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Some complementary results on asymptotic behavior of solutions of neutral difference equations

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Abstract: This paper focuses on examining the boundedness and asymptotic behavior of all solutions of the neutral difference equations

$$\Delta[x_n - p_n x_{n-\kappa}] + q_n x_{n-\ell} = 0 \quad \text{for } n = 0, 1, \cdots \tag{$\star$}$$

and

$$\Delta[x_n - p x_{n-\kappa}] + q_n x_{n-\ell} = 0 \quad \text{for } n = 0, 1, \cdots, \tag{$\star\star$}$$

where $\kappa, \ell \in \mathbb{N}$, $\{p_n\} \subset [0, 1)$, $p \in [0, 1)$ and $\{q_n\} \subset [0, \infty)$. Diverging from much of the existing literature, our results accommodate the scenario where $\{p_n\} \subset [\frac{1}{2}, 1)$ and $p \in [\frac{1}{2}, 1)$ for ($\star$) and ($\star\star$), respectively. Furthermore, we underscore the practical implications of our results through the presentation of numerical examples.

Key words: Neutral difference equations, variable coefficients, boundedness, asymptotic behavior

1. Introduction

In recent years, stability theory has developed as a versatile and effective tool for comprehending the dynamics embedded in difference equations (DEs). Throughout the decades, several academics have worked carefully to develop and formulate sufficient conditions to ensure that every solution of a delay difference equation (DDE) converges to zero as the time variable approaches infinity. Some researchers have since extended and generalized these analytical frameworks to examine neutral delay difference equations (NDDEs). It is important to note that delays occur in both DDEs and NDDEs, with the latter adding complication due to the presence of neutral terms. In this work, consider the following NDDE

$$\Delta[x_n - p_n x_{n-\kappa}] + q_n x_{n-\ell} = 0, \quad n = 0, 1, 2, \cdots, \tag{1.1}$$

where $\kappa, \ell \in \mathbb{N}$, $\{p_n\} \subset \mathbb{R}$ and $\{q_n\} \subset \mathbb{R}^+ \cup \{0\}$. When $p_n \equiv p \in \mathbb{R}$, (1.1) takes the form

$$\Delta[x_n - p x_{n-\kappa}] + q_n x_{n-\ell} = 0, \quad n = 0, 1, 2, \cdots \tag{1.2}$$

and if $\kappa = 0$, then (1.2) can be written in the form

$$\Delta x_n + \frac{q_n}{1-p} x_{n-\ell} = 0, \quad n = 0, 1, 2, \cdots. \tag{1.3}$$

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Furthermore, for $p_n \equiv 0$, the asymptotic behavior of the solutions for simplified delay difference equation by (1.1), which can be expressed as

$$\Delta x_n + q_n x_{n-\ell} = 0, \quad n = 0, 1, 2, \cdots, \tag{1.4}$$

has been thoroughly examined in many literature, refer to [1–8, 10, 11] for more details. We shall provide an overview of the most notable results for (1.1), (1.2) and (1.4). Let us begin with the theorem that yields one of the best results for the asymptotic behavior of (1.4), achieving the best possible stability with the constant $(\frac{3}{2} + \frac{1}{2[\ell+1]})$, as mentioned in references [3, 6]. This emphasizes the importance and strength of the following theorem in ensuring substantial stability in the absence of neutral elements in the equation.

Theorem A ([3]) *Assume that*

$$\sum_{j=0}^{\infty} q_j = \infty \tag{1.5}$$

and

$$\limsup_{n\to\infty} \sum_{j=n-\ell}^{n} q_j < \frac{3}{2} + \frac{1}{2[\ell+1]}. \tag{1.6}$$

Then, every solution $\{x_n\}$ *of* (1.4) *tends to zero as* $n \to \infty$.

The upper bound $(\frac{3}{2} + \frac{1}{2[\ell+1]})$ in (1.6) emerges naturally from the monotonicity property of a function of the form $\Phi(\lambda) := a\lambda - \frac{b}{2}\lambda^2$, which plays a major role in the proof of [3]. Several authors later extended and generalized this result by replacing (1.6) with a weaker condition or a condition of a slightly different type. In [12], Zhou et al. make enhancements to Theorem A to achieve a better result for (1.2), which is stated as follows:

Theorem B ([12]) *Let* $p \in (-\frac{1}{2}, \frac{1}{2})$. *Assume* (1.5) *and*

$$\limsup_{n\to\infty} \sum_{j=n-\ell}^{n} q_j < \frac{3}{2} + \frac{(1-2|p|)^2}{2[\ell+1]} - 2|p|(2-|p|). \tag{1.7}$$

Then, every solution $\{x_n\}$ *of* (1.2) *tends to zero as* $n \to \infty$.

It should be noted that in [12] the authors assume $p \in (-1, 1)$. However, (1.7) cannot hold when $p \in (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$. Additionally, condition (1.7) reduces to (1.6) for (1.4), i.e. when $p = 0$. This highlights some restriction on the applicability of the Theorem B for certain ranges of p. Applying Theorem A to (1.3), which is also a particular case of (1.2), one can see that Theorem B is not the best result of asymptotic behavior for (1.1). Subsequently, in [9], Tang identifies an opportunity to address some of this gap and enhances Theorem A, ultimately achieving the best result to date for (1.1) involving the summation of $(\ell+1)$ consecutive terms of the coefficient $\{q_n\}$, which is quoted below.

Theorem C (**[9]**) *Assume that there exists* $p \in [0, \frac{1}{2})$ *such that* $|p_n| \leq p$ *for all large* n. *Further, assume* (1.5) *and*

$$\limsup_{n\to\infty} \sum_{j=n-\ell}^{n} q_j < \varphi(p) := \begin{cases} \dfrac{3}{2} - 2p + \dfrac{1}{2[\ell+1]}, & p \in \left[0, \frac{\ell}{4[\ell+1]}\right) \\ \sqrt{2\dfrac{\ell+2}{\ell+1}(1-2p)}, & p \in \left[\frac{\ell}{4[\ell+1]}, \frac{1}{2}\right). \end{cases}$$

Then, every solution $\{x_n\}$ *of* (1.1) *tends to zero as* $n \to \infty$.

Technically, very similar to that of Theorem A, the upper bound function φ originates due to the proof technique used in [9, Theorem 2.2]. The separation point $\frac{\ell}{4[\ell+1]}$ relates to the function Φ again. Examining some fundamental properties of the function φ, we see that it is continuous, decreasing and concave down on $[0, \frac{1}{2})$ with $\varphi(0) = \frac{3}{2} + \frac{1}{2[\ell+1]}$ and $\lim_{p\to\frac{1}{2}} \varphi(p) = 0$. It is worth mentioning that for $\ell \in \mathbb{N}$, $(\frac{3}{2} - 2p + \frac{1}{2[\ell+1]})$ is tangent to $\sqrt{2\frac{\ell+2}{\ell+1}(1-2p)}$ at $p = \frac{\ell}{4[\ell+1]}$. More precisely, $\sqrt{2\frac{\ell+2}{\ell+1}(1-2p)} \leq \frac{3}{2} - 2p + \frac{1}{2[\ell+1]}$ holds for $p \in [0, \frac{1}{2}]$ and $\ell \in \mathbb{N}$ with equality if and only if $p = \frac{\ell}{4[\ell+1]}$.

Therefore, in the following theorems, Tang [9] proceeds establishing two new results, wherein $p \in [\frac{1}{2}, 1)$ is admissible. This is accomplished by using a smaller initial limit for the sum, while simultaneously increasing the right-hand side to obtain more desirable results.

Theorem D (**[9]**) *Assume that there exists* $p \in [0, 1)$ *such that* $0 \leq p_n \leq p$ *for all large* n. *Further, assume* (1.5) *and*

$$\limsup_{n\to\infty} \sum_{j=n-(3\ell+(m-1)\kappa+1)}^{n} q_j < (1-p)\left(1 + \frac{\ell+2}{2[\ell+1]}(1-p)\right), \tag{1.8}$$

where $m \in \mathbb{N}$ *satisfies* $p + \frac{3}{2}p^m \leq 1$. *Then, every solution* $\{x_n\}$ *of* (1.1) *tends to zero as* $n \to \infty$.

Following this, Tang [9] employs an iterative technique to advance the result of Theorem C for (1.2) in the following theorem.

Theorem E (**[9]**) *Assume that* $p \in [0, 1)$ *and* (1.5) *holds. Further, assume*

$$\limsup_{n\to\infty} \sum_{j=n-(\ell+(m-1)\kappa)}^{n} q_j < \frac{1-p}{1-p^m}\left(\frac{3-4p^m}{2} + \frac{1}{2[\ell+(m-1)\kappa+1]}\right), \tag{1.9}$$

where $m \in \mathbb{N}$ *satisfies* $4p^m \leq 1$. *Then, every solution* $\{x_n\}$ *of* (1.2) *tends to zero as* $n \to \infty$.

The objective of this work is to replace the conditions $p + \frac{3}{2}p^m \leq 1$ and $4p^m \leq 1$, as well as the right-hand side conditions (1.8) and (1.9) outlined in Theorems D and E by weaker conditions $p + \big(1 + (1-p)^2\big)p^m \leq 1$ and $2p^m < 1$, along with improved right-hand side conditions, respectively. Our theorems are expected to provide enhanced insights into both boundedness and asymptotic behavior of all solutions of (1.1) and (1.2). Additionally, we aim to rectify certain inaccuracies given in Theorems D and E.

The structure of the paper is as follows: In Section 2, we provide the results concerning the boundedness and asymptotic behavior of solutions of (1.1) and (1.2). In Section 3, we offer numerical examples to illustrate the practical relevance and novelty of our main results. Section 4 contains discussion on the proofs of the results presented in Section 2. Lastly, Section 5 comprises remarks and amendments for Theorems D and E, serving as the concluding section of the paper.

2. Main results

We now present our main results on the boundedness and asymptotic behavior of all solutions of (1.1) and (1.2).

2.1. Results for boundedness

This subsection presents novel conditions that ensure the boundedness of all solutions of (1.1) and (1.2).

Theorem 2.1 *Assume that there exists* $p \in [0, \frac{\ell}{\ell+2}]$ *such that* $0 \le p_n \le p$ *for all large* n*. Further, assume*

$$\sum_{j=n-(3\ell+(m-1)\kappa+1)}^{n} q_j \le (1-p)\left(1+\frac{\ell+2}{2[\ell+1]}(1-p)\right) \quad \textit{for all large } n,$$

where $m \in \mathbb{N}$ *satisfies* $p + \big(1+(1-p)^2\big)p^m \le 1$*. Then, every solution of* (1.1) *is bounded.*

Let us define a number

$$\varphi_m(p,\kappa,\ell) := \begin{cases} \dfrac{1-p}{1-p^m}\left(\dfrac{3-4p^m}{2}+\dfrac{1}{2[\ell+(m-1)\kappa+1]}\right), & p \in \left[0, \sqrt[m]{\frac{\ell+(m-1)\kappa}{4[\ell+(m-1)\kappa+1]}}\right) \\ \dfrac{1-p}{1-p^m}\sqrt{2\dfrac{\ell+(m-1)\kappa+2}{\ell+(m-1)\kappa+1}(1-2p^m)}, & p \in \left[\sqrt[m]{\frac{\ell+(m-1)\kappa}{4[\ell+(m-1)\kappa+1]}}, \frac{1}{\sqrt[m]{2}}\right). \end{cases}$$

Theorem 2.2 *Assume that* $p \in [0,1)$*. Further, assume*

$$\sum_{j=n-(\ell+(m-1)\kappa)}^{n} q_j \le \varphi_m(p,\kappa,\ell) \quad \textit{for all large } n, \tag{2.1}$$

where $m \in \mathbb{N}$ *satisfies* $2p^m < 1$*. Then, every solution of* (1.2) *is bounded.*

2.2. Results for asymptotic behavior

This subsection outlines new conditions that govern the asymptotic behavior of all solutions of (1.1) and (1.2) toward zero.

Theorem 2.3 *Assume that there exists* $p \in [0, \frac{\ell}{\ell+2}]$ *such that* $0 \le p_n \le p$ *for all large* n*. Further, assume* (1.5) *and*

$$\limsup_{n\to\infty} \sum_{j=n-(3\ell+(m-1)\kappa+1)}^{n} q_j < (1-p)\left(1+\frac{\ell+2}{2[\ell+1]}(1-p)\right), \tag{2.2}$$

where $m \in \mathbb{N}$ *satisfies* $p + \big(1+(1-p)^2\big)p^m \le 1$*. Then, every solution of* (1.1) *tends to zero at infinity.*

Theorem 2.4 *Assume that* $p \in [0, 1)$ *and* (1.5) *holds. Further, assume*

$$\limsup_{n\to\infty} \sum_{j=n-(\ell+(m-1)\kappa)}^{n} q_j < \varphi_m(p, \kappa, \ell), \tag{2.3}$$

where $m \in \mathbb{N}$ *satisfies* $2p^m < 1$. *Then, every solution of* (1.2) *tends to zero at infinity.*

3. Numerical examples

In this section, we present two neutral difference equations for which, to the best of our knowledge, none of the existing results can address the asymptotic nature of their solutions. However, our results in Theorems 2.3 and 2.4 can be applied to obtain positive outcomes.

Example 3.1 *Consider the neutral difference equation*

$$\Delta\left[x_n - \frac{7n}{10n+1}x_{n-1}\right] + \frac{11(2+(-1)^n)}{1250}x_{n-5} = 0 \quad \text{for } n = 0, 1, \cdots . \tag{3.1}$$

- ***Theorem D****: It is observed that*

$$p_n = \frac{7n}{10n+1} \le \frac{7}{10} =: p < 1 \quad \text{for } n = 0, 1, \cdots$$

and

$$p + \frac{3}{2}p^m = \frac{7}{10} + \frac{3}{2}\left(\frac{7}{10}\right)^m \le 1 \implies m = 5, 6, \cdots .$$

Upon calculation, it is determined that

$$\begin{aligned}\limsup_{n\to\infty} \sum_{j=n-(3\ell+(m-1)\kappa+1)}^{n} q_j &= \limsup_{n\to\infty} \sum_{j=n-(m+15)}^{n} q_j \\ &= \limsup_{n\to\infty} \frac{11}{2500}\Big(64 + 4m + (-1)^n\big(1-(-1)^m\big)\Big) \\ &\ge \frac{451}{1250} \approx 0.36 \quad \text{for } m = 5, 6, \cdots ,\end{aligned}$$

and

$$(1-p)\left(1 + \frac{\ell+2}{2[\ell+1]}(1-p)\right) = \frac{141}{400} \approx 0.3525.$$

Consequently, it is readily apparent that

$$\limsup_{n\to\infty} \frac{11}{2500}\Big(64 + 4m + (-1)^n\big(1-(-1)^m\big)\Big) \not< \frac{141}{400} \approx 0.3525 \quad \text{for } m = 5, 6, \cdots .$$

Therefore, Theorem D does not provide any answers in this context.

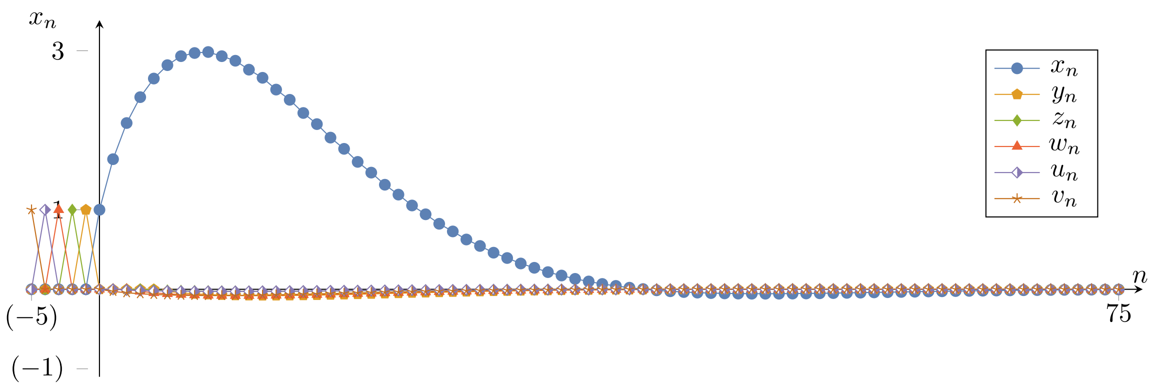

Figure 1. Some solutions of (3.1).

- ***Theorem 2.3***: *It is evident with $m := 4$ and $\ell = 5$ that*

$$p_n = \frac{7n}{10n+1} \leq \frac{7}{10} = 0.7 =: p \leq \frac{\ell}{\ell+2} = \frac{5}{7} \approx 0.714286 \quad \textit{for } n = 0, 1, \cdots$$

and

$$p + \left(1 + (1-p)^2\right)p^4 = \frac{7}{10} + \left(1 + \left(1 - \frac{7}{10}\right)^2\right)\left(\frac{7}{10}\right)^4 = \frac{961709}{1000000} \approx 0.96 \leq 1.$$

Furthermore, it is calculated that

$$\limsup_{n\to\infty} \sum_{j=n-(3\ell+(m-1)\kappa+1)}^{n} q_j = \limsup_{n\to\infty} \sum_{j=n-19}^{n} q_j = \frac{44}{125} = 0.352 < 0.3525.$$

This demonstrates that the conditions stipulated in Theorem 2.3 are satisfied.

Furthermore, it is obvious to note that the condition (1.5) holds. Consequently, it is evident that Theorem 2.3 is invoked, whereas Theorem D is not invoked in this context. In Figure 1, a total of six (linearly independent) particular solutions of (3.1) are plotted.

Example 3.2 *Consider the neutral difference equation*

$$\Delta\left[x_n - \frac{4}{5}x_{n-2}\right] + \frac{9 + (-1)^n 5}{400} x_{n-1} = 0 \quad \textit{for } n = 0, 1, \cdots . \tag{3.2}$$

- ***Theorem E***: *We observe that the inequality*

$$\left(\frac{4}{5}\right)^m \leq \frac{1}{4} \quad \textit{for } m = 7, 8, \cdots ,$$

holds. Upon computation, it is found that

$$\limsup_{n\to\infty} \sum_{j=n-(\ell+(m-1)\kappa)}^{n} q_j = \limsup_{n\to\infty} \sum_{j=n-(2m-1)}^{n} \frac{9 + (-1)^j 5}{400} = \frac{9m}{200}$$

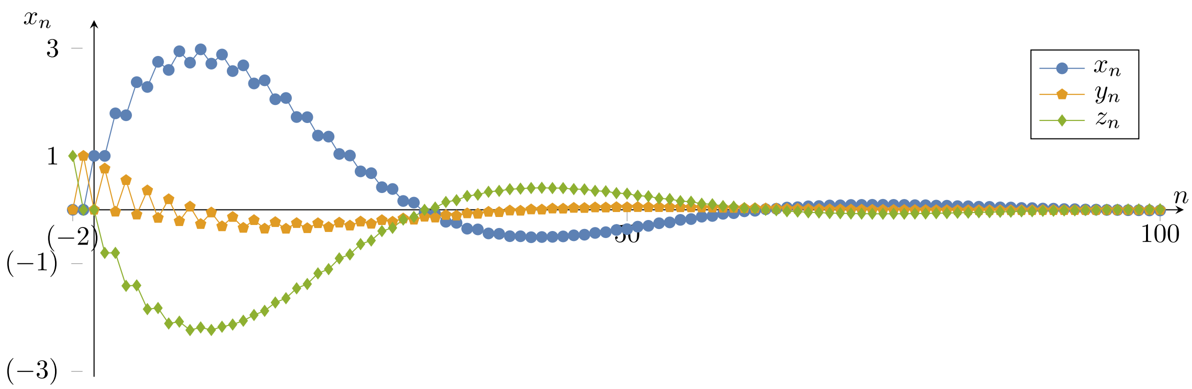

Figure 2. Some solutions of (3.2).

and

$$\frac{1-p}{1-p^m}\left(\frac{3-4p^m}{2}+\frac{1}{2[\ell+(m-1)\kappa+1]}\right)=\frac{5^m(6m+1)-8m4^m}{20m(5^m-4^m)} \quad \text{for } m=7,8,\cdots.$$

It is noted that

$$\frac{9m}{200}\not<\frac{5^m(6m+1)-8m4^m}{20m(5^m-4^m)} \quad \text{for } m=7,8,\cdots.$$

Consequently, Theorem E does not apply.

- ***Theorem* 2.4*: It is apparent with* $m:=4$ *that*

$$\left(\frac{4}{5}\right)^4=\frac{256}{625}=0.4096 \quad \text{and} \quad \frac{\ell+(m-1)\kappa}{4[\ell+(m-1)\kappa+1]}=\frac{7}{32}=0.21875\leq\left(\frac{4}{5}\right)^4<\frac{1}{2},$$

and further, computation reveals that

$$\begin{aligned}\limsup_{n\to\infty}\sum_{j=n-(\ell+(m-1)\kappa)}^{n}q_j&=\limsup_{n\to\infty}\sum_{j=n-7}^{n}\frac{9+(-1)^j5}{400}=\frac{9}{50}=0.18\\&<\frac{1-p}{1-p^m}\sqrt{2\frac{\ell+(m-1)\kappa+2}{\ell+(m-1)\kappa+1}(1-2p^m)}\\&=\frac{5\sqrt{113}}{246}\approx 0.21606,\end{aligned}$$

satisfying the conditions of Theorem 2.4.

Therefore, while Theorem E does not apply, Theorem 2.4 does hold. Additionally, it is easy to see the condition (1.5) *is satisfied. In Figure 2, the graphic of solutions of* (3.2) *with the initial values* $x_{-2}=0$, $x_{-1}=0$, $x_0=1$, $y_{-2}=0$, $y_{-1}=1$, $y_0=0$, $z_{-2}=1$, $z_{-1}=0$, $z_0=0$ *are plotted.*

4. Proof of the main results

The proofs of Theorems 2.1 and 2.2 can be inferred from that of [9, Theorem 2.3] and [9, Theorem 3.1], respectively. Hence, we will provide the details of the proofs for Theorems 2.3 and 2.4 in this section.

Proof [Proof of Theorem 2.3] Let $\{x_n\}$ be any solution of Equation (1.1). We shall prove that

$$\lim_{n\to\infty} x_n = 0. \tag{4.1}$$

Consider the following two cases for the sequence $\{x_n\}$.

Case 1. Let $\{x_n\}$ be nonoscillatory. That is, $\{x_n\}$ is either eventually positive or eventually negative. Without loss of generality, we may assume that $\{x_n\}$ is eventually positive. We can find $n_0 \in \mathbb{N}$ such that $x_n > 0$, $x_{n-\kappa} > 0$ and $x_{n-\ell} > 0$ for $n = n_0, n_0+1, \cdots$. Defining $\{z_n\}$ as

$$z_n := x_n - p_n x_{n-\kappa} \quad \text{for } n = n_0, n_0+1, \cdots, \tag{4.2}$$

we see from (1.1) that $\{z_n\}$ is eventually nonincreasing. Set $L_z := \lim_{n\to\infty} z_n$. Since $\{x_n\}$ is bounded by Theorem 2.1, $\{z_n\}$ is also bounded, i.e. L_z is finite. Then,

$$\sum_{j=n_0}^{\infty} q_j x_{j-\ell} = z_{n_0} - L_z < \infty,$$

which shows by (1.5) that $\liminf_{n\to\infty} x_n = 0$. Let $L_x := \limsup_{n\to\infty} x_n$. We select two increasing divergent sequences $\{u_k\} \subset \mathbb{N}$ and $\{v_k\} \subset \mathbb{N}$ such that $x_{u_k} \to 0$ and $x_{v_k} \to L_x$ as $k \to \infty$. Note that $\{x_{u_k-\kappa}\}$ and $\{x_{v_k-\kappa}\}$ are bounded sequences, they have convergent subsequences by the Bolzano–Weierstrass theorem. Without loss of generality, we denote these convergent subsequences by $\{x_{u_k-\kappa}\}$ and $\{x_{v_k-\kappa}\}$, respectively. Furthermore, note that the limit of the sequence $\{x_{v_k-\kappa}\}$ cannot exceed L_x. Then, we estimate

$$\begin{aligned} z_{v_k} - z_{u_k} &= [x_{v_k} - p_{v_k} x_{v_k-\kappa}] - [x_{u_k} - p_{u_k} x_{u_k-\kappa}] \\ &\geq x_{v_k} - x_{u_k} - p x_{v_k-\kappa} \quad \text{for } k \in \mathbb{N}. \end{aligned}$$

Letting $k \to \infty$ yields $0 = L_z - L_z \geq (1-p)L_x$, which implies $L_x = 0$, i.e. (4.1) holds. Thus, the proof is complete for Case 1.

Case 2. Let $\{x_n\}$ be oscillatory. We set

$$L_x := \limsup_{n\to\infty} |x_n|.$$

Clearly, $L_x \geq 0$. We claim that $L_x = 0$. Assume the contrary that $L_x > 0$. Let $\{u_k\} \subset \mathbb{N}$ be an increasing divergent sequence such that $x_{u_k} \to L_x$ as $k \to \infty$. Note that $\{x_{u_k-\kappa}\}$ is a bounded sequence, it has a convergent subsequence by the Bolzano–Weierstrass theorem. Without loss of generality, we will denote this convergent subsequence by $\{x_{u_k-\kappa}\}$. Further, note that the limit of the sequence $\{x_{u_k-\kappa}\}$ cannot exceed L_x. Consequently, by (4.2), we get

$$|z_{u_k}| \geq |x_{u_k}| - p|x_{u_k-\kappa}| \quad \text{for } k \in \mathbb{N}.$$

Letting $k \to \infty$, we obtain

$$L_z \geq L_x - pL_x = (1-p)L_x > 0, \tag{4.3}$$

where $L_z := \limsup_{n\to\infty} |z_n|$. Let us define

$$\beta(p,\ell) := (1-p)\left(1 + \frac{\ell+2}{2[\ell+1]}(1-p)\right) \quad \text{for } n = n_0, n_0+1, \cdots. \tag{4.4}$$

From (2.2) and (4.4), there exist $n_0 \in \mathbb{N}$ and $\frac{\ell+2}{\ell+1}(1-p) < B < \beta(p,\ell)$ such that $n_0 \geq 3\ell + (m-1)\kappa + 1$, and

$$\sum_{j=n-(3\ell-(m-1)\kappa+1)}^{n} q_j \leq B \quad \text{for } n = n_0, n_0+1, \cdots . \tag{4.5}$$

For every $\varepsilon \in (0, \frac{1-p-p^m}{1-p+p^m} L_x)$, there exists $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$,

$$|x_n| \leq L_x + \varepsilon \quad \text{for } n = n_1, n_1+1, \cdots . \tag{4.6}$$

Note that $\{\Delta z_n\}$ is oscillatory, and there exists an increasing divergent sequence $\{v_k\} \subset \mathbb{N}$ such that $v_k > n_1 + 2[3\ell+(m-1)\kappa+1]+\kappa$ for $k \in \mathbb{N}$, $v_k \to \infty$ and $|z_{v_k}| \to L_z$ as $k \to \infty$ with $|z_{v_k}| > \max\{(1-p)(L_x-\varepsilon), |z_{v_k-1}|\}$ for $k \in \mathbb{N}$. Without loss of generality, we may assume that $\{z_{v_k}\}$ is a positive sequence. Now, we consider the following two subcases.

Subcase 1. Let

$$z_n > 0 \quad \text{for } v_k - [3\ell + (m-1)\kappa + 1] \leq n \leq v_k.$$

Then, we have

$$-x_n < p^i(L_x + \varepsilon) \quad \text{for } v_k - [3\ell + (m-i)\kappa + 1] \leq n \leq v_k \text{ and } i = 1, 2, \cdots, m. \tag{4.7}$$

Define the sequence $\{y_n\}$ by

$$y_n := z_n - p^m(L_x + \varepsilon) \quad \text{for } n = n_1, n_1+1, \cdots . \tag{4.8}$$

Then, it follows from (1.1), (4.2) and (4.8) that

$$\Delta y_n = \Delta z_n = q_n(-z_{n-\ell} - p_{n-\ell}x_{n-\ell-\kappa}) \leq -q_n y_{n-\ell} \quad \text{for } v_k - (2\ell+1) \leq n \leq v_k. \tag{4.9}$$

Then, for $k \in \mathbb{N}$, we deduce

$$\begin{aligned} y_{v_k} &= z_{v_k} - p^m(L_x + \varepsilon) \\ &> (1-p)(L_x - \varepsilon) - p^m(L_x + \varepsilon) \\ &= (1 - p - p^m)L_x - (1 - p + p^m)\varepsilon > 0. \end{aligned}$$

On the other hand, from the fact that $\Delta y_{v_k-1} = \Delta z_{v_k-1} > 0$ and (4.9), we conclude that $y_{v_k-\ell-1} < 0$. Hence, there exists $s_k \in \mathbb{N}$ such that $v_k - \ell \leq s_k \leq v_k$,

$$y_{s_k-1} < 0 \quad \text{and} \quad y_{s_k} \geq 0.$$

Then, there exists $\xi \in [0,1)$ such that

$$\xi(y_{s_k} - y_{s_k-1}) = y_{s_k} \quad \text{or equivalently} \quad y_{s_k} = \xi \Delta y_{s_k-1}. \tag{4.10}$$

Using (1.1), (4.7), and (4.8), we obtain

$$\Delta y_n \leq (L_x + \varepsilon)p^m q_n \quad \text{for } n_1 \leq n \leq v_k. \tag{4.11}$$

We deduce from (4.5), (4.10), and (4.11) that

$$\begin{aligned}-y_{n-\ell} &= \sum_{j=n-\ell}^{s_k-1}\Delta y_j - \xi\Delta y_{s_k-1}\\ &\le p^m(L_x+\varepsilon)\left(\sum_{j=n-\ell}^{s_k-1} q_j - \xi q_{s_k-1}\right)\\ &\le p^m(L_x+\varepsilon)\left(B-\sum_{j=s_k-1}^{n}\omega_{j,s_k-1}(\xi)q_j\right) \quad \text{for } s_k-1\le n\le v_k-1,\end{aligned} \tag{4.12}$$

where, for $\lambda\in\mathbb{R}^+$, the weight sequence $\{\omega_{n,s}(\lambda)\}$ is defined by

$$\omega_{n,s}(\lambda) := \begin{cases}1, & n\ne s\\ \lambda, & n=s.\end{cases} \tag{4.13}$$

Substituting (4.12) into (4.9), we have

$$\Delta y_n \le p^m(L_x+\varepsilon)q_n\left(B-\sum_{j=s_k-1}^{n}\omega_{j,s_k-1}(\xi)q_j\right) \quad \text{for } s_k-1\le n\le v_k-1.$$

Set

$$\Lambda_1 := p^m\left(\max\{B(1-p), B-p(1-p)\} - \frac{\ell+2}{2[\ell+1]}(1-p)^2\right).$$

Hence, $\Lambda_1 < p^m(1-p)^2 \le 1-p-p^m$. For this subcase, there are two possibilities for the number $d := \sum_{j=s_k-1}^{v_k-1}\omega_{j,s_k-1}(\xi)q_j$. Replacing $|x_{n^*}|$ by $(L_x+\varepsilon)$ and using similar arguments to that in the proof of [9, Theorem 2.3], we can conclude that

$$y_{v_k} \le (L_x+\varepsilon)\Lambda_1 \quad \text{for } k\in\mathbb{N}. \tag{4.14}$$

It follows from (4.8) and (4.14) that

$$z_{v_k} = y_{v_k} + p^m(L_x+\varepsilon) \le (\Lambda_1+p^m)(L_x+\varepsilon) \quad \text{for } k\in\mathbb{N}.$$

Letting $k\to\infty$ and $\varepsilon\to 0$ in the above equation, we have

$$L_z = \limsup_{k\to\infty} z_{v_k} \le (\Lambda_1+p^m)L_x < (1-p)L_x,$$

which leads to contradiction with (4.3). Hence, $L_x=0$. Thus, the proof is complete for Subcase 1.

Subcase 2. Let $r_k\in\mathbb{N}$ satisfy $v_k-[3\ell+(m-1)\kappa+1]\le r_k\le v_k$,

$$z_{r_k-1}\le 0 \quad \text{and} \quad z_{r_k}>0.$$

It should be noted that $\Delta z_{r_k-1}\ge 0$. Then, there exists $\xi\in(0,1]$ such that

$$\xi(z_{r_k}-z_{r_k-1}) = z_{r_k} \quad \text{or equivalently} \quad z_{r_k} = \xi\Delta z_{r_k-1}.$$

From (1.1), (4.2) and (4.6), we have

$$\Delta z_n \leq q_n(L_x + \varepsilon) \quad \text{for } n_0 + \ell \leq n \leq v_k - 1,$$

and

$$\Delta z_n \leq q_n p(L_x + \varepsilon) \quad \text{for } r_k + \ell \leq n \leq v_k - 1.$$

Then, using similar arguments to that in the proof of [9, Theorem 2.3], we obtain

$$\Delta z_n \leq (L_x + \varepsilon) q_n \left(B + p - \sum_{j=r_k-1}^{n} \omega_{j,r_k-1}(\xi) q_j \right) \quad \text{for } r_k - 1 \leq n \leq r_k + \ell - 1.$$

Setting

$$\Lambda_2 := B - \frac{\ell+2}{2[\ell+1]}(1-p)^2.$$

Consequently, $\Lambda_2 < 1 - p$. There are also two possible subcases for the number $h := \sum_{j=r_k-1}^{r_k+\ell-1} \omega_{j,r_k-1}(\xi) q_j$. Replacing $|x_{n^*}|$ by $(L_x + \varepsilon)$ and using similar arguments to that in the proof of [9, Theorem 2.3], we can conclude that

$$z_{v_k} \leq (L_x + \varepsilon)\Lambda_2 \quad \text{for } k \in \mathbb{N}.$$

Letting $k \to \infty$ and $\varepsilon \to 0$ in the above equation, we have

$$L_z = \limsup_{k \to \infty} z_{v_k} \leq \Lambda_2 L_x < (1-p) L_x,$$

which leads to contradiction with (4.3). Hence, we obtain $L_x = 0$ which is finalized the proof for Subcase 2.

Hence, the proof for Case 2 is also complete.

□

Proof [Proof of Theorem 2.4] Let $\{x_n\}$ be any solution of Equation (1.2). We will prove that (4.1) holds. Consider the following two cases for the sequence $\{x_n\}$.

Case 1. Let $\{x_n\}$ be nonoscillatory. Replacing p_n by p and then using the same arguments to that in Case 1 in the proof of Theorem 2.3.

Case 2. Let $\{x_n\}$ be oscillatory. Set

$$L_x := \limsup_{n \to \infty} |x_n|.$$

It is easy to see that $L_x \geq 0$. We claim that $L_x = 0$. Assume the contrary that $L_x > 0$. Let $\{u_k\} \subset \mathbb{N}$ be an increasing divergent sequence such that $x_{u_k} \to L_x$ as $k \to \infty$. Note that $\{x_{u_k - \kappa}\}$ is a bounded sequence, it has a convergent subsequence by the Bolzano–Weierstrass theorem. Without loss of generality, we will denote this convergent subsequence by $\{x_{u_k-\kappa}\}$. Further, note that the limit of the sequence $\{x_{u_k-\kappa}\}$ cannot exceed L_x. Consequently, we obtain

$$|z_{u_k}| \geq |x_{u_k}| - p|x_{u_k-\kappa}| \quad \text{for } k \in \mathbb{N},$$

where

$$z_n := x_n - px_{n-\kappa} \quad \text{for } n = n_0, n_0+1, \cdots. \tag{4.15}$$

Letting $k \to \infty$, we obtain

$$L_z \geq L_x - pL_x = (1-p)L_x > 0, \tag{4.16}$$

where $L_z := \limsup_{n\to\infty} |z_n|$. Let us define

$$\begin{aligned} \alpha_m(p,\kappa,\ell) &:= \frac{1-p}{1-p^m}\bigg(\frac{3-4p^m}{2} + \frac{1}{2[\ell+(m-1)\kappa+1]}\bigg), \\ \beta_m(p,\kappa,\ell) &:= \frac{1-p}{1-p^m}\sqrt{2\frac{\ell+\kappa(m-1)+2}{\ell+\kappa(m-1)+1}(1-2p^m)}. \end{aligned} \tag{4.17}$$

From (2.3) and (4.17), there exist $n_0 \in \mathbb{N}$, $\frac{\ell+(m-1)\kappa+2}{\ell+(m-1)\kappa+1}\frac{1-p}{1-p^m} < A < \alpha_m(p,\kappa,\ell)$ and $0 < B < \beta_m(p,\kappa,\ell)$ such that $n_0 \geq \ell + (m-1)\kappa$,

$$\sum_{j=n-\ell-(m-1)\kappa}^{n} q_j \leq \psi := \begin{cases} A, & p \in \Big[0, \sqrt[m]{\frac{\ell+(m-1)\kappa}{4[\ell+(m-1)\kappa+1]}}\Big) \\ B, & p \in \Big[\sqrt[m]{\frac{\ell+(m-1)\kappa}{4[\ell+(m-1)\kappa+1]}}, \frac{1}{\sqrt[m]{2}}\Big) \end{cases} \quad \text{for } n = n_0, n_0+1, \cdots.$$

For every $\varepsilon \in \big(0, (1-2p^m)L_x\big)$, there exists $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ such that

$$|x_n| \leq L_x + \varepsilon \quad \text{for } n = n_1, n_1+1, \cdots. \tag{4.18}$$

Define the sequence $\{y_n\}$ by

$$y_n := z_n - (1-p)\frac{p^m}{1-p^m}(L_x+\varepsilon) \quad \text{for } n = n_1, n_1+1, \cdots. \tag{4.19}$$

Then, we can derive

$$-x_{n-\ell} \leq -\sum_{i=0}^{m-1} p^i y_{n-\ell-i\kappa} \quad \text{for } n = n_2, n_2+1, \cdots \tag{4.20}$$

where $n_2 \geq n_1 + \ell + m\kappa$. From (1.2), (4.19) and (4.20), it follows that

$$\Delta y_n = \Delta z_n = -q_n x_{n-\ell} \leq -q_n \sum_{i=0}^{m-1} p^i y_{n-\ell-i\kappa} \quad \text{for } n = n_2, n_2+1, \cdots. \tag{4.21}$$

It is noteworthy that $\{\Delta z_n\}$ is oscillatory, and there exists an increasing divergent sequence $\{v_k\} \subset \mathbb{N}$ such that $v_k > n_0+2[(m-1)\kappa+\ell]$ for $k \in \mathbb{N}$, $v_k \to \infty$ and $|z_{v_k}| \to L_z$ as $k \to \infty$ with $|z_{v_k}| > \max\{(1-p)(L_x-\varepsilon), |z_{v_k-1}|\}$ for $k \in \mathbb{N}$. Without loss of generality, we may assume that $\{z_{v_k}\}$ is a positive sequence. Then, for $k \in \mathbb{N}$, we deduce

$$\begin{aligned} y_{v_k} &= z_{v_k} - (1-p)\frac{p^m}{1-p^m}(L_x+\varepsilon) \\ &> (1-p)(L_x-\varepsilon) - (1-p)\frac{p^m}{1-p^m}(L_x+\varepsilon) \\ &= (1-p)\frac{(1-2p^m)L_x-\varepsilon}{1-p^m} > 0. \end{aligned}$$

On the other hand, from the fact that $\Delta z_{v_k-1} > 0$ and (4.21), we conclude that $y_{v_k-\ell-i_0\kappa-1} < 0$ for some $i_0 \in \{0, 1, \cdots, m-1\}$. Hence, there exists $s_k \in \mathbb{N}$ such that $v_k - \ell - i_0\kappa \leq s_k \leq v_k$,

$$y_{s_k-1} < 0 \quad \text{and} \quad y_{s_k} \geq 0.$$

Then, we can find $\xi \in [0,1)$ such that

$$\xi(y_{s_k} - y_{s_k-1}) = y_{s_k} \quad \text{or equivalently} \quad y_{s_k} = \xi \Delta y_{s_k-1}.$$

Considering (1.2) and (4.18), we conclude that

$$\Delta y_n \leq (L_x + \varepsilon) q_n \quad \text{for } n_1 \leq n \leq v_k.$$

We can show that

$$-y_{n-\ell-i_0\kappa} \leq (L_x + \varepsilon)\left(\psi - \sum_{j=s_k-1}^{n} \omega_{j,s_k-1}(\xi) q_j\right) \quad \text{for } s_k - 1 \leq n \leq v_k - 1. \tag{4.22}$$

Substituting (4.22) into (4.21), we have

$$\Delta y_n \leq \frac{1-p^m}{1-p}(L_x + \varepsilon) q_n \left(\psi - \sum_{j=s_k-1}^{n} \omega_{j,s_k-1}(\xi) q_j\right) \quad \text{for } s_k - 1 \leq n \leq v_k - 1.$$

Set

$$\Lambda := \begin{cases} A - \frac{\ell+(m-1)\kappa+2}{2[\ell+(m-1)\kappa+1]} \frac{1-p}{1-p^m}, & p \in \left[0, \sqrt[m]{\frac{\ell+(m-1)\kappa}{4[\ell+(m-1)\kappa+1]}}\right) \\ \frac{\ell+(m-1)\kappa+1}{2[\ell+(m-1)\kappa+2]} \frac{1-p^m}{1-p} B^2, & p \in \left[\sqrt[m]{\frac{\ell+(m-1)\kappa}{4[\ell+(m-1)\kappa+1]}}, \frac{1}{\sqrt[m]{2}}\right). \end{cases}$$

We simply have

$$\Lambda < (1-p)\frac{1-2p^m}{1-p^m}.$$

There are three possible cases for the number $d := \sum_{j=s_k-1}^{v_k-1} \omega_{j,s_k-1}(\xi) q_j$. Replacing $|x_{n^*}|$ by $(L_x + \varepsilon)$ and using similar arguments to that in the proof of [9, Theorem 3.1], we can conclude that

$$y_{v_k} \leq \Lambda(L_x + \varepsilon) \quad \text{for } k \in \mathbb{N}. \tag{4.23}$$

Thus, for $k \in \mathbb{N}$, we estimate from (4.19) and (4.23) that

$$z_{v_k} \leq \Lambda(L_x + \varepsilon) + (1-p)\frac{p^m}{1-p^m}(L_x + \varepsilon) \tag{4.24}$$

Letting $k \to \infty$ and $\varepsilon \to 0$ in above equation, we have

$$L_z = \limsup_{k\to\infty} z_{v_k} < (1-p)L_x,$$

which contradicts (4.16), i.e. $L_x = 0$, and hence the proof is done.

□

5. Discussion

Now, we focus our attention to the proofs of [9, Theorem 2.3, Theorem 2.4, Theorem 3.1 and Theorem 3.2]. Let us first consider the proofs of [9, Theorem 2.3 and Theorem 2.4] by comparing them with the proofs of [9, Theorem 2.1 and Theorem 2.2] together. It is apparent that the main idea in the proof of [9, Theorem 2.3] is very similar to that in the proof of [9, Theorem 2.1]. In the proof of [9, Theorem 2.3], the authors utilize functions of the forms

$$f(x,p,\ell) := \beta(p,\ell)x - \frac{\ell+2}{2(\ell+1)}x^2 \quad \text{for } x \in [0,\infty),\ p \in [0,1] \text{ and } \ell \in \mathbb{N}$$

and

$$g(x,p,\ell) := f(x,p,\ell) + p\big(\beta(p,\ell) - (1-p)\big) \quad \text{for } x \in [0,\infty),\ p \in [0,1] \text{ and } \ell \in \mathbb{N},$$

where $\beta(p,\ell)$ is defined in (4.4). Readily, $f(\cdot,p,\ell)$ and $g(\cdot,p,\ell)$ are both increasing on $[0,\frac{\ell+1}{\ell+2}\beta(p,\ell)]$. For instance, in Subcase 1 in the proof of [9, Theorem 2.3], Tang uses the inequality $f(x,p,\ell) \leq f(1-p,p,\ell)$ for $0 \leq x \leq 1-p$. However, $p \in [0,1)$ does not imply $1-p \in [0,\frac{\ell+1}{\ell+2}\beta(p,\ell)]$, which is essential. Especially, $1-p \notin [0,\frac{\ell+1}{\ell+2}\beta(p,\ell)]$ when $p \in (\frac{\ell}{\ell+2},1)$. To guarantee $f(\cdot,p,\ell)$ and $g(\cdot,p,\ell)$ are both increasing on $[0,1-p]$, we require that $[0,1-p] \subset [0,\frac{\ell+1}{\ell+2}\beta(p,\ell)]$, i.e. $p \in [0,\frac{\ell}{\ell+2}]$. Hence, [9, Theorem 2.3 and Theorem 2.4] can be corrected as follows:

Theorem F (Correction of [9, Theorem 2.3]) *Assume that there exists $p \in [0,\frac{\ell}{\ell+2}]$ such that $0 \leq p_n \leq p$ for all large n. Further, assume*

$$\sum_{j=n-(3\ell+(m-1)\kappa+1)}^{n} q_j \leq (1-p)\left(1 + \frac{\ell+2}{2[\ell+1]}(1-p)\right) \quad \textit{for all large } n,$$

where $m \in \mathbb{N}$ satisfies $p + \frac{3}{2}p^m \leq 1$. Then, every solution of (1.1) is bounded.

Theorem G (Correction of Theorem D) *Assume that there exist $p \in [0,\frac{\ell}{\ell+2}]$ and $m \in \mathbb{N}$ such that $p + \frac{3}{2}p^m \leq 1$ and $0 \leq p_n \leq p$ for all large n. Further, assume (1.5) and (1.8). Then, every solution $\{x_n\}$ of (1.1) tends to zero as $n \to \infty$.*

In the following remark, we compare Theorem 2.3 and Theorem G.

Remark 5.1 *Readily, $p + \frac{3}{2}p^m \leq 1$ holds for $m = \lceil \log_p(\frac{2}{3}(1-p)) \rceil, \lceil \log_p(\frac{2}{3}(1-p)) \rceil + 1, \cdots$, while $p + \big(1 + (1-p)^2\big)p^m \leq 1$ holds for $m = \lceil \log_p(\frac{1-p}{1+(1-p)^2}) \rceil, \lceil \log_p(\frac{1-p}{1+(1-p)^2}) \rceil + 1, \cdots$.*

From Figure 3, it is clear that $\lceil \log_p(\frac{1-p}{1+(1-p)^2}) \rceil \leq \lceil \log_p(\frac{2}{3}(1-p)) \rceil$ for $p \in [0,1)$ (which covers $[0,\frac{\ell}{\ell+2}] \subset [0,1)$ for any $\ell \in \mathbb{N}$). The condition (1.8) appears in both of these theorems but its left-hand side becomes smaller for the smaller m values. This verifies that Theorem 2.3 improves Theorem G.

On the other hand, if we consider [9, Theorem 3.1] when $m = 1$, it is just a restriction of [9, Theorem 2.1] to (1.2), i.e. there is nothing new in its proof for this particular case. However, with $m = 1$ and $p \in [\frac{\ell}{4[\ell+1]},\frac{1}{4}]$,

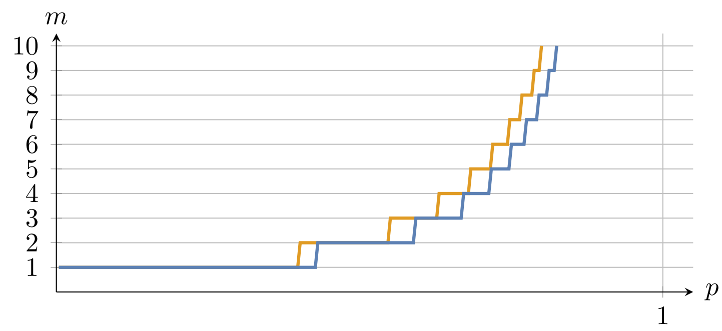

Figure 3. Graphics of $\lceil \log_p(\frac{1-p}{1+(1-p)^2})\rceil$ and $\lceil \log_p(\frac{2}{3}(1-p))\rceil$.

it appears that [9, Theorem 3.1] improves [9, Theorem 2.1]. This is due to a similar technical mistake mentioned above, which also lies in the proof of [9, Theorem 3.1]. In the proof of [9, Theorem 3.1], we encounter a function of the form

$$f_m(x,p,\kappa,\ell) := \alpha_m(p,\kappa,\ell)x - \frac{\ell+(m-1)\kappa+2}{2[\ell+(m-1)\kappa+1]}x^2$$

for $x \in [0,\infty)$, $p \in [0,1]$ and $m,\kappa,\ell \in \mathbb{N}$, where $\alpha_m(p,\kappa,\ell)$ is defined in (4.17). Note that $f_m(\cdot,p,\kappa,\ell)$ is increasing on $[0, \frac{\ell+(m-1)\kappa+1}{\ell+(m-1)\kappa+2}\alpha_m(p,\ell,\kappa)]$. As we see that $\frac{1-p}{1-p^m} \notin [0, \frac{\ell+(m-1)\kappa+1}{\ell+(m-1)\kappa+2}\alpha_m(p,\ell,\kappa)]$ if $p \in [0,1)$. For the condition $\frac{1-p}{1-p^m} \in [0, \frac{\ell+(m-1)\kappa+1}{\ell+(m-1)\kappa+2}\alpha_m(p,\ell,\kappa)]$ to be held, it has to be assumed that $4p^m \leq \frac{\ell+(m-1)\kappa}{\ell+(m-1)\kappa+1}$. Hence, corrections of [9, Theorem 3.1 and Theorem 3.2] are as follows:

Theorem H (Correction of [9, Theorem 3.1]) *Assume that $p \in [0,1)$. Further, assume*

$$\sum_{j=n-(\ell+(m-1)\kappa)}^{n} q_j \leq \frac{1-p}{1-p^m}\left(\frac{3-4p^m}{2} + \frac{1}{2[\ell+(m-1)\kappa+1]}\right) \quad \text{for all large } n, \tag{5.1}$$

where $m \in \mathbb{N}$ satisfies $4p^m \leq \frac{\ell+(m-1)\kappa}{\ell+(m-1)\kappa+1}$. Then, every solution of (1.2) is bounded.

In the following remark, we compare Theorem 2.2 and Theorem H.

Remark 5.2 *Consider the autonomous neutral difference equation*

$$\Delta[x_n - px_{n-1}] + qx_{n-2} = 0 \quad \text{for } n = 0,1,\cdots, \tag{5.2}$$

where $p \in [0,1)$ and $q \in [0,\infty)$, $\kappa = 1$ and $\ell = 2$.

The conditions of Theorem 2.2 for (5.2) reduce to

$$q \leq \frac{1}{m+2}\frac{1-p}{1-p^m}\begin{cases}\dfrac{3-4p^m}{2} + \dfrac{1}{2(m+2)}, & p \in \left[0, \sqrt[m]{\frac{m+1}{4(m+2)}}\right) \\ \sqrt{2\frac{m+3}{m+2}(1-2p^m)}, & p \in \left[\sqrt[m]{\frac{m+1}{4(m+2)}}, \frac{1}{\sqrt[m]{2}}\right)\end{cases} \tag{5.3}$$

for $m = \lfloor \log_p(\frac{1}{2})\rfloor + 1, \lfloor \log_p(\frac{1}{2})\rfloor + 2, \cdots$ since the condition $2p^m < 1$ hold for these m values.

On the other hand, the conditions of Theorem H for (5.2) reduce to

$$4p^m \leq \frac{m+1}{m+2} \quad \text{and} \quad q \leq \frac{1}{m+2}\frac{1-p}{1-p^m}\left(\frac{3-4p^m}{2}+\frac{1}{2(m+2)}\right) \tag{5.4}$$

for suitable values of $m\in\mathbb{N}$. *Note that these* m *values are greater than* $\left(\lfloor \log_p(\frac{1}{2})\rfloor + 1\right)$.

Figure 4 shows the region for possible values of the pairs (p,q) *satisfying the conditions of Theorem 2.2 and Theorem H.*

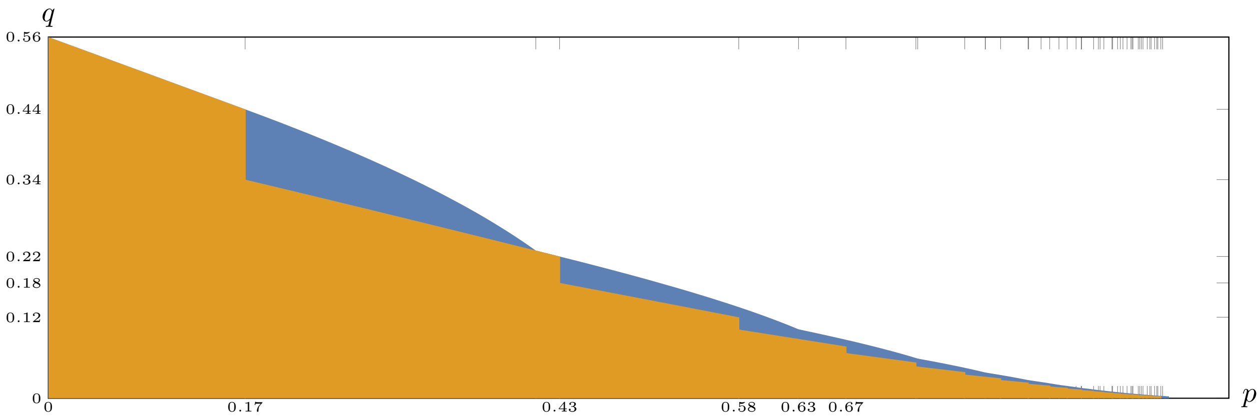

Figure 4. Theorem 2.2 supplies the union of the blue region and the yellow region for the (p,q) pairs satisfying (5.3), and Theorem H only supplies the yellow region for the (p,q) pairs satisfying (5.4).

Theorem I (Correction of Theorem E) *Assume that* $p\in[0,1)$, *and there exists* $m\in\mathbb{N}$ *such that* $4p^m \leq \frac{\ell+(m-1)\kappa}{\ell+(m-1)\kappa+1}$. *Further, assume (1.5) and (1.9). Then, every solution* $\{x_n\}$ *of (1.2) tends to zero as* $n\to\infty$.

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