieberbach [8] published a paper in which the famous coefficient hypothesis was proposed. There were a lot of papers devoted to this conjecture and its related coefficient problems (see [5–7, 9–11, 13, 15, 17–23, 25, 26]).

It is well known that the starlike and convex function classes defined on the open unit disk U are defined analytically as follows:

$$S^* = \left\{ f \in S : \; Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \; z \in U \right\} \text{ and } C = \left\{ f \in S : \; Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, \; z \in U \right\}.$$

Now let us define some subclasses of S.

Definition 1.1 *For $\beta \in (0,1]$, $\delta \in (0,1]$, and $\tau \in \mathbb{C} - \{0\}$, the function $f \in S$ is said to be in the class $S^*(\delta, \beta, \tau)$, which we will call $\delta-$starlike function class with respect to symmetric points of complex order $\tau\,(\tau \in \mathbb{C} - \{0\})$, if the following condition is satisfied:*

$$\left| \frac{1}{\tau} \left[\frac{2\left[(1-\delta) f(z) + \delta z f'(z)\right]}{f(z) - f(-z)} - 1 \right] \right| < \beta, z \in U;$$

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that is,

$$S^*(\delta,\beta,\tau)=\left\{f\in S:\ \left|\frac{1}{\tau}\left[\frac{2\left[(1-\delta)f(z)+\delta zf'(z)\right]}{f(z)-f(-z)}-1\right]\right|<\beta, z\in U\right\}.$$

Definition 1.2 *For $\beta\in(0,1]$, $\delta\in(0,1]$, and $\tau\in\mathbb{C}-\{0\}$, the function $f\in S$ is said to be in the class $C(\delta,\beta,\tau)$, which we will call $\delta-$convex function class with respect to symmetric points of complex order $\tau(\tau\in\mathbb{C}-\{0\})$, if the following condition is satisfied:*

$$\left|\frac{1}{\tau}\left[\frac{2\left[(1-\delta)f(z)+\delta zf'(z)\right]'}{(f(z)-f(-z))'}-1\right]\right|<\beta, z\in U;$$

that is,

$$C(\delta,\beta,\tau)=\left\{f\in S:\ \left|\frac{1}{\tau}\left[\frac{2\left[(1-\delta)f(z)+\delta zf'(z)\right]'}{(f(z)-f(-z))'}-1\right]\right|<\beta, z\in U\right\}.$$

Definition 1.3 *For $\beta\in(0,1]$, $\delta\in(0,1]$, and $\tau\in\mathbb{C}-\{0\}$, the function $f\in S$ is said to be in the class $KS^*(\delta,\beta,\tau)$, which we will call $\delta-$close-to-starlike function class with respect to symmetric points of complex order $\tau(\tau\in\mathbb{C}-\{0\})$, if the following condition is satisfied:*

$$\left|\frac{1}{\tau}\left(\frac{2f(z)}{g(z)-g(-z)}-1\right)\right|<\beta, z\in U$$

that is,

$$KS^*(\delta,\beta,\tau)=\left\{f\in S:\ \left|\frac{1}{\tau}\left(\frac{2f(z)}{g(z)-g(-z)}-1\right)\right|<\beta, z\in U\right\}, g\in S^*(\delta,\beta,\tau).$$

Definition 1.4 *For $\beta\in(0,1]$, $\delta\in(0,1]$, and $\tau\in\mathbb{C}-\{0\}$, the function $f\in S$ is said to be in the class $KC(\delta,\beta,\tau)$, which we will call $\delta-$close-to-convex function class with respect to symmetric points of complex order $\tau(\tau\in\mathbb{C}-\{0\})$, if the following condition is satisfied:*

$$\left|\frac{1}{\tau}\left[\frac{2f'(z)}{(g(z)-g(-z))'}-1\right]\right|<\beta, z\in U;$$

that is,

$$KC(\delta,\beta,\tau)=\left\{f\in S:\ \left|\frac{1}{\tau}\left[\frac{2f'(z)}{(g(z)-g(-z))'}-1\right]\right|<\beta, z\in U\right\}, g\in C(\delta,\beta,\tau).$$

