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Some free boundary problems arising in rock mechanics

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Abstract: An initial boundary value problem for the in situ leaching is considered. We describe physical processes at the microscopic level with a pore size $\varepsilon \ll 1$ by model $\mathbb{A}^{\varepsilon}$, where dynamics of the incompressible solid skeleton is described by the Lamé equations and the physical process in the pore space by the Stokes equations for the incompressible fluid with diffusion equations for the concentration of acid and product of chemical reactions. Since the solid skeleton changes its geometry upon dissolution, the "pore space – solid skeleton" boundary is a free boundary. The goal of the present manuscript is a model $\mathbb{H}$, which is the homogenization of the model $\mathbb{A}^{\varepsilon}$. That is, the limit as ε tend to zero, of the model $\mathbb{A}^{\varepsilon}$. As usual, free boundary problems are only solvable locally in time. On the other hand, in situ leaching has a very long process duration and there is still no correct microscopic model that describes this process for an arbitrary time interval. To avoid this contradiction, we propose correct approximate microscopic models $\mathbb{B}^{\varepsilon}(r)$ for this process with a given solid skeleton structure depending on some function r from the set $\mathfrak{M}_{(0,T)}$. Problem $\mathbb{B}^{\varepsilon}(r)$ is the model $\mathbb{A}^{\varepsilon}$ without an additional boundary condition at the free boundary that defines this boundary, but with some additional terms in the Stokes and Lame equations that depend linearly on the velocities and disappear upon homogenization. To derive a macroscopic mathematical model $\mathbb{H}(r)$ and separately the additional boundary condition at free boundary we use Nguenseng's two-scale convergence method as ε tends to zero. As a result, we obtain a homogenized model $\mathbb{H}(r)$ and an additional equation, possesses construct an operator, which fixed point uniquely defines function r^* from the set $\mathfrak{M}_{(0,T)}$ and prove the existence and uniqueness theorem for the macroscopic mathematical model $\mathbb{H}$.

Key words: Free boundary problems, structures with special periodicity, homogenization, fixed point theorem

1. Introduction: the problem statement

1.1. The problem statement and main result

The extraction of rare metals by leaching is a very important problem of the national economy. Natural deposits of uranium, nickel, and other rare metals are complex geologically heterogeneous objects. Inhomogeneity means that the properties of an object of interest change in space. Analyses of wells and cores show that the geological properties (porosity, permeability, etc.) of ore bodies are heterogeneous even within a single deposit. Very often insufficient taking into account the consequences of inhomogeneities at the stage of operation planning becomes obvious too late, when the acid solution uploaded to soil through injection wells appears far from the intended location. In addition, an important role is played by the concentration of the injected acid, the injection modes of acid solutions, and other factors.

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Hence, understanding the dynamics of fluids in heterogeneous porous media and the mechanism of dissolution of rocks by acids is of fundamental importance for the effective management of rare metal mining.

Here we follow the ideas of R. Burridge, J. B. Keller [2] and E. Sánchez-Palencia [16], who were the first to explain that an exact description of fluid filtration and seismic waves in rocks at the macroscopic level is possible if and only if:

(a) the physical process under consideration is described at the microscopic level by equations of Newtonian classical continuum mechanics (exact model);

(b) a set of small dimensionless parameters is selected;

(c) the macroscopic mathematical models is an exact asymptotic limits (homogenization) of exact mathematical models at the microscopic level, when the selected small parameters tend to zero.

The liquid motion in a pore space $\Omega^{\varepsilon}_{f}(r)\subset\Omega$ for $t>0$ is governed by the Stokes equations for the incompressible viscous fluid

$$\nabla\cdot\mathbb{P}^{\varepsilon}_{f}=0,\ \ \mathbb{P}^{\varepsilon}_{f}=\alpha^{\varepsilon}_{\mu}\mathbb{D}(x,\frac{\partial\boldsymbol{w}^{\varepsilon}_{f}}{\partial t})-\frac{\partial\pi^{\varepsilon}}{\partial t}\mathbb{I},\ \ \nabla\cdot\boldsymbol{w}^{\varepsilon}_{f}=0 \tag{1.1}$$

for the liquid displacements $\boldsymbol{w}^{\varepsilon}_{f}$, the velocity $\boldsymbol{v}^{\varepsilon}_{f}=\dfrac{\partial\boldsymbol{w}^{\varepsilon}_{f}}{\partial t}$ and the pressure $p^{\varepsilon}=\dfrac{\partial\pi^{\varepsilon}}{\partial t}$.

The motion of the incompressible solid skeleton in the domain $\Omega^{\varepsilon}_{s}(r)$ is described by Lamé equations

$$\nabla\cdot\mathbb{P}^{\varepsilon}_{s}=0,\ \ \mathbb{P}^{\varepsilon}_{s}=\lambda_{0}\mathbb{D}(x,\boldsymbol{w}^{\varepsilon}_{s})-p^{\varepsilon}\mathbb{I},\ \ \nabla\cdot\boldsymbol{w}^{\varepsilon}_{s}=0 \tag{1.2}$$

for the solid displacements $\boldsymbol{w}^{\varepsilon}_{s}$ and pressure p^{ε}.

Diffusion of the acid and products of chemical reactions in the pore space Ω^{ε}_{f} for $t>0$ is described by diffusion equations

$$\frac{\partial c^{\varepsilon}}{\partial t}=\alpha_{c}\,\triangle c^{\varepsilon}, \tag{1.3}$$

$$\frac{\partial c^{\varepsilon}_{j}}{\partial t}=\alpha_{c,j}\,\triangle c^{\varepsilon}=0,\ \ j=1,...,k \tag{1.4}$$

for ***Acid concentration*** $c^{\varepsilon}(\boldsymbol{x},t)$ and for ***Concentrations of products of chemical reactions***$c^{\varepsilon}_{j}(\boldsymbol{x},t),\ j=1,...,k$.

Differential equations at a free boundary $\Gamma^{\varepsilon}(r)$ between liquid and solid components are supplemented by boundary conditions

$$\boldsymbol{w}^{\varepsilon}_{f}=\boldsymbol{w}^{\varepsilon}_{s}, \tag{1.5}$$

$$\mathbb{P}^{\varepsilon}_{f}<\mathbf{N}^{\varepsilon}>=\mathbb{P}^{\varepsilon}_{s}<\mathbf{N}^{\varepsilon}>, \tag{1.6}$$

$$(D^{\varepsilon}_{N}+\beta^{\varepsilon})c^{\varepsilon}+\alpha_{c}\,\frac{\partial c^{\varepsilon}}{\partial\,N}=0, \tag{1.7}$$

$$(D^{\varepsilon}_{N}+\beta^{\varepsilon}_{j})c^{\varepsilon}_{j}+\alpha_{c,j}\,\frac{\partial c^{\varepsilon}_{j}}{\partial\,N}=0,\ \ j=1,...,k, \tag{1.8}$$

expressing the laws of conservation of mass and momentum [8] (Appendix A, section A6), and additional boundary condition

$$D^{\varepsilon}_{N} = \alpha^{\varepsilon}\, c^{\varepsilon}(\boldsymbol{x}, t),\ \ \boldsymbol{x} \in\ \Gamma^{\varepsilon}(r),\ \ t > 0, \tag{1.9}$$

postulated in theoretical chemistry [3].

The last boundary condition should allow us to find a free boundary $\Gamma^{\varepsilon}(r)$.

At the given boundaries with injection wells S^1 and production wells S^2, and at the impenetrable boundary S^0, the following conditions

$$\mathbb{P}^{\varepsilon} < \boldsymbol{n} >= -p_0\boldsymbol{n},\ \ \boldsymbol{x} \in\ S^1 \cup S^2,\ \ t > 0, \tag{1.10}$$

$$\chi^{\varepsilon}_{r}\boldsymbol{w}^{\varepsilon}_{f} + (1-\chi^{\varepsilon}_{r})\boldsymbol{w}^{\varepsilon}_{f} = 0\ \ \boldsymbol{x} \in S^0,\ \ t > 0, \tag{1.11}$$

$$\frac{\partial c^{\varepsilon}}{\partial n} = 0,\ \ \boldsymbol{x} \in S^0,\ \ t > 0, \tag{1.12}$$

$$\frac{\partial c^{\varepsilon}_{j}}{\partial n} = 0,\ \ j = 1, ..., k,\ \ \boldsymbol{x} \in S^0,\ \ t > 0, \tag{1.13}$$

$$c^{\varepsilon}(\boldsymbol{x}, t) = c^{0}(\boldsymbol{x}, t),\ \boldsymbol{x} \in S^1 \cup S^2,\ \ t > 0,\ \ j = 1, ..., k, \tag{1.14}$$

are met.