Definition 1.5 *For $\beta\in(0,1]$, $\delta\in(0,1]$, and $\tau\in\mathbb{C}-\{0\}$, the function $f\in S$ is said to be in the class $QS^*(\delta,\beta,\tau)$, which we will call $\delta-$quasistarlike function class with respect to symmetric points of complex order $\tau(\tau\in\mathbb{C}-\{0\})$, if the following condition is satisfied:*

$$\left|\frac{1}{\tau}\left[\frac{2zf'(z)}{g(z)-g(-z)}-1\right]\right|<\beta, z\in U$$

that is,

$$QS^*(\delta,\beta,\tau)=\left\{f\in S:\ \left|\frac{1}{\tau}\left[\frac{2zf'(z)}{g(z)-g(-z)}-1\right]\right|<\beta, z\in U\right\}, g\in S^*(\delta,\beta,\tau).$$

Definition 1.6 *For $\beta \in (0,1]$, $\delta \in (0,1]$, and $\tau \in \mathbb{C} - \{0\}$, the function $f \in S$ is said to be in the class $QC(\delta, \beta, \tau)$, which we will call $\delta-$ quasiconvex function class with respect to symmetric points of complex order τ $(\tau \in \mathbb{C} - \{0\})$, if the following condition is satisfied:*

$$\left| \frac{1}{\tau} \left[\frac{2(zf'(z))'}{(g(z) - g(-z))'} - 1 \right] \right| < \beta, z \in U;$$

that is,

$$QC(\delta, \beta, \tau) = \left\{ f \in S : \left| \frac{1}{\tau} \left[\frac{2(zf'(z))'}{(g(z) - g(-z))'} - 1 \right] \right| < \beta, z \in U \right\}, g \in C(\delta, \beta, \tau).$$

Remark 1.7 *In the case $\delta = 1$ from the Definition 1.1 and Definition 1.2, we obtain the following classes, which we will call starlike and convex function class with respect to symmetric points of complex order τ $(\tau \in \mathbb{C} - \{0\})$, respectively:*

$$S^*(\beta, \tau) = \left\{ f \in S : \left| \frac{1}{\tau} \left[\frac{2zf'(z)}{f(z) - f(-z)} - 1 \right] \right| < \beta, z \in U \right\},$$
$$C(\beta, \tau) = \left\{ f \in S : \left| \frac{1}{\tau} \left[\frac{2(zf'(z))'}{(f(z) - f(-z))'} - 1 \right] \right| < \beta, z \in U \right\}.$$

Remark 1.8 *Taking $\tau = 1$ in the Definition 1.1 and Definition 1.2, we obtain the following classes, which we will call $\delta-$ starlike and $\delta-$ convex function class with respect to symmetric points, respectively:*

$$S^*(\delta, \beta) = \left\{ f \in S : \left| \frac{2\left[(1-\delta)f(z) + \delta zf'(z)\right]}{f(z) - f(-z)} - 1 \right| < \beta, z \in U \right\},$$
$$C(\delta, \beta) = \left\{ f \in S : \left| \frac{2\left[(1-\delta)f(z) + \delta zf'(z)\right]'}{(f(z) - f(-z))'} - 1 \right| < \beta, z \in U \right\}.$$

Remark 1.9 *Setting $\tau = 1$ and $\delta = 1$ in the Definition 1.1 and Definition 1.2, we obtain the classes*

$$S^*(\beta) = \left\{ f \in S : \left| \frac{2zf'(z)}{f(z) - f(-z)} - 1 \right| < \beta, z \in U \right\},$$
$$C(\beta) = \left\{ f \in S : \left| \frac{2(zf'(z))'}{(f(z) - f(-z))'} - 1 \right| < \beta, z \in U \right\},$$

which we will call starlike and convex function class with respect to symmetric points, respectively.

Remark 1.10 *In the case $\delta = 1$ from the Definition 1.3 and Definition 1.4, we obtain the classes*

$$KS^*(\beta, \tau) = \left\{ f \in S : \left| \frac{1}{\tau} \left[\frac{2f(z)}{g(z) - g(-z)} - 1 \right] \right| < \beta, z \in U \right\}, g \in S^*(\beta, \tau),$$

$$KC(\beta, \tau) = \left\{ f \in S : \left| \frac{1}{\tau} \left[\frac{2f'(z)}{(g(z) - g(-z))'} - 1 \right] \right| < \beta, z \in U \right\}, g \in C(\beta, \tau),$$

which we will call close-to-starlike and close-to-convex function class with respect to symmetric points of complex order τ $(\tau \in \mathbb{C} - \{0\})$, respectively.

Remark 1.11 *In the case $\tau = 1$ from the Definition 1.3 and Definition 1.4, we obtain the classes*

$$KS^*(\beta) = \left\{ f \in S : \ \left| \frac{2f(z)}{g(z) - g(-z)} - 1 \right| < \beta, z \in U \right\}, g \in S^*(\delta, \beta),$$

$$KC(\beta) = \left\{ f \in S : \ \left| \frac{2f'(z)}{(g(z) - g(-z))'} - 1 \right| < \beta, z \in U \right\}, g \in C(\delta, \beta),$$

which we will call $\delta-$close-to-starlike and $\delta-$close-to-convex function class with respect to symmetric points, respectively.

Remark 1.12 *In the case $\delta = 1$ and $\tau = 1$ from the Definition 1.3 and Definition 1.4, we obtain the classes*

$$KS^*(\beta) = \left\{ f \in S : \ \left| \frac{2f(z)}{g(z) - g(-z)} - 1 \right| < \beta, z \in U \right\}, g \in S^*(\beta),$$

$$KC(\beta) = \left\{ f \in S : \ \left| \frac{2f'(z)}{(g(z) - g(-z))'} - 1 \right| < \beta, z \in U \right\}, g \in C(\beta),$$

which we will call close-to-starlike and close-to-convex function class with respect to symmetric points, respectively.

Remark 1.13 *In the case $\delta = 1$ from the Definition 1.5 and Definition 1.6, we obtain the following classes, which we will call quasistarlike and quasiconvex function class with respect to symmetric points of complex order $\tau \, (\tau \in \mathbb{C} - \{0\})$, respectively*

$$QS^*(\beta, \tau) = \left\{ f \in S : \ \left| \frac{1}{\tau} \left[\frac{2zf'(z)}{g(z) - g(-z)} - 1 \right] \right| < \beta, z \in U \right\}, g \in S^*(\beta, \tau),$$

$$QC(\beta, \tau) = \left\{ f \in S : \ \left| \frac{1}{\tau} \left[\frac{2(zf'(z))'}{(g(z) - g(-z))'} - 1 \right] \right| < \beta, z \in U \right\}, g \in C(\beta, \tau).$$

Remark 1.14 *Setting $\tau = 1$ in the Definition 1.5 and Definition 1.6, we obtain the classes*

$$QS^*(\delta, \beta) = \left\{ f \in S : \ \left| \frac{2zf'(z)}{g(z) - g(-z)} - 1 \right| < \beta, z \in U \right\}, g \in S^*(\delta, \beta),$$

$$QC(\delta, \beta) = \left\{ f \in S : \ \left| \frac{2(zf'(z))'}{(g(z) - g(-z))'} - 1 \right| < \beta, z \in U \right\}, g \in C(\delta, \beta),$$

which we will call $\delta-$quasistarlike and $\delta-$quasiconvex function class with respect to symmetric points, respectively.

Remark 1.15 *In the case $\delta = 1$ and $\tau = 1$ from the Definition 1.5 and Definition 1.6, we obtain the classes*

$$QS^*(\beta) = \left\{ f \in S : \ \left| \frac{2zf'(z)}{g(z) - g(-z)} - 1 \right| < \beta, z \in U \right\}, g \in S^*(\beta),$$

$$QC(\beta)=\left\{f\in S:\ \left|\frac{2\left(zf'(z)\right)'}{\left(g(z)-g(-z)\right)'}-1\right|<\beta, z\in U\right\}, g\in C(\beta),$$

which we will call quasistarlike and quasiconvex function class with respect to symmetric points, respectively.

The classes of starlike, convex, close-to-starlike, quasistarlike, close-to-convex, quasiconvex function classes are studied in [1–4]. The casses of starlike, convex, close-to-convex, quasiconvex functions with respect to symmetric points are studied in [12, 14].

In this paper, we give some coefficient estimates for the classes $S^*(\delta,\beta,\tau)$, $C(\delta,\beta,\tau)$, $QS^*(\delta,\beta,\tau)$, $QC(\delta,\beta,\tau)$, $KS^*(\delta,\beta,\tau)$ and $KC(\delta,\beta,\tau)$. Additionally, the results obtained for specific values of the parameters in our study are compared with the results in the literature.

2. Coefficient estimates for the classes $S^*(\delta,\beta,\tau)$ and $C(\delta,\beta,\tau)$

In this section, we examine the coefficient estimates problem for the classes $S^*(\delta,\beta,\tau)$ and $C(\delta,\beta,\tau)$.

Firstly, we give the following theorem on the coefficient estimates for the class $S^*(\delta,\beta,\tau)$.

Theorem 2.1 *If $f\in S^*(\delta,\beta,\tau)$, then for each $n=1,2,3,4,...$ are provided the following inequalities*

$$|a_{2n}|\le\frac{2\prod_{k=1}^{n}\left[|\tau|\beta+(k-1)\delta\right]}{\left[1+(2n-1)\delta\right](n-1)!\delta^{n-1}}\ \textit{and}\ |a_{2n+1}|\le\frac{\prod_{k=1}^{n}\left[|\tau|\beta+(k-1)\delta\right]}{n!\delta^{n}}. \tag{2.1}$$

Proof Let $f\in S^*(\delta,\beta,\tau)$, $\delta\in(0,1]$, $\beta\in(0,1]$, and $\tau\in\mathbb{C}-\{0\}$. Then,

$$Re\frac{2\left[(1-\delta)f(z)+\delta zf'(z)\right]}{f(z)-f(-z)}>1-|\tau|\beta.$$

Let's

$$\frac{2\left[(1-\delta)f(z)+\delta zf'(z)\right]}{f(z)-f(-z)}=h(z)=1+h_1z+h_2z^2+h_3z^3+h_4z^4+\cdots.$$

From this equality, we can write for each $n=1,2,3,...$

$$\left[1+(2n-1)\delta\right]a_{2n}=\sum_{k=1}^{n}h_{2(n-k)+1}a_{2k-1}\ \text{and}\ 2n\delta a_{2n+1}=\sum_{k=1}^{n}h_{2(n-k+1)}a_{2k-1}. \tag{2.2}$$

Using triangle inequality and inequalities $|h_n|\le 2|\tau|\beta$, which are provided for all $n=1,2,3,4,...$ (see [24]), to the equalities (2.2), we obtain the following inequalities for each $n=1,2,3,...$

$$\left[1+(2n-1)\delta\right]|a_{2n}|\le 2|\tau|\beta\sum_{k=1}^{n}|a_{2k-1}|\ \text{and}\ n\delta|a_{2n+1}|\le|\tau|\beta\sum_{k=1}^{n}|a_{2k-1}|. \tag{2.3}$$

From these inequalities, we get

$$\left[1+(2n-1)\delta\right]|a_{2n}|\le 2|\tau|\beta\left(\frac{\prod_{k=2}^{n-1}\left[|\tau|\beta+(k-1)\delta\right]}{(n-2)!\delta^{n-2}}+|a_{2n-1}|\right), \tag{2.4}$$

$$n\delta\left|a_{2n+1}\right|\leq\left|\tau\right|\beta\left(\frac{\prod_{k=2}^{n-1}\left[\left|\tau\right|\beta+\left(k-1\right)\delta\right]}{(n-2)!\delta^{n-2}}+\left|a_{2n-1}\right|\right)\tag{2.5}$$

for each $n=1,2,3,...$.

Then, inductively we obtain that

$$\left|a_{2n}\right|\leq\frac{2\prod_{k=1}^{n}\left[\left|\tau\right|\beta+\left(k-1\right)\delta\right]}{\left[1+\left(2n-1\right)\delta\right](n-1)!\delta^{n-1}}$$

for each $n=1,2,3,...$

Similarly from the inequality (2.5), we obtain the following inequalities for each $n=1,2,3,...$

$$\left|a_{2n+1}\right|\leq\frac{\prod_{k=1}^{n}\left[\left|\tau\right|\beta+\left(k-1\right)\delta\right]}{n!\delta^{n}}.$$

Thus, the proof of theorem is completed. □

For the specific values of the parameters in Theorem 2.1, we obtain the results for the classes $S^{*}\left(\beta,\tau\right)$, $S^{*}\left(\delta,\beta\right)$ and $S^{*}\left(\beta\right)$.

Since $f\in C\left(\delta,\beta,\tau\right)\Leftrightarrow zf'^{*}\left(\delta,\beta,\tau\right)$, the following theorem is the result of Theorem 2.1.

Theorem 2.2 *If $f\in C\left(\delta,\beta,\tau\right)$, then*

$$\left|a_{2n}\right|\leq\frac{\prod_{k=1}^{n}\left[\left|\tau\right|\beta+\left(k-1\right)\delta\right]}{\left[1+\left(2n-1\right)\delta\right]n!\delta^{n-1}}\ \text{and}\ \left|a_{2n+1}\right|\leq\frac{\prod_{k=1}^{n}\left[\left|\tau\right|\beta+\left(k-1\right)\delta\right]}{\left(2n+1\right)n!\delta^{n}}$$

for each $n=1,2,3,4,...$.

From the Theorem 2.1, we obtain the results for the classes $C\left(\beta,\tau\right)$, $C\left(\delta,\beta\right)$, and $C\left(\beta\right)$.

3. Main results

In this section, we included the main results of our study.

Theorem 3.1 *If $f\in KS^{*}\left(\delta,\beta,\tau\right)$, then*

$$\left|a_{2n}\right|\leq\frac{2\prod_{k=1}^{n}\left[\left|\tau\right|\beta+\left(k-1\right)\delta\right]}{(n-1)!\delta^{n-1}}\ \text{and}\ \left|a_{2n+1}\right|\leq\frac{\left(2\delta n+1\right)\prod_{k=1}^{n}\left[\left|\tau\right|\beta+\left(k-1\right)\delta\right]}{n!\delta^{n}}$$

for each $n=1,2,3,...$.

Proof Let $f\in KS^{*}\left(\delta,\beta,\tau\right)$, $\beta\in[0,1]$ and $\tau\in\mathbb{C}-\{0\}$. Then, we have

$$Re\frac{2f\left(z\right)}{g\left(z\right)-g\left(-z\right)}>1-\left|\tau\right|\beta,$$

where $g\in S^{*}\left(\delta,\beta,\tau\right)$ and given in the following form:

$$g\left(z\right)=z+b_{2}z^{2}+b_{3}z^{3}+\cdots=z+\sum_{n=2}^{\infty}b_{n}z^{n},\ b_{n}\in\mathbb{C}.$$

For the function $\lambda : \mathbb{C} \to \mathbb{C}$ defined as follows:

$$\frac{2f(z)}{g(z) - g(-z)} = \lambda(z) = 1 + \lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \lambda_4 z^4 + \cdots, \tag{3.1}$$

we have $Re\lambda(z) > 1 - |\tau|\beta$ and $|\lambda_n| \leq 2|\tau|\beta$ (see [24]).
From the equality (3.1) for each $n = 1, 2, 3, \ldots$, we can write

$$a_{2n} = \sum_{k=1}^{n} \lambda_{2(n-k)+1} b_{2k-1} \text{ and } a_{2n+1} = \sum_{k=1}^{n} \lambda_{2(n-k+1)} b_{2k-1} + b_{2n+1}\ . \tag{3.2}$$

Using triangle inequality to the equalities (3.2), we obtain

$$|a_{2n}| \leq 2|\tau|\beta \sum_{k=1}^{n} |b_{2k-1}| \text{ and } |a_{2n+1}| \leq 2|\tau|\beta \sum_{k=1}^{n} |b_{2k-1}| + |b_{2n+1}| \tag{3.3}$$

for each $n = 1, 2, 3, \ldots$.
From the inequalities (3.3) and (2.1), we get

$$|a_{2n}| \leq 2|\tau|\beta \left(\frac{\prod_{k=2}^{n-1} [|\tau|\beta + (k-1)\delta]}{(n-2)!\delta^{n-2}} + |b_{2n-1}| \right), \tag{3.4}$$

$$|a_{2n+1}| \leq 2|\tau|\beta \left(\frac{\prod_{k=2}^{n-1} [|\tau|\beta + (k-1)\delta]}{(n-2)!\delta^{n-2}} + |b_{2n-1}| \right) + |b_{2n+1}| \tag{3.5}$$

for each $n = 1, 2, 3, \ldots$.

Using the inequalities (2.1) for the function $g \in S^*(\delta, \beta, \tau)$ in the inequality (3.4), we conclude that the following inequality is provided

$$|a_{2n}| \leq \frac{2\prod_{k=1}^{n} [|\tau|\beta + (k-1)\delta]}{(n-1)!\delta^{n-1}}$$

for each $n = 1, 2, 3, \ldots$.

Similarly from the inequality (3.5), we can obtain

$$|a_{2n+1}| \leq \frac{(2\delta n + 1)\prod_{k=1}^{n} [|\tau|\beta + (k-1)\delta]}{n!\delta^{n}}\ .$$

Thus, the proof of theorem is completed. □

From the Theorem 3.1, we obtain the following results, respectively.

Corollary 3.2 *If* $f \in KS^*(\beta, \tau)$, *then*

$$|a_{2n}| \leq \frac{2\prod_{k=1}^{n} [|\tau|\beta + k - 1]}{(n-1)!} \text{ and } |a_{2n+1}| \leq \frac{(2n+1)\prod_{k=1}^{n} [|\tau|\beta + k - 1]}{n!}$$

for each $n = 1, 2, 3, 4, \ldots$

Corollary 3.3 *If* $KS^*(\delta,\beta)$, *then*

$$|a_{2n}| \leq \frac{2\prod_{k=1}^{n}[\beta+(k-1)\delta]}{(n-1)!\delta^{n-1}} \text{ and } |a_{2n+1}| \leq \frac{(2\delta n+1)\prod_{k=1}^{n}[\beta+(k-1)\delta]}{n!\delta^{n}}$$

for each $n = 1,2,3,4,\ldots$

Corollary 3.4 *If* $f \in KS^*(\beta)$, *then*

$$|a_{2n}| \leq \frac{2\prod_{k=1}^{n}(\beta+k-1)}{(n-1)!} \text{ and } |a_{2n+1}| \leq \frac{(2n+1)\prod_{k=1}^{n}(\beta+k-1)}{n!}$$

for each $n = 1,2,3,4,\ldots$

Since $f \in KC(\delta,\beta,\tau) \Leftrightarrow zf'^*(\delta,\beta,\tau)$,the following theorem is result of Theorem 3.1.

Theorem 3.5 *If* $f \in KC(\delta,\beta,\tau)$, *then,*

$$|a_{2n}| \leq \frac{\prod_{k=1}^{n}[|\tau|\beta+(k-1)\delta]}{n!\delta^{n-1}} \text{ and } |a_{2n+1}| \leq \frac{(2n\delta+1)\prod_{k=1}^{n}[|\tau|\beta+(k-1)\delta]}{(2n+1)n!\delta^{n}}$$

for each $n = 1,2,3,4,\ldots$.

From Theorem 3.5, we obtain the results for the classes $KC(\beta,\tau)$, $KC(\delta,\beta)$ and $KC(\beta)$, respectively. Now, let us give coefficient estimates for the functions belonging to the class $QS^*(\beta,\tau)$.

Theorem 3.6 *If* $f \in QS^*(\beta,\tau)$, *then*

$$|a_{2n}| \leq \frac{\prod_{k=1}^{n}[|\tau|\beta+(k-1)\delta]}{n!\delta^{n-1}} \text{ and } |a_{2n+1}| \leq \frac{(2\delta n+1)\prod_{k=1}^{n}[|\tau|\beta+(k-1)\delta]}{(2n+1)n!\delta^{n}}$$

for each $n = 1,2,3,4,\ldots$.

Proof Let $f \in QS^*(\beta,\tau)$, $\beta \in [0,1]$ and $\tau \in \mathbb{C}-\{0\}$. Then, we have

$$Re\frac{2zf'(z)}{g(z)-g(-z)} > 1-|\tau|\beta,$$

where $g \in S^*(\delta,\beta,\tau)$ and given in the following form:

$$g(z) = z + b_2z^2 + b_3z^3 + \cdots = z + \sum_{n=2}^{\infty} b_n z^n,\ b_n \in \mathbb{C}.$$

For the function $\gamma : \mathbb{C} \to \mathbb{C}$ defined as follows:

$$\frac{2zf'(z)}{g(z)-g(-z)} = \gamma(z) = 1 + \gamma_1 z + \gamma_2 z^2 + \gamma_3 z^3 + \gamma_4 z^4 + \cdots \tag{3.6}$$

we have $Re\gamma(z) > 1-|\tau|\beta$ and $|\gamma_n| \leq 2|\tau|\beta$ (see [24]).

From the equality (3.6) for each $n = 1, 2, 3, \ldots$, we can write

$$2na_{2n} = \sum_{k=1}^{n} \gamma_{2(n-k)+1} b_{2k-1} \text{ and } (2n+1)\, a_{2n+1} = \sum_{k=1}^{n} \gamma_{2(n-k+1)} b_{2k-1} + b_{2n+1}. \tag{3.7}$$

Using triangle inequality to the equalities (3.7), we obtain

$$n\,|a_{2n}| \le |\tau|\,\beta \sum_{k=1}^{n} |b_{2k-1}| \text{ and } (2n+1)\,|a_{2n+1}| \le 2\,|\tau|\,\beta \sum_{k=1}^{n} |b_{2k-1}| + |b_{2n+1}| \tag{3.8}$$

for each $n = 1, 2, 3, \ldots$.

From the inequalities (3.8) and (2.1), we get

$$n\,|a_{2n}| \le |\tau|\,\beta \left(\frac{\prod_{k=2}^{n-1} [|\tau|\,\beta + (k-1)\,\delta]}{(n-2)!\delta^{n-2}} + |b_{2n-1}| \right), \tag{3.9}$$

$$(2n+1)\,|a_{2n+1}| \le 2\,|\tau|\,\beta \left(\frac{\prod_{k=2}^{n-1} [|\tau|\,\beta + (k-1)\,\delta]}{(n-2)!\delta^{n-2}} + |b_{2n-1}| \right) + |b_{2n+1}| \tag{3.10}$$

for each $n = 1, 2, 3, \ldots$.

Using the inequalities (2.1) for the function $g \in S^*(\delta, \beta, \tau)$ in the inequality (3.9), we conclude that the following inequality is provided for each $n = 1, 2, 3, \ldots$

$$n\,|a_{2n}| \le |\tau|\,\beta \left(\frac{\prod_{k=2}^{n-1}[|\tau|\beta+(k-1)\delta]}{(n-2)!\delta^{n-2}} + \frac{\prod_{k=1}^{n-1}[|\tau|\beta+(k-1)\delta]}{(n-1)!\delta^{n-1}} \right) = \frac{|\tau|\beta\prod_{k=2}^{n-1}[|\tau|\beta+(k-1)\delta][|\tau|\beta+(n-1)\delta]}{(n-1)!\delta^{n-1}} = \frac{\prod_{k=1}^{n}[|\tau|\beta+(k-1)\delta]}{(n-1)!\delta^{n-1}};$$

that is,

$$|a_{2n}| \le \frac{\prod_{k=1}^{n} [|\tau|\,\beta + (k-1)\,\delta]}{n!\delta^{n-1}}.$$

Also, similarly from the inequality (3.10) we can write:

$$(2n+1)\,|a_{2n+1}| \le \frac{2\prod_{k=1}^{n}[|\tau|\beta+(k-1)\delta]}{(n-1)!\delta^{n-1}} + |b_{2n+1}| \le \frac{(2\delta n+1)\prod_{k=1}^{n}[|\tau|\beta+(k-1)\delta]}{n!\delta^{n}},$$

which equivalent to

$$|a_{2n+1}| \le \frac{(2\delta n+1)\prod_{k=1}^{n} [|\tau|\,\beta + (k-1)\,\delta]}{(2n+1)\,n!\delta^{n}}.$$

Thus, the proof of theorem is completed. □

From Theorem 3.6, we obtain the following results, respectively.

Corollary 3.7 *If $f \in QS^*(\beta, \tau)$, then*

$$|a_{2n}| \le \frac{\prod_{k=1}^{n} (|\tau|\,\beta + k - 1)}{n!} \text{ and } |a_{2n+1}| \le \frac{\prod_{k=1}^{n} (\beta + k - 1)}{n!}$$

for each $n = 1, 2, 3, 4, \ldots$

Corollary 3.8 *If* $f \in QS^*(\delta, \beta)$, *then*

$$|a_{2n}| \leq \frac{\prod_{k=1}^{n} [\beta + (k-1)\delta]}{n!\delta^{n-1}} \text{ and } |a_{2n+1}| \leq \frac{(2\delta n + 1) \prod_{k=1}^{n} [\beta + (k-1)\delta]}{(2n+1)\, n!\delta^{n}}$$

for each $n = 1, 2, 3, 4, ...$

Corollary 3.9 *If* $f \in QS^*(\beta)$, *then*

$$|a_{2n}| \leq \frac{\prod_{k=1}^{n} (\beta + k - 1)}{n!} \text{ and } |a_{2n+1}| \leq \frac{\prod_{k=1}^{n} (\beta + k - 1)}{n!}$$

for each $n = 1, 2, 3, 4, ...$

Since $f \in QC(\delta, \beta, \tau) \Leftrightarrow zf'^*(\beta, \tau)$, the following theorem is result of Theorem 3.6.

Theorem 3.10 *If* $QC(\delta, \beta, \tau)$, *then*

$$|a_{2n}| \leq \frac{\prod_{k=1}^{n} [|\tau|\beta + (k-1)\delta]}{2n \cdot n!\delta^{n-1}} \text{ and } |a_{2n+1}| \leq \frac{(2n\delta + 1) \prod_{k=1}^{n} [|\tau|\beta + (k-1)\delta]}{(2n+1)^2\, n!\delta^{n}}$$

for each $n = 1, 2, 3, 4, ...$.

From the Theorem 3.10, we obtain the results for the classes $QC(\beta, \tau)$, $QC(\delta, \beta)$, and $QC(\beta)$.

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