The problem ended with initial conditions

$$c^{\varepsilon}(\boldsymbol{x}, 0) = c^{0}(\boldsymbol{x}, 0),\ \ \boldsymbol{x} \in \Omega^{\varepsilon}_{f}(0), \tag{1.15}$$

$$\Gamma^{\varepsilon}(r(\boldsymbol{x}, 0)) = \Gamma^{\varepsilon}_{0} \tag{1.16}$$

$$c^{\varepsilon}_{j}(\boldsymbol{x}, t) = 0,\ \ \boldsymbol{x} \in S^1 \cup S^2,\ \ c^{\varepsilon}_{j}(\boldsymbol{x}, 0) = 0,\ j = 1, ..., k,\ \ \boldsymbol{x} \in \Omega^{\varepsilon}_{f}(0). \tag{1.17}$$

$$\boldsymbol{w}^{\varepsilon}_{f}(\boldsymbol{x}, 0) = 0,\ \ \pi^{\varepsilon}(\boldsymbol{x}, 0) = 0,\ \ \boldsymbol{x} \in \Omega. \tag{1.18}$$

In (1.1) – (1.18) $\mathbb{P}^{\varepsilon} = \chi^{\varepsilon}\mathbb{P}^{\varepsilon}_{f} + (1-\chi^{\varepsilon})\mathbb{P}^{\varepsilon}_{s}$ is the stress tensor, $\mathbb{P}_{f} = \chi^{\varepsilon}\mathbb{P}^{\varepsilon}$ is the stress tensor in the liquid component, $(1-\chi^{\varepsilon})\mathbb{P}^{\varepsilon}_{s}$ is the stress tensor in the solid component, $\mathbb{D}(x, \dfrac{\partial \boldsymbol{w}^{\varepsilon}_{f}}{\partial t})$ is a strain tensor in the liquid component, $\mathbb{D}(x, \boldsymbol{w}^{\varepsilon})$ is a strain tensor in the solid component, χ^{ε} is the characteristic function of the solid skeleton, D^{ε}_{N} is the normal velocity of the boundary Γ^{ε} in the direction of the unit normal $\boldsymbol{N}^{\varepsilon}(r)$ outward to Ω^{ε}_{f}, $v^{\varepsilon}_{f,N}$ is the normal component of the liquid velocity at the free boundary, $p_0(\boldsymbol{x}, t)$ and $c_0(\boldsymbol{x})$ are given presser and acid concentration at the wells.

The absolutely rigid solid skeleton has been considered in [11], where the key role in the microscopic description was played by the function $r(\boldsymbol{x}, t)$ from the set

$$\mathfrak{M}_{(0,T)} = \{r\ \in \mathbb{H}^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T),\ 0 \leqslant r(\boldsymbol{x}, t) < \frac{1}{2},\ -\theta \leqslant \frac{\partial r}{\partial t}(\boldsymbol{x}, t) \leqslant 0,$$
$$0 < \gamma < 1,\ \ \theta = \text{const} > 0;\ \ |r|^{(2+\gamma)}_{\Omega_T} \leqslant M_0\}, \tag{1.19}$$

which determined the structure of the pore space.

We call the problem (1.1) – (1.18) as a problem $\mathbb{A}^\varepsilon$. As we have mentioned in the abstract, we first consider the approximate problem $\mathbb{B}^\varepsilon(r)$ with given structure of the pore space. In this auxiliary problem, for a fixed $\varepsilon > 0$, the solid skeleton is a union of disjoint sets sufficiently close to balls of radius $\varepsilon\, r$, slowly decreasing in volume, which simplifies the geometry of the original pore space and allows us to prove the existence of approximate solutions. As usual, almost every new problem has multiple choice. For example, for our case, we may consider nonstationary Stokes equations, but then we somehow must find a priori estimates for the liquid velocities keeping in mind the difficulties with free boundary separating liquid and solid components. For the stationary Stokes equations we have the same problem and we must decide, what of the problem we may solve. In either case, we must first find some approximations that can make the problem easier. This is the path we have decided to take.

As the problem $\mathbb{B}^\varepsilon(r)$ we will call the problem $\mathbb{A}^\varepsilon$ without an additional boundary condition at the free boundary that defines this boundary, but with known structure of the pore space and with modified dynamic equation

$$\nabla\cdot\big(\alpha^\varepsilon_\mu\mathbb{D}(x,\frac{\partial \boldsymbol{w}^\varepsilon_f}{\partial t}) - \frac{\partial \pi^\varepsilon}{\partial t}\mathbb{I}\big) - \varepsilon\frac{\partial \boldsymbol{w}^\varepsilon_f}{\partial t} = 0,\ \ \nabla\cdot\boldsymbol{w}^\varepsilon_f = 0 \tag{1.20}$$

instead of dynamic equations (1.1).

We call the problem (1.2), (1.5), (1.6), (1.10), (1.11), (1.18), (1.20) as a ***Dynamic problem*** $\mathbb{B}^\varepsilon(r)$ and the problem (1.3), (1.7), (1.12), (1.14), (1.15) as a ***Diffusion problem*** $\mathbb{B}^\varepsilon(r)$ and the homogenization of the problem $\mathbb{B}^\varepsilon(r)$ as the problem $\mathbb{H}(r)$.

Finally, the homogenization of the boundary condition (1.9) gives us the operator from $\mathfrak{M}_{(0,T)}$ into $\mathfrak{M}_{(0,T)}$, which unique fixed point r^* will define desired unique homogenization of the problem $\mathbb{A}^\varepsilon$ as the problem $\mathbb{H}=\mathbb{H}(r^*)$.

To homogenize the problem $\mathbb{B}^\varepsilon(r)$ we will use Nguetseng's two-scale convergent method [13].

We neglected the convection of acid and products of chemical reaction, because the diffusion rate is an order of magnitude greater than the convection rate due to the very low fluid filtration rate. This speed does not exceed 4-7 m per year.

The concentrations of products of chemical reaction will be found after solving the problem $\mathbb{H}$.

1.2. Main result.

Theorem 1.1 *Let $r\in\mathfrak{M}_{(0,T)}$, $p_0\in\mathbb{W}^{1,1}_2(\Omega_T)$ and $c_0\in\mathbb{H}^{2+\alpha}(\overline{\Omega})$ be given functions and*

$$\alpha^\varepsilon_\mu = \varepsilon^2\mu_1,\ \ 0<\mu_1<\infty. \tag{1.21}$$

Then the problem $\mathbb{B}^\varepsilon(r)$ has an unique weak solution $\boldsymbol{w}^\varepsilon_f\in\mathbb{W}^{1,1}_2(\Omega_{f,T})$, $\boldsymbol{w}^\varepsilon_s\in\mathbb{W}^{1,0}_2(\Omega_{s,T})$, $p^\varepsilon,\pi^\varepsilon\in\mathbb{W}^{1,1}_2(\Omega_T)$ and $c^\varepsilon\in\mathbb{W}^{1,0}_2(\Omega_{f,T})$.

For definition of the weak solution to the problem $\mathbb{B}^\varepsilon(r)$ see section 3.

Theorem 1.2 *Under conditions of the Theorem 1.1 the problem $\mathbb{H}(r)$ has an unique solution $\boldsymbol{w}_f\in\mathbb{W}^{1,0}_2(\Omega_T)$, $\boldsymbol{w}_s\in\mathbb{W}^{1,0}_2(\Omega_T)$, $p,\pi\in\mathbb{W}^{1,1}_2(\Omega_T)$ and $c\in\mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T)$.*

Theorem 1.3 *Under conditions of the Theorem 1.1 the problem* $\mathbb{H}$ *has an unique solution* $\boldsymbol{w}_f \in \mathbb{W}_2^{1,0}(\Omega_T)$, $\boldsymbol{w}_s \in \mathbb{W}_2^{2,0}(\Omega_T)$, $p, \pi \in \mathbb{W}_2^{1,1}(\Omega_T)$ *and* $c \in \mathbb{H}^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T)$.

In our manuscript, we use the notation adopted in [7].

2. Notations and auxiliary results

2.1. Dimensionless parameters

The dimensionless parameter $\varepsilon = \dfrac{l}{L}$ is taken as a small parameter.

The dimensionless parameter λ_0 is the Lame coefficient.

The dimensionless parameter α_μ^ε characterizes the viscosity of the liquid in pores:

$$\alpha_\mu^\varepsilon = \frac{2\,\mu}{L\,g\,t_*\,\rho_0}$$

The dimensionless parameter α^ε characterizes the speed of dissolution of the solid skeleton.

Diffusion of acid is characterized by dimensionless coefficient

$$\alpha_c = \frac{D\,T}{L^2}.$$

Here l is the characteristic pore size and L is the characteristic size of the physical domain under consideration, t_* is the characteristic time of the duration of a physical process, ρ_0 is the density of water, g is the acceleration of gravity, μ is the dynamic viscosity of the liquid, ϱ_f is the dimensionless density of the liquid component related to the density of water ρ_0, α_c and $\alpha_{c,j}$, $j = 1, ..., k$ are the acid and products of chemical reactions diffusion coefficients. Parameters α^ε, β^ε, α_μ^ε and β_j^ε $j = 1, ..., k$, may depend on the small parameter ε and parameters α_c, $\alpha_{c,j}$, are given positive constants that do not depend on the small parameter ε.

2.2. Domains and boundaries

$\Omega \subset \mathbb{R}^3$ is a bounded domain with piecewise smooth boundary $S = \partial\Omega = \bar{S}^0 \cup \bar{S}^1 \cup \bar{S}^2$.

The boundary $S^0 \subset \mathbb{R}^3$ is impermeable to liquid in the pore space, the boundary $S^1 \subset \mathbb{R}^3$ simulates injection wells and boundary $S^2 \subset \mathbb{R}^3$ simulates production wells.

We will assume that Ω is the unit square,

$$S^0 = \{\boldsymbol{x} : x_3 = \pm\frac{1}{2}, -\frac{1}{2} \leqslant x_1,\, x_2 \leqslant \frac{1}{2}\},$$

$$S^1 = \{\boldsymbol{x} : x_1 = -\frac{1}{2}, -\frac{1}{2} \leqslant x_2,\, x_3 \leqslant \frac{1}{2}\},$$

$$S^2 = \{\boldsymbol{x} : x_1 = \frac{1}{2}, -\frac{1}{2} \leqslant x_2,\, x_3 \leqslant \frac{1}{2}\}.$$

By construction $\Omega_f^\varepsilon(t) \cup \Gamma^\varepsilon(t) \cup \Omega_f^\varepsilon(t) = \Omega$ and the free boundary $\Gamma^\varepsilon(t) = \partial\Omega_f^\varepsilon(t) \bigcap \partial\Omega_s^\varepsilon(t)$ divides liquid and solid components in Ω.

Let $\Omega_T = \Omega \times (0,T) \subset \mathbb{R}^3$, $\Omega_{f,T}^\varepsilon = \displaystyle\bigcup_{t=0}^{t=T} \Omega_f^\varepsilon(t)$, $\Omega_{s,T}^\varepsilon = \displaystyle\bigcup_{t=0}^{t=T} \Omega_s^\varepsilon(t)$, $\Gamma_T^\varepsilon = \displaystyle\bigcup_{t=0}^{t=T} \Gamma^\varepsilon(t)$ and

$$\Omega = \bigcup_{\boldsymbol{k}\in\mathbb{Z}} \overline{\Omega}^{\boldsymbol{k},\varepsilon}, \ \ \Omega^{\boldsymbol{k},\varepsilon} = \{\boldsymbol{x} \in \Omega : \boldsymbol{x} = \varepsilon \boldsymbol{k} + \varepsilon \boldsymbol{y}\},$$

$$\Omega_f^{\boldsymbol{k},\varepsilon}(t) = \Omega_f^{\varepsilon}(t) \cap \Omega^{\boldsymbol{k},\varepsilon}, \ \ \Omega_s^{\boldsymbol{k},\varepsilon}(t) = \Omega_s^{\varepsilon}(t) \cap \Omega^{\boldsymbol{k},\varepsilon}, \ \ \Gamma^{\boldsymbol{k},\varepsilon}(t) = \Gamma^{\varepsilon}(t) \cap \Omega^{\boldsymbol{k},\varepsilon}$$

for all $\boldsymbol{k} = (k_1, k_2, k_3)$, $k_1, k_2, k_3 \in \mathbb{Z}$ (integer numbers) and for all $\boldsymbol{y} \in Y = (-\frac{1}{2}, \frac{1}{2})^3 \subset \mathbb{R}^3$.

2.3. The structure of the pore space

In what follows all functions of the type $\varphi(\boldsymbol{y}; \boldsymbol{x}, t)$, where $(\boldsymbol{x}, t) \in \Omega$ and $\boldsymbol{y} \in \mathbb{R}^3$ are considered 1 - periodic in variable $\boldsymbol{y}$:

$$\varphi(\boldsymbol{y}; \boldsymbol{x}, t) = \varphi\big(\varsigma(\boldsymbol{y}); \boldsymbol{x}, t)\big), \ \ \boldsymbol{y} = [|\boldsymbol{y}|] + \varepsilon\, \varsigma(\boldsymbol{y}), \ \ [|\boldsymbol{y}|] = ([|y_1|], [|y_2|], [|y_3|]). \tag{2.1}$$

Here $[|a|]$ is the integer part of the number a.

We put $\mathbf{Y} = \{\boldsymbol{y} \in \mathbb{R}^3 : -\frac{1}{2} < y_k < \frac{1}{2}, \ k = 1, 2, 3\}$,

$$Y_f(r) = \{\boldsymbol{y} \in \mathbf{Y} : |\boldsymbol{y}| > r\}, \ Y_s(r) = \{\boldsymbol{y} \in \mathbf{Y} : |\boldsymbol{y}| < r\}. \tag{2.2}$$

and

$$\chi_r(\boldsymbol{y}) = \frac{sgn(|\boldsymbol{y}| - r) + 1}{2}, \ \chi_r^{\varepsilon}(\boldsymbol{x}, t) = \chi_r(\frac{\boldsymbol{x}}{\varepsilon}). \tag{2.3}$$

Thus,

$$\Omega_f^{\varepsilon}(r) = \mathrm{Int}\{\boldsymbol{x} \in \Omega : \chi_r^{\varepsilon}(\boldsymbol{x}, t) > 0\}, \ \Omega_s^{\varepsilon}(r) = \mathrm{Int}\{\boldsymbol{x} \in \Omega : \chi_r^{\varepsilon}(\boldsymbol{x}, t) < 0\};$$

$$\Omega_j^{\boldsymbol{k},\varepsilon}(r) = \Omega^{\boldsymbol{k},\varepsilon} \cap \Omega_j^{\varepsilon}(r), \ \Omega_{j,T}^{\varepsilon}(r) = \bigcup_{t=0}^{t=T} \Omega_j^{\varepsilon}\big(r(\boldsymbol{x}, t)\big) \ j = f, s; \tag{2.4}$$

$$\Gamma^{\varepsilon}(r) = \overline{\Omega}_f^{\varepsilon}(r) \bigcap \overline{\Omega}_s^{\varepsilon}(r), \ \Gamma^{\varepsilon,k}(r) = \Gamma^{\varepsilon}(r) \bigcap \Omega^{\boldsymbol{k},\varepsilon}, \ k = 1, ..., n^3. \tag{2.5}$$

We call the structure, defined by the formula (2.3) as ***Structure with special periodicity***.

2.4. Matrices and differential operators

We fix the standard Cartesian basis $\{\boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3\}$ in $\mathbb{R}^3$ and $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are ***Tensors*** (linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$). The action of the tensor $\mathbb{A}$ on the vector $\boldsymbol{b}$ is denoted as the vector $\boldsymbol{c} = \mathbb{A} < \boldsymbol{b} >$. As $(\boldsymbol{a} \cdot \boldsymbol{b})$ we denote the ***Scalar product*** of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The product $\mathbb{C} = \mathbb{A} \cdot \mathbb{B}$ is a transformation $\mathbb{A} : \mathbb{B}(\mathbb{R}^3) \to \mathbb{R}^3$, where $\mathbb{B}(\mathbb{R}^3) = \{\boldsymbol{y} \in \mathbb{R}^3 : \boldsymbol{y} = \mathbb{B}(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathbb{R}^3\}$. $\mathbb{I}$ is a unit tensor: $\mathbb{I} \cdot \mathbb{A} = \mathbb{A} \cdot \mathbb{I} = \mathbb{A}$ for any tensor $\mathbb{A}$.

For any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ as $\boldsymbol{a} \otimes \boldsymbol{b}$ we denote the ***Diad*** (second-order tensor), where $(\boldsymbol{a} \otimes \boldsymbol{b}) < \boldsymbol{c} >= \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c})$.

As $\mathbb{J}_{ij}$ we denote the tensor $\frac{1}{2}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i)$. Then $\mathbb{A} = \sum_{i,j=1}^{3} a_{ij} \mathbb{J}_{ij}$. Tensor $\mathbb{A}$ is symmetric, if $(\mathbb{A} < \boldsymbol{e}_j > \cdot \boldsymbol{e}_i) = (\mathbb{A} < \boldsymbol{e}_i > \cdot \boldsymbol{e}_j)$.

The tensor $\mathbb{C}$ is symmetric if $(\mathbb{C} < \boldsymbol{a} > \cdot \boldsymbol{b}) = (\mathbb{C} < \boldsymbol{b} > \cdot \boldsymbol{a})$.

By (A), (B), and (C) we denote the corresponding to tensors $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ matrices in the chosen Cartesian coordinate system:

$$(A)=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix},\ (B)=\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix},\ (C)=\begin{pmatrix} c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{pmatrix},$$

For matrices the usual operations of sum $(A)+(B)$, multiplication by scalars $\alpha(B)$ and product $(A)\cdot(B)$ are defined.

Let $\boldsymbol{u}(\boldsymbol{x},t)=\big(u_1(\boldsymbol{x},t),u_2(\boldsymbol{x},t),u_3(\boldsymbol{x},t)\big)$ and $\mathbb{D}(x,\boldsymbol{u})=\dfrac{1}{2}\big(\nabla_x\boldsymbol{u}+(\nabla_x\boldsymbol{u})^*\big)$.

Then the second-order tensor

$$\mathbb{D}(x,\boldsymbol{u})=\sum_{i,j=1}^{3}d_{ij}(x,\boldsymbol{u})\boldsymbol{e}_i\otimes\boldsymbol{e}_j,\ \ d_{ik}(x,\boldsymbol{u})=\frac{1}{2}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}),\ i,\,j=1,\,2,\,3,$$

is called the ***Symmetric gradient*** of the vector $\boldsymbol{u}$.

We put as the definition

$$\mathbb{D}(x,\boldsymbol{u})<\boldsymbol{a}>\overset{def.}{=}\sum_{i=1}^{3}d_{ij}(x,\boldsymbol{u})a_i(\boldsymbol{e}_i\otimes\boldsymbol{e}_j+\boldsymbol{e}_j\otimes\boldsymbol{e}_i). \tag{2.6}$$

Then

$$\mathbb{D}(x,\boldsymbol{u}):\mathbb{D}(x,\boldsymbol{v})=\sum_{i,j=1}^{3}d_{ij}(x,\boldsymbol{u})d_{ij}(x,\boldsymbol{v}),\ \ \mathbb{D}(x,\boldsymbol{u}):\mathbb{D}(x,\boldsymbol{u})=|\mathbb{D}(x,\boldsymbol{u})|^2=\sum_{i,j=1}^{3}|d_{ij}(x,\boldsymbol{u})|^2,$$

$$\|\mathbb{D}\big(x,\boldsymbol{u}(.,t_0)\big)\|^2_{2,\Omega}=\sum_{i,j=1}^{3}\|d_{ij}\big(x,\boldsymbol{u}(.,t_0)\big)\|^2_{2,\Omega},\ \ \|\mathbb{D}\big(x,\boldsymbol{u})\big)\|^2_{2,\Omega_T}=\sum_{i,j=1}^{3}\|d_{ij}(x,\boldsymbol{u})\|^2_{2,\Omega_T}, \tag{2.7}$$

$$\big(\mathbb{D}(x,\boldsymbol{u})<\boldsymbol{a}>\cdot\boldsymbol{b}\big)=\big(\mathbb{D}(x,\boldsymbol{u})<\boldsymbol{b}>\cdot\boldsymbol{a}\big)\overset{def.}{=}\mathbb{D}(x,\boldsymbol{u})<\boldsymbol{a},\boldsymbol{b}>, \tag{2.8}$$

$$|\mathbb{D}\big(x,\boldsymbol{u}(.,t_0)\big)|^2=\int_0^{t_0}|\mathbb{D}\big(x,\frac{\partial\boldsymbol{u}}{\partial t}(.,t)\big)|^2dt. \tag{2.9}$$

2.5. Moving boundaries and strong gaps

Let $[A]=A_f-A_s$ be a jump of A over the $\mathbb{C}^1$ boundary $\Gamma^\varepsilon(r)$.

Lemma 2.1 *(Integration by parts[8], Appendix A, section A.6, formula (A.6.13).)*

Let $\mathbb{C}^1$ boundary $\Gamma^\varepsilon(r)$ divides Ω_T into two subdomains $\Omega_{f,T}$ and $\Omega_{f,T}$.

Then for any smooth function η, vanishing at $\partial\Omega$, holds true the integral identity

$$\int_0^{t_0}\int_\Omega\eta\,\big(\frac{\partial A_f}{\partial t}\chi^\varepsilon_r+\frac{\partial A_s}{\partial t}(1-\chi^\varepsilon_r)+\nabla\cdot(\chi^\varepsilon_r\boldsymbol{B}_f+(1-\chi^\varepsilon_r)\boldsymbol{B}_f\big)dxdt=$$

$$\int_0^{t_0}\int_{\Gamma^\varepsilon(r)}\eta\left((A_s-A_f)\,D_N^\varepsilon+(\boldsymbol{B}_f-\boldsymbol{B}_s)\cdot\boldsymbol{N}^\varepsilon\right)\sin\psi\,d\sigma dt+$$
$$\int_\Omega\eta(\boldsymbol{x},t_0)\Big(\chi_r^\varepsilon(\boldsymbol{x},t_0)\big(A_f(\boldsymbol{x},t_0)\chi_r^\varepsilon(\boldsymbol{x},t_0)+A_s(\boldsymbol{x},t_0)(1-\chi_r^\varepsilon(\boldsymbol{x},t_0))\big)\Big)dx-$$
$$\int_\Omega\eta(\boldsymbol{x},0)\Big(\chi_r^\varepsilon(\boldsymbol{x},0)\big(A_f(\boldsymbol{x},0)\chi_r^\varepsilon(\boldsymbol{x},t_0)+A_s(\boldsymbol{x},t_0)(1-\chi_r^\varepsilon(\boldsymbol{x},t_0))\big)\Big)dx-$$
$$\int_0^{t_0}\int_\Omega(\frac{\partial\eta}{\partial t}(A_f\chi_r^\varepsilon+A_s(1-\chi_r^\varepsilon)+\big((\chi_r^\varepsilon\boldsymbol{B}_f+(1-\chi_r^\varepsilon)\boldsymbol{B}_s\big)\cdot\nabla\eta)dxdt.$$

Here $0<t_0<T$, $\boldsymbol{N}^\varepsilon\in\mathbb{R}^3$ *is the unit normal vector to* $\Gamma^\varepsilon(r)$, *pointing outward to* $\Omega_f^\varepsilon(r)$, D_N^ε *is the normal velocity of the boundary* $\Gamma^\varepsilon(r)$ *in the direction of the normal* $\boldsymbol{N}^\varepsilon$, *and* ψ *is the angle between unit vector* $\boldsymbol{l}$ *of the time axis and unit normal vector* $\boldsymbol{\nu}\in\mathbb{R}^4$ *to* Γ_T^ε, *pointing outward to* $\Omega_{f,T}^\varepsilon$, *such that* $\sin\psi=\boldsymbol{\nu}\cdot\boldsymbol{N}$ *and* $\cos\psi=\boldsymbol{\nu}\cdot\boldsymbol{l}$.

In particular,

$$\int_0^{t_0}\int_{\Omega_s^\varepsilon(r)}\eta\,\frac{\partial A_s}{\partial t}dxdt=\int_0^{t_0}\int_{\Gamma^\varepsilon(r(.,t))}\eta\,A_sD_N^\varepsilon\sin\psi\,d\sigma dt-\int_0^{t_0}\int_{\Omega_s^\varepsilon(r)}A_s\frac{\partial\eta}{\partial t}dxdt,$$
$$\int_0^{t_0}\int_{\Omega_f^\varepsilon(r)}\eta\,\frac{\partial A_f}{\partial t}dxdt=-\int_0^{t_0}\int_{\Gamma^\varepsilon(r(.,t))}\eta\,A_fD_N^\varepsilon\sin\psi\,d\sigma dt-\int_0^{t_0}\int_{\Omega_f^\varepsilon(r)}A_f\frac{\partial\eta}{\partial t}dxdt.\quad(2.10)$$

2.6. Poincaré inequality

Lemma 2.2 *Let* $Q\subset\mathbb{R}^3$ *be a bounded domain with Lipschitz piecewise smooth boundary. Then for any function* $\boldsymbol{w}\in\overset{\circ}{W}{}_2^1(Q)$ *holds true the inequality* [15]*:*

$$\|\boldsymbol{w}\|_{2,Q}\leqslant M(Q)\|\mathbb{D}(x,\boldsymbol{w})\|_{2,Q},$$

where $M(Q)<\infty$ *for bounded* Q.

If $\Omega\subset\bigcup_{|\boldsymbol{k}|=1}^{n^3}\Omega^{\boldsymbol{k},\varepsilon}$ *and* $\boldsymbol{w}\in\overset{\circ}{W}{}_2^1(\Omega^{\boldsymbol{k},\varepsilon})$ $\boldsymbol{k}=(k_1,k_2,k_3),\in\mathbb{Z}$, *then*

$$\int_{\Omega^{\boldsymbol{k},\varepsilon}}|\boldsymbol{w}|^2dx\leqslant\varepsilon^2M(\Omega^{\boldsymbol{k},\varepsilon})\int_{\Omega^{\boldsymbol{k},\varepsilon}}|\mathbb{D}(x,\boldsymbol{w})|^2dx$$

and

$$\int_\Omega|\boldsymbol{w}|^2dx\leqslant\varepsilon^2M(\Omega)\int_\Omega|\mathbb{D}(x,\boldsymbol{w})|^2dx.\quad(2.11)$$

2.7. The simplest embedding theorem

Lemma 2.3 *Let* $\Omega\subset\mathbb{R}^3$ *with peace-wice* $\mathbb{C}^1$ *boundary.*

Then for any function $u\in\mathbb{W}_2^1(\Omega)$ *identically equal zero on some part of the boundary* $\partial\Omega$ *with strictly positive surface measure holds true the estimate*

$$\|u\|_{2,\Omega}\leqslant M(\Omega)\|\nabla u\|_{2,\Omega}.\quad(2.12)$$

The constant $M(\Omega)$ *is bounded for bounded* Ω.

(See inequality (2.14), Theorem 2.1, §2, chapter II in [7]).

Remark 2.4 *The condition that* $u(\boldsymbol{x})$ *vanishes at* $\partial\Omega$ *can be replaced by the condition*

$$\int_{\Omega} u(\boldsymbol{x})dx = 0.$$

(see Remark 2.1., Theorem 2.2, § 2, Chapter II, [7]).

2.8. Mollifiers

Let $J(s) \geqslant 0$, $J(s) = 0$ for $|s| > 1$, $J(s) = J(-s)$, $J \in C^{\infty}(-\infty, +\infty)$, and

$$\int_{\mathbb{R}^3} J(|\boldsymbol{x}|)dx = 1, \;\; \boldsymbol{x} \in \mathbb{R}^3. \tag{2.13}$$

Definition 2.5 *The operator* $\boldsymbol{M}_h : \mathbb{L}_2(\Omega) \to \mathbb{C}^{\infty}(\overline{\Omega})$

$$\boldsymbol{M}_h(\boldsymbol{u})(\boldsymbol{x}) = \frac{1}{h^3}\int_{\mathbb{R}^3} J(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{h})\boldsymbol{u}(\boldsymbol{y})dy, \tag{2.14}$$

is called a ***Mollifier*** *and the function* $\boldsymbol{M}_h(\boldsymbol{u})$ *is called the* ***Mollification***.

Lemma 2.6 *Let* $\boldsymbol{u} \in \mathbb{L}_p(\Omega)$ *and* $p \geqslant 1$.

Then

$$\int_{\Omega} \boldsymbol{M}_h(\boldsymbol{u})\boldsymbol{v}dx = \int_{\Omega} \boldsymbol{u}\boldsymbol{M}_h(\boldsymbol{v})dx \tag{2.15}$$

and

$$\|\boldsymbol{M}_h(\boldsymbol{u})\|_{p,\Omega} \leqslant \|\boldsymbol{u}\|_{p,\Omega}, \;\; \lim_{h\to 0} \|\boldsymbol{M}_h(\boldsymbol{u}) - \boldsymbol{u}\|_{p,\Omega} = 0 \tag{2.16}$$

For proof see [1], Lemma 2.18.

2.9. Extension Lemma

Extension results are very important in homogenization (Zhikov at al.[17]). For example, some sequence has different properties in different domains and only properties of the sequence in the first domain permits to choose convergent subsequence. Therefore, we must preserve the best properties of the sequence and apply the extension from the first domain onto the second one. Fortunately all the indicated results apply for our case for structure with special periodicity because in each cell of periodicity $\Omega^{\boldsymbol{k},\varepsilon}$ we may directly use the method suggested in [17] for soft inclusions (see chapter 3 Elementary Soft and Stiff Problems, section 3.1, pp. 86-95).

Lemma 2.7 *Let* $\{\boldsymbol{w}_s^{\varepsilon}\}$ *be a bounded sequence in* $\mathbb{W}_2^{1,0}\big(\Omega_{s,T}^{\varepsilon}(r)\big)$

1) Then for all $\varepsilon > 0$ *there exist extensions* $\widetilde{\boldsymbol{w}}_s^{\varepsilon} = \mathbb{E}_s(\boldsymbol{w}_s^{\varepsilon})$, $\mathbb{E}_s : \mathbb{W}_{2,s}^{1,0}(\Omega_T) \to \mathbb{W}_2^{1,0}(\Omega_T)$, *such that*

$$(\widetilde{\boldsymbol{w}}_s^{\varepsilon}-\boldsymbol{w}_s^{\varepsilon})(1-\chi_r^{\varepsilon})=0,\ \big(\mathbb{D}(x,\widetilde{\boldsymbol{w}}_s^{\varepsilon})-\mathbb{D}(x,\boldsymbol{w}_s^{\varepsilon})\big)(1-\chi_r^{\varepsilon})=0;$$
$$\|\widetilde{\boldsymbol{w}}_s^{\varepsilon}\|_{2,\Omega_T}\leqslant M\|\boldsymbol{w}_s^{\varepsilon}\|_{2,\Omega_{s,T}(r)},\ \ \|\mathbb{D}(x,\widetilde{\boldsymbol{w}}_s^{\varepsilon})\|_{2,\Omega_T}\leqslant M\|\mathbb{D}(x,\boldsymbol{w}_s^{\varepsilon})\|_{2,\Omega_{s,T}(r)},$$

where M *is independent of* ε*.*

2) For each elementary cell $\Omega^{\boldsymbol{k},\varepsilon}$ *such that the measure of the intersection* $\Omega^{\boldsymbol{k},\varepsilon}\bigcap\Omega_{s,T}(r)$ *is positive* $\widetilde{\boldsymbol{w}}_s^{\varepsilon}(\boldsymbol{x},t)=0$ *for* $\boldsymbol{x}\in\partial\big(\Omega^{\boldsymbol{k},\varepsilon}\bigcap\Omega_{s,T}(r)\big)$ *and* $\widetilde{\boldsymbol{w}}_s^{\varepsilon}\in$

3) Let $\{\boldsymbol{w}_f^{\varepsilon}\}$ *be a bounded sequence in* $\mathbb{W}_2^{1,1}\big(\Omega_{f,T}^{\varepsilon}(r)\big)$*.*

Then for all $\varepsilon>0$ *there exist extensions* $\widetilde{\boldsymbol{w}}_f^{\varepsilon}\in\mathbb{W}_2^{1,1}(\Omega_T)$ *from* $\Omega_{f,T}^{\varepsilon}(r)$ *onto* Ω_T *of functions* $\boldsymbol{w}_f^{\varepsilon}$ *such that* $\widetilde{\boldsymbol{w}}_f^{\varepsilon}\in\mathbb{W}_2^{1,1}(\Omega_T)$*,* $\widetilde{\boldsymbol{w}}_s^{\varepsilon}\in\overset{\circ}{\mathbb{W}}{}_2^{1,0}(\Omega^{\boldsymbol{k},\varepsilon})\cap\overset{\circ}{\mathbb{W}}{}_2^{1,0}\big(\Omega_{s,T}^{\varepsilon}(r)\big)$ *and*

$$(\widetilde{\boldsymbol{w}}_f^{\varepsilon}-\boldsymbol{w}_f^{\varepsilon})\chi_r^{\varepsilon}=0,\ \big(\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^{\varepsilon})-\mathbb{D}(x,\boldsymbol{w}_f^{\varepsilon})\big)\chi_r^{\varepsilon}=0,\ (\widetilde{\boldsymbol{w}}_f^{\varepsilon}-\boldsymbol{w}_f^{\varepsilon})\chi_r^{\varepsilon}=0,\ (\frac{\partial\widetilde{\boldsymbol{w}}_f^{\varepsilon}}{\partial t}-\frac{\partial\boldsymbol{w}_f^{\varepsilon}}{\partial t})\chi_r^{\varepsilon}=0;$$
$$\|\widetilde{\boldsymbol{w}}_f^{\varepsilon}\|_{2,\Omega_T}\leqslant M\|\boldsymbol{w}_f^{\varepsilon}\|_{2,\Omega_{f,T}(r)},\ \ \|\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^{\varepsilon})\|_{2,\Omega_T}\leqslant M\|\mathbb{D}(x,\boldsymbol{w}_f^{\varepsilon})\|_{2,\Omega_{f,T}(r)},$$
$$\|\frac{\partial\widetilde{\boldsymbol{w}}_f^{\varepsilon}}{\partial t}\|_{2,\Omega_T}\leqslant M\|\frac{\partial\boldsymbol{w}_f^{\varepsilon}}{\partial t}\|_{2,\Omega_{f,T}}. \tag{2.17}$$

Proof First of all, we note that displacements $\boldsymbol{w}_s^{\varepsilon}$ of the elastic skeleton enter the differential equations and boundary conditions only through their derivatives and, excluding elementary cells intersecting with the boundary S^0, all other cells are surrounded completely (for cells without intersection with S^0) or partially (for intersection with $S^1\cup S^2$) by a liquid component. That is, in the cells we have marked, the displacements $\boldsymbol{w}_s^{\varepsilon}$ of the elastic skeleton are determined with an accuracy of an arbitrary constant. To avoid this possibility we set

$$\widetilde{\boldsymbol{w}}_s^{\varepsilon}=\mathbf{M}_h(\boldsymbol{w}_s^{\varepsilon})\ \text{ for } h<\frac{1}{2}\big(\frac{1}{2}-\max_{\boldsymbol{x}\in\Omega}r_0(\boldsymbol{x})\big). \tag{2.18}$$

By construction $\widetilde{\boldsymbol{w}}_s^{\varepsilon}\in\mathbb{W}_2^{1,0}(\Omega_s^{\boldsymbol{k},\varepsilon}(r))$. It is easy to see that $\widetilde{\boldsymbol{w}}_s^{\varepsilon}\in\overset{\circ}{\mathbb{W}}{}_2^{1,0}(\Omega^{\boldsymbol{k},\varepsilon})\cap\overset{\circ}{\mathbb{W}}{}_2^{1,0}\big(\Omega_{s,T}^{\varepsilon}(r)\big)$.

Thus, to prove the second statement we first apply embedding theorem (estimate (2.12)) in each cell $\Omega^{\boldsymbol{k},\varepsilon}$ and then in Ω_T.

To prove the last statement we note that there are several options for extensions of $\boldsymbol{w}_f^{\varepsilon}$. We choose the extension

$$\widetilde{\boldsymbol{w}}_f^{\varepsilon}=\chi_r^{\varepsilon}\boldsymbol{w}_f^{\varepsilon}-(1-\chi_r^{\varepsilon})\boldsymbol{w}_s^{\varepsilon} \tag{2.19}$$

for which

$$\widetilde{\boldsymbol{w}}_f^{\varepsilon}(\boldsymbol{x},t)=0,\ \ \boldsymbol{x}\in\Gamma\big(r(\boldsymbol{x},t)\big),\ 0<t<T. \tag{2.20}$$

□

2.10. Hölder's inequality

Lemma 2.8 *For any* $f,\ g\in\mathbb{L}_2(\Omega)$ *holds true the Hölder's inequality*

$$\|f\,g\|_{1,\Omega}\leqslant\|f\|_{2,\Omega}\|g\|_{2,\Omega}. \tag{2.21}$$

For details see [5], section 37, chapter 10, p.384.

2.11. Orthogonal decomposition of the space $\mathbb{L}_2(\Omega)$

Let $\Omega \subset \mathbb{R}^3$ and $\overset{\circ}{\mathbb{J}}(\Omega_T)$ be the set of all infinitely smooth solenoidal vector functions $\boldsymbol{u}^J$ in Ω_T, and $\overset{\circ}{\mathbb{G}}(\Omega_T)$ be the set of the gradients $\boldsymbol{u}^G = \nabla u^G$ of all infinitely smooth real functions u^J in Ω_T.

As $\overset{\circ}{\mathbb{J}}{}^{(1)}(\Omega_T)$ we denote the closure of the set $\overset{\circ}{\mathbb{J}}(\Omega_T)$ in $\mathbb{L}_2(\Omega_T)$ and as the set $\overset{\circ}{\mathbb{G}}{}^{(1)}(\Omega_T)$ we denote the closure of the set $\overset{\circ}{\mathbb{G}}(\Omega_T)$ in $\mathbb{L}_2(\Omega)$.

Lemma 2.9 *The space $\mathbb{L}_2(\Omega_T)$ is the direct sum of the subspaces $\overset{\circ}{\mathbb{J}}{}^{(1)}(\Omega_T)$ and $\overset{\circ}{\mathbb{G}}{}^{(1)}(\Omega_T)$ in $\mathbb{L}_2(\Omega_T)$.*

For proof see [6], § 2.

3. Equivalent formulation of the problem $\mathbb{B}^\varepsilon(r)$ as systems of integral identities

First of all, we note that for incompressible media, the pressure p^ε ceases to be a dynamic variable and instead of the basic space $\mathbb{L}_2(\Omega_T)$ we can choose its subspace $\overset{\circ}{\mathbb{J}}{}^{(1)}(\Omega_T)$ as the basic space (see Lemma 2.9).

In what follows we will leave the same notations $\mathbb{W}_2^{1,0}(\Omega_T)$ and $\mathbb{W}_2^{1,1}(\Omega_T)$ for the subspaces of all solenoidal functions in $\mathbb{W}_2^{1,0}(\Omega_T)$ and $\mathbb{W}_2^{1,1}(\Omega_T)$.

Note also that for the selected structure the rigid skeleton is a union of disjoint sets, sufficiently close to spheres of radius εr, slowly decreasing in volume, which simplifies the geometry of the original pore space and allows us to prove the existence of weak solutions to the problem $\mathbb{B}^\varepsilon(r)$.

3.1. Equivalent formulation of the dynamic problem $\mathbb{B}^\varepsilon(r)$ as systems of integral identities

Definition 3.1 *Let the structure $\chi_r^\varepsilon(\boldsymbol{x},t)$ of the pore space $\Omega_{f,T}^\varepsilon$ be given by the function $r \in \mathfrak{M}_{(0,T)}$ and*

$$I_1^\varepsilon(\boldsymbol{\varphi}) = \int_0^{t_0}\int_{\Omega_f^\varepsilon r(.,t)} \mu_1\varepsilon^2 \mathbb{D}(x, \frac{\partial \widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}) : \mathbb{D}(x,\boldsymbol{\varphi})dxdt,$$

$$I_2^\varepsilon(\boldsymbol{\varphi}) = \lambda_0 \int_0^{t_0}\int_\Omega \big((1-\chi_r^\varepsilon)(\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_s^\varepsilon\big)\big) : \mathbb{D}\big(x,\boldsymbol{\varphi})dxdt,$$

$$I_3^\varepsilon(\boldsymbol{\varphi}) = \varepsilon\int_0^{t_0}\int_\Omega \chi_r^\varepsilon(\frac{\partial \boldsymbol{w}_f^\varepsilon}{\partial t}\cdot\boldsymbol{\varphi})dxdt.$$

Then functions $\widetilde{\boldsymbol{w}}_f^\varepsilon$ and $\widetilde{\boldsymbol{w}}_s^\varepsilon$, where $\widetilde{\boldsymbol{w}}_f^\varepsilon \in \mathbb{W}_2^{1,1}(\Omega_T)$ $\widetilde{\boldsymbol{w}}_s^\varepsilon \in \mathbb{W}_2^{1,0}(\Omega_T)$ are called a weak solution to the dynamic problem $\mathbb{B}^\varepsilon(r)$, if they satisfy boundary conditions (1.11), the dynamic integral identity

$$-\int_0^{t_0}\int_\Omega \big((\nabla p_0\cdot\boldsymbol{\varphi})\big)dxdt = \int_0^{t_0}\int_\Omega \chi_r^\varepsilon\big(\varepsilon^2\mu_1\mathbb{D}(x,\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})\big) : \mathbb{D}(x,\boldsymbol{\varphi})dxdt +$$
$$\int_0^{t_0}\int_\Omega \big((1-\chi_r^\varepsilon)(\lambda_0\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_s^\varepsilon\big)\big) : \mathbb{D}\big(x,\boldsymbol{\varphi})dxdt + \varepsilon\int_0^{t_0}\int_\Omega \chi_r^\varepsilon(\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}\cdot\boldsymbol{\varphi})dxdt \equiv$$
$$I_1^\varepsilon(\boldsymbol{\varphi}) + I_2^\varepsilon(\boldsymbol{\varphi}) + I_3^\varepsilon(\boldsymbol{\varphi}),\ \ 0<t_0<T \quad (3.1)$$

for arbitrary smooth functions $\boldsymbol{\varphi}$, *vanishing at the boundary* $S^0 \times (0,T)$, *and the continuity integral identity*

$$\int_0^{t_0} \int_\Omega \big(\chi_r \widetilde{\boldsymbol{w}}_f^\varepsilon + (1-\chi_r^\varepsilon)\widetilde{\boldsymbol{w}}_s^\varepsilon\big) \cdot \nabla\zeta dxdt = 0, \;\; 0 < t_0 < T \tag{3.2}$$

for arbitrary smooth functions ζ, *vanishing at the boundary* $(S^1 \cup S^2) \times (0,T)$.

3.2. Equivalent formulation of the diffusion problem $\mathbb{B}^\varepsilon(r)$ as an integral identity

Definition 3.2 *Let the structure* $\chi_r^\varepsilon(\boldsymbol{x},t)$ *of the pore space* $\Omega_{f,T}^\varepsilon$ *be given by the function* $r \in \mathfrak{M}_{(0,T)}$.

Then the function $\widetilde{c}^\varepsilon \in \mathbb{W}_2^{1,0}(\Omega_T)$ *is called a weak solution to the diffusion problem* $\mathbb{B}^\varepsilon(r)$ *if it satisfy boundary condition (1.14) and the following integral identity*

$$\int_\Omega \chi^\varepsilon(.,t_0)\big(\widetilde{c}^\varepsilon(.,t_0) + \frac{\beta^\varepsilon}{\alpha^\varepsilon}\big)\xi(.,t_0)dx - \int_\Omega \chi^\varepsilon(.,0)\big(c_0 + \frac{\beta^\varepsilon}{\alpha^\varepsilon}\big)\,\xi(.,0)\,dx +$$
$$\int_0^{t_0}\int_\Omega \chi^\varepsilon\big(-(\widetilde{c}^\varepsilon + \frac{\beta^\varepsilon}{\alpha^\varepsilon})\,\frac{\partial\xi}{\partial t} + \nabla\xi\cdot(\alpha_c\nabla\widetilde{c}^\varepsilon\big)dxdt = 0. \tag{3.3}$$

Here ξ *is an arbitrary smooth function vanishing at the boundary* $\big(S^1 \cup S^2\big) \times (0,T)$ *and* $0 < t < T$.

Remark 3.3 *In deriving the integral identity, we used the boundary condition (1.9) on the free boundary, so that the term containing the integral over this boundary vanishes.*

3.3. Equivalent formulation of the boundary condition (1.9)

Lemma 3.4 [11] *The boundary condition* (1.9) *is equivalent to the integral identity*

$$\int\int_\Omega \chi_r^\varepsilon\big(-\frac{\partial}{\partial t}(\zeta\,\boldsymbol{a}^\varepsilon\cdot\boldsymbol{\xi}_0^\varepsilon) + \varepsilon\,\nabla\cdot(\zeta\,\widetilde{c}^\varepsilon\,\boldsymbol{\xi}_0^\varepsilon)\big)dxdt = 0 \tag{3.4}$$

which is valid for any smooth functions $\boldsymbol{\xi}_c^\varepsilon(r,\boldsymbol{x}) = \boldsymbol{\xi}_c(r,\dfrac{\boldsymbol{x}}{\varepsilon})$*, functions* ζ*, vanishing at* $t=0$ *and at* $t=$ *and at boundary* $\partial\Omega$*, and functions* $\boldsymbol{a}_c^\varepsilon(r,\boldsymbol{x}) = \boldsymbol{a}_c(r,\dfrac{\boldsymbol{x}}{\varepsilon})$*, such that* $\boldsymbol{a}_c$ *vanishes outside of some small neighbourhood of* $\gamma_c(r)$ *and* $\boldsymbol{a}_c(r,\boldsymbol{y}) = \boldsymbol{n}_c(r)$*, where* $\boldsymbol{n}_c(r)$ *is the unit normal to* $\gamma_c(r)$*, outward to the domain* $Y_f(r)$*.*

4. Proof of Theorem 1.1.

Due to the linearity of the problem $\mathbb{B}^\varepsilon(r)$ it is sufficient to simply derive the corresponding a priori estimates.

4.1. Existence of the weak solution to the dynamic problem $\mathbb{B}^\varepsilon(r)$

Lemma 4.1 *Under conditions of Theorem 1.1 the dynamic problem* $\mathbb{B}^\varepsilon(r)$ *has a unique weak solution such that*

$$\max_{0<t<T}\|\chi_r^\varepsilon(.,t)\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t)\|_{2,\Omega} \leqslant M(p_0)M(\Omega), \tag{4.1}$$

$$\max_{0<t<T}\|\big(1-\chi_r^\varepsilon(.,t)\big)\widetilde{\boldsymbol{w}}_s^\varepsilon(.,t)\|_{2,\Omega} \leqslant M(p_0)M(\Omega), \tag{4.2}$$

$$\sqrt{\varepsilon}\|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}\|_{2,\Omega_T} + \max_{0<t<T}\sqrt{\alpha_\mu^\varepsilon}\|\chi_r^\varepsilon(.,t)\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t)\big)\|_{2,\Omega} + \sqrt{\alpha_\mu^\varepsilon}\mathbb{D}(x,\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})\|_{2,\Omega_T} \leqslant M(p_0), \tag{4.3}$$

$$\max_{0<t<T}\|\big(1-\chi_r^\varepsilon(.,t)\big)\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_s^\varepsilon(.,t)\big)\|_{2,\Omega} \leqslant M(p_0), \tag{4.4}$$

where $M(p_0)$ *and* $M(\Omega)$ *do not depend on* ε *and* T *and* $M(\Omega)<\infty$ *for bounded* Ω*.*

Proof Let in (3.1) $\boldsymbol{\varphi} = (1-\chi_r^\varepsilon)\widetilde{\boldsymbol{w}}_s^\varepsilon - \chi_r^\varepsilon\widetilde{\boldsymbol{w}}_f^\varepsilon$.

Then using the simplest embedding theorem (Lemma 2.6) and Holder's inequality (Lemma 2.8) we obtain

$$\lambda_0\int_\Omega\big((1-\chi_r^\varepsilon(.,t_0)\big)|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_s^\varepsilon(.,t_0)\big)|^2dx = |\int_\Omega\big((1-\chi_r^\varepsilon(.,t_0)\big)\big(\nabla p_0\cdot\widetilde{\boldsymbol{w}}_s^\varepsilon(.,t_0)\big)dx| \leqslant$$
$$\delta\int_\Omega\big((1-\chi_r^\varepsilon(.,t_0)\big)|\widetilde{\boldsymbol{w}}_s^\varepsilon(.,)|^2 + M(\delta)\int_\Omega\big((1-\chi_r^\varepsilon)(.,t_0)\big)|\nabla p_0|^2dx \leqslant$$
$$\delta M(\Omega)\int_\Omega\big((1-\chi_r^\varepsilon)(.,t_0)\big)|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_s^\varepsilon(.,t_0)|^2dx + M(\delta)\int_\Omega\big((1-\chi_r^\varepsilon)(.,t_0)\big)|\nabla p_0|^2dx.$$

Estimates (4.2) and (4.4) now follow from the last estimate for $\delta = \dfrac{\lambda_0}{2M(\Omega)}$ and the embedding theorem.

To estimate liquid displacements we put in (3.1) $\boldsymbol{\varphi} = \chi_r^\varepsilon\dfrac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}$ and use integration by parts

$$|\int_{\Omega_f^\varepsilon(r)(.,t_0))}(\nabla p_0\cdot\widetilde{\boldsymbol{w}}_f^\varepsilon)dxdt| = \int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\varepsilon^2\mu_1|\mathbb{D}(x,\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})|^2dxdt + \varepsilon\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}|^2dxdt$$

and arrive at the equality

$$|\int_{\Omega_f^\varepsilon(r)(.,t_0))}(\nabla p_0\cdot\widetilde{\boldsymbol{w}}_f^\varepsilon)dxdt|=\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\varepsilon^2\mu_1|\mathbb{D}(x,\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})|^2dxdt+\varepsilon\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}|^2dxdt.$$

Taking into account the relation (2.9) we obtain

$$\int_{\Omega_f^\varepsilon(r(.,t_0))}\alpha_\mu^\varepsilon|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)\big)|^2dx+\varepsilon\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t)+\nu)}|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}|^2dxdt\leqslant$$
$$|\int_{\Omega_f^\varepsilon(r(.,t_0))}\big((\nabla p_0\cdot\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)\big)dx|\leqslant\delta\int_{\Omega_f^\varepsilon(r(.,t_0))}|\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)|^2dx+M(\delta,p_0)$$

for any positive δ.

The estimate for $\chi_r^\varepsilon\widetilde{\boldsymbol{w}}_f^\varepsilon$ now follows from Poincaré inequality (2.11):

$$\|\chi_r^\varepsilon(.,t)\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t)\|_{2,\Omega}\leqslant\varepsilon M\|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t)\big)\|_{2,\Omega}=M\frac{\varepsilon}{\sqrt{\varepsilon^2\mu_1}}\sqrt{\varepsilon^2\mu_1}\|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t)\big)\|_{2,\Omega}\leqslant\frac{1}{\mu_1}M(p_0)M(\Omega).$$

□

4.2. Existence of the weak solution to the diffusion problem $\mathbb{B}^\varepsilon(r)$

Lemma 4.2 *Under conditions of Theorem 1.1 the diffusion problem $\mathbb{B}^\varepsilon(r)$ has a unique weak solution $\widetilde{c}^\varepsilon$ such that*

$$\|(\widetilde{c}^\varepsilon-c_0)\|_{2,\Omega_T}+\|\nabla(\widetilde{c}^\varepsilon-c_0)\|_{2,\Omega_T}\leqslant MT\|\nabla c_0\|_{2,\Omega}. \tag{4.5}$$

Proof To prove it we only need to obtain a priori estimates to the solution of the diffusion problem $\mathbb{B}^\varepsilon(r)$, written in the equivalent form of the integral identity (3.3). To do this we repeat the proof of the Lemma 2.1 in §2, chapter III [7] with test function $\xi=\widetilde{c}^\varepsilon-c_0$ using trivial inequality $|ab|\leqslant\delta a^2+\dfrac{b^2}{4\delta}$ for any $\delta>0$, the Hölder inequality

$$|\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}u\,v\,dxdt|\leqslant\frac{\delta}{2}\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}|u|^2dxdt+\frac{1}{2\delta}\int_0^{t_0}\int_{\Omega_f^\varepsilon(t)}|u|^2dxdt$$

and integration by parts we obtain the chain of inequalities

$$0=\int_{\Omega_f^\varepsilon(r(.,t_0))}\big(\widetilde{c}^\varepsilon(\boldsymbol{x},t_0)-c_0(\boldsymbol{x})+\frac{\beta^\varepsilon}{\alpha^\varepsilon}+c_0(\boldsymbol{x})\big)\big(\widetilde{c}^\varepsilon(\boldsymbol{x},t_0)-c_0(\boldsymbol{x})\big)dx-$$
$$\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}(\widetilde{c}^\varepsilon-c_0+\frac{\beta^\varepsilon}{\alpha^\varepsilon}+c_0)\frac{\partial}{\partial t}(\widetilde{c}^\varepsilon-c_0+c_0+\frac{\beta^\varepsilon}{\alpha^\varepsilon})dxdt+$$
$$\alpha_c\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\big(\nabla(\widetilde{c}^\varepsilon-c_0)\cdot\nabla(\widetilde{c}^\varepsilon-c_0+c_0)\big)dxdt=$$
$$\int_{\Omega_f^\varepsilon(r(.,t_0))}\Big(\big(\widetilde{c}^\varepsilon(.,t_0)-c_0\big)^2+(\frac{\beta^\varepsilon}{\alpha^\varepsilon}+c_0)\big(\widetilde{c}^\varepsilon(.,t_0)-c_0\big)\Big)dx-$$

$$- \frac{1}{2}\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\frac{\partial}{\partial t}(\widetilde{c}^\varepsilon(.,t_0) - c_0 + c_0 + \frac{\beta^\varepsilon}{\alpha^\varepsilon})^2 dxdt + \alpha_c\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}|\nabla(\widetilde{c}^\varepsilon - c_0)|^2 dxdt +$$

$$\alpha_c\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\big(\nabla(\widetilde{c}^\varepsilon - c_0)\cdot\nabla c_0\big)dxdt =$$

$$\int_{\Omega_f^\varepsilon(r(.,t_0))}\Big(\big(\widetilde{c}^\varepsilon(.,t_0) - c_0\big)^2 + (\frac{\beta^\varepsilon}{\alpha^\varepsilon} + c_0)\big(\widetilde{c}^\varepsilon(.,t_0) - c_0\big) + \frac{1}{2}(\widetilde{c}_0 + \frac{\beta^\varepsilon}{\alpha^\varepsilon})^2\Big)\Big)dx +$$

$$\frac{1}{2}\int_0^{t_0}\int_{\Gamma_f^\varepsilon(r(.,t))}(\widetilde{c}^\varepsilon + \frac{\beta^\varepsilon}{\alpha^\varepsilon})^2 D_N^\varepsilon \sin\psi\, d\sigma dt + \alpha_c\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}|\nabla(\widetilde{c}^\varepsilon - c_0|^2 dxdt +$$

$$\alpha_c\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\big(\nabla(\widetilde{c}^\varepsilon - c_0)\cdot\nabla c_0\big)dxdt \geqslant$$

$$\int_{\Omega_f^\varepsilon(r(.,t_0))}\Big(\big(\widetilde{c}^\varepsilon(.,t_0) - c_0\big)^2 + (\frac{\beta^\varepsilon}{\alpha^\varepsilon} + c_0)\big(\widetilde{c}^\varepsilon(.,t_0) - c_0\big) + \frac{1}{2}(c_0 + \frac{\beta^\varepsilon}{\alpha^\varepsilon})^2\Big)\Big)dx +$$

$$\alpha_c\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}|\nabla(\widetilde{c}^\varepsilon - c_0)|^2 dxdt + \alpha_c\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\big(\nabla(\widetilde{c}^\varepsilon - c_0)\cdot\nabla c_0\big)dxdt \geqslant$$

$$\int_{\Omega_f^\varepsilon(r(.,t_0))}\Big(\big(\widetilde{c}^\varepsilon(.,t_0) - c_0\big)^2 + \frac{\alpha_c}{2}\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}|\nabla(\widetilde{c}^\varepsilon - c_0)|^2 dxdt - \frac{\alpha_c}{2}\int_\Omega|\nabla c_0|^2 dxdt \tag{4.6}$$

which proves the statement of the lemma. □

5. Homogenization of the problem $\mathbb{H}^\varepsilon(r)$

The homogenization procedure itself is well explained in many publications. For dynamic problems the reader can follow the proof of the formula (3.12) in §2, chapter VII, [17] or the proof Theorem 1, case $c_f^\varepsilon = \infty$ §1.1., chapter 1 in [8], and for diffusion problem- section 10.1, chapter 10 in [8].

Here we use G. Nguetseng's two – scale convergent method [13]:

Theorem 5.1 ***1.*** *Any bounded in $\mathbb{L}_2(\Omega_T)$ sequence $\{\boldsymbol{u}^\varepsilon\}$ contains some subsequence two – scale convergent to some 1-periodic in the variable $\boldsymbol{y}$ function $\boldsymbol{U}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{U}\in\mathbb{L}_2(\Omega_T\times Y)$.*

2. *Let sequences $\{\boldsymbol{u}^\varepsilon\}$ and $\{\varepsilon\mathbb{D}(x,\boldsymbol{u}^\varepsilon)\}$ are uniformly bounded in $\mathbb{L}_2(\Omega_T)$.*

Then there exists the 1 – periodic in $\boldsymbol{y}$ function $\boldsymbol{U}=\boldsymbol{U}(\boldsymbol{x},t,\boldsymbol{y})$ and the sequence $\{\boldsymbol{u}^\varepsilon\}$ such that $\boldsymbol{U}$, $\nabla_y\boldsymbol{U}\in\mathbb{L}_2(\Omega_T\times Y)$, and sequences $\{\boldsymbol{u}^\varepsilon\}$ and $\{\varepsilon\mathbb{D}(x,\boldsymbol{u}^\varepsilon)\}$ (for simplicity we keep the same indices for subsequences) two – scale converge in $\mathbb{L}_2(\Omega_T)$ to $\boldsymbol{U}$ and $\mathbb{D}(y,\boldsymbol{U})$ correspondingly.

3. *Let sequences $\{\boldsymbol{u}^\varepsilon\}$ and $\{D(x,\boldsymbol{u}^\varepsilon)\}$ are bounded in $\mathbb{L}_2(\Omega_T)$.*

Then there exist functions $\boldsymbol{u}(\boldsymbol{x},t)$ and $\boldsymbol{U}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{u}\in\mathbb{W}_2^{1,0}(\Omega_T)$, $\boldsymbol{U}\in\mathbb{L}_2(\Omega_T\times Y)\cap\mathbb{W}_2^{1,0}(Y)$ and subsequences from $\{\boldsymbol{u}^\varepsilon\}$ and $\{\mathbb{D}(x,\boldsymbol{u}^\varepsilon)\}$ such that the function $\boldsymbol{U}$ is 1 – periodic in $\boldsymbol{y}$, $\mathbb{D}(x,\boldsymbol{u})\in\mathbb{L}_2(\Omega_T)$, $D(y,\boldsymbol{U})\in\mathbb{L}_2(\Omega_T\times Y)$, and the subsequence from $\{\mathbb{D}(x,\boldsymbol{u}^\varepsilon)\}$ two – scale converges to the function $\mathbb{D}(x,\boldsymbol{u})+D(y,\boldsymbol{U})$.

Note, that weak and two – scale convergence are connected by the relation:

$$\text{if } u^{\varepsilon} \stackrel{2-sc.}{\to} U(\boldsymbol{x},t,\boldsymbol{y}) \text{ then } u^{\varepsilon}(\boldsymbol{x}) \rightharpoonup \int_Y U(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y} \ \text{(weakly converges)}.$$

Lemma 5.2 *Under the conditions of Theorem 1.1 there exist 1-periodic in the variable* $\boldsymbol{y}$ *functions* $\boldsymbol{W}(\boldsymbol{y};\boldsymbol{x},t)$, $\mathbb{D}\big(y,\boldsymbol{W}(\boldsymbol{y};\boldsymbol{x},t)\big)$, $\boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t)$ *and* $\mathbb{D}\big(y,\boldsymbol{U}(\boldsymbol{y};\boldsymbol{x},t)\big)$ *such that*

1) the sequence $\{\widetilde{\boldsymbol{w}}_f^{\varepsilon}\}$ *converges weakly to the function* $\boldsymbol{w}_f$ *and two-scale to the 1-periodic in variable* $\boldsymbol{y}$ *function* $\boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t)$;

2) the sequence $\{\varepsilon\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^{\varepsilon})\}$ *converges two-scale to the 1-periodic in variable* $\boldsymbol{y}$ *function* $\mathbb{D}(y,\boldsymbol{W}_f)$;

3) the sequence $\{(1-\chi^{\varepsilon})\widetilde{\boldsymbol{w}}_s^{\varepsilon}\}$ *converges two-scale and weakly to the function* $\boldsymbol{w}_s \in \mathbb{L}_2(\Omega_T)$;

4) the sequence $\{(1-\chi^{\varepsilon})\mathbb{D}(x,\widetilde{\boldsymbol{w}}_s^{\varepsilon})\}$ *converges two-scale to the function* $\mathbb{D}(x,\boldsymbol{w}_s)+D(y,\boldsymbol{W}_s)$;

5) the sequence $\{\widetilde{c}^{\varepsilon}\}$ *converges two-scale and weakly to the function* $c \in \mathbb{W}_2^{1,0}(\Omega_T)$;

6) the sequence $\{\nabla\widetilde{c}^{\varepsilon}\}$ *converges two-scale to the function* $\nabla c + \nabla_y C$, *where* $c \in \mathbb{W}_2^{1,0}(\Omega_T)$.

Here$\boldsymbol{W}_f \in \mathbb{L}_2(0,T;\Omega\times Y)$, $\boldsymbol{W}_s \in \mathbb{L}_2(\Omega_T\times Y)\cap\mathbb{W}_2^{1,0}(Y)$ *and* $C \in \mathbb{L}_2(\Omega_T\times Y)\cap\mathbb{W}_2^{1,0}(Y)$.

7) hold true the following a priori estimates

$$\|\boldsymbol{w}_f\|_{2,\Omega_T} \leqslant M(p_0)M(\Omega), \tag{5.1}$$

$$\|\mathbb{D}\big(y,\boldsymbol{W}_f)\|_{2,Y_f\times\Omega_T} \leqslant M(p_0)M(\Omega), \tag{5.2}$$

$$\|\boldsymbol{w}_s\|_{2,\Omega_T} + \|\mathbb{D}\big(x,\boldsymbol{w}_s)\|_{2,\Omega_T} \leqslant M(p_0)M(\Omega), \tag{5.3}$$

$$\|(c-c_0)\|_{2,\Omega_T}^{(1,0)} \leqslant M(c_0)M(\Omega), \tag{5.4}$$

where $M(p_0)$, $M(c_0)$, *and* $M(\Omega)$ *do not depend on* ε *and* T *and* $M(\Omega)<\infty$ *for bounded* Ω.

The proof is straightforward and based on the estimates (4.1)-(4.5).

We only note that

$$\sqrt{\varepsilon}\|\frac{\partial\widetilde{\boldsymbol{w}}_f^{\varepsilon}}{\partial t}\|_{2,\Omega_T} \leqslant M(p_0)$$

and

$$\lim_{\varepsilon\to 0}\varepsilon\frac{\partial\widetilde{\boldsymbol{w}}_f^{\varepsilon}}{\partial t} = 0.$$

5.1. Homogenization of the dynamic problem $\mathbb{B}^{\varepsilon}(r)$ for the liquid component

Now we restore the pressure

$$p(\boldsymbol{x},t) = \lim_{\varepsilon\to} p^{\varepsilon}(\boldsymbol{x},t)$$

and the antiderivative of the pressure

$$\pi(\boldsymbol{x},t) = \lim_{\varepsilon\to}\pi^{\varepsilon}(\boldsymbol{x},t) = \int_0^t p(\boldsymbol{x},s)ds$$

and add it to the limiting dynamic equations.

Namely, holds true the following

Lemma 5.3 *Under the conditions of the Theorem 1.1 the limiting procedure in the integral identities (3.1) and (3.2) results the following dynamic problem* $\mathbb{H}(r)$ *velocities and pressure, consisting of Darcy law of filtration*

$$\boldsymbol{w}_f = -\frac{1}{\mu_1}\mathbb{B}^v(r) < \nabla(\pi - p_0 t) > \tag{5.5}$$

for the liquid displacements $\boldsymbol{w}_f$ *and the antiderivative* π *of the presser* p *in the domain* Ω_T*, completed with boundary conditions*

$$\pi(\boldsymbol{x},t) = 0, \;\; \boldsymbol{x} \in S^1 \cup S^2, \;\; 0 < t < T, \tag{5.6}$$

$$\boldsymbol{w}_f \cdot \boldsymbol{n} = 0, \;\; \boldsymbol{x} \in S^0, \;\; 0 < t < T \tag{5.7}$$

and the Lame equations

The symmetric strictly positively definite matrix $\mathbb{B}^v(r)$ *is given by formula (1.1.27) in [8].*

Proof Let in (3.1) $\nabla \cdot \boldsymbol{\varphi} = 0$ and

$$I_1^\varepsilon(\boldsymbol{\varphi}) = \int_0^{t_0}\int_{\Omega_f^\varepsilon r(.,t)} \mu_1\varepsilon^2\mathbb{D}(x, \frac{\partial \widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}) : \mathbb{D}(x,\boldsymbol{\varphi})dxdt,$$

$$I_2^\varepsilon(\boldsymbol{\varphi}) = -\int_0^{t_0}\int_{\Omega_f^\varepsilon r(.,t)} \frac{\partial \pi^\varepsilon}{\partial t}\mathbb{I} : \mathbb{D}(x,\boldsymbol{\varphi})dxdt = -\int_0^{t_0}\int_\Omega \chi_r^\varepsilon \frac{\partial \boldsymbol{\varphi}}{\partial t}\cdot \nabla\pi^\varepsilon dxdt,$$

$$I_3^\varepsilon(\boldsymbol{\varphi}) = \varepsilon\int_0^{t_0}\int_\Omega \chi_r^\varepsilon(\frac{\partial \boldsymbol{w}_f^\varepsilon}{\partial t}\cdot\boldsymbol{\varphi})dxdt. \tag{5.8}$$

We rewrite the term $I_1^\varepsilon(\boldsymbol{\varphi})$, as

$$I_1^\varepsilon(\boldsymbol{\varphi}) = \int_0^{t_0}\int_{\Omega_f^\varepsilon\Omega_f^\varepsilon(r(.,t))} \big(\mu_1\varepsilon^2\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)\big) : \mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})dxdt +$$
$$\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\frac{\partial}{\partial t}\Big(\mu_1\varepsilon^2\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)\big) : \mathbb{D}(x,\boldsymbol{\varphi})\big)\Big)dxdt - \int_0\int_{\Omega_f^\varepsilon(t)}\big(\mu_1\varepsilon^2\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)\big) : \mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})dxdt =$$
$$-\int_0^{t_0}\int_{\Omega_f^\varepsilon(r(.,t))}\big(\mu_1\varepsilon^2\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)\big) : \mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})dxdt. \tag{5.9}$$

Thus, the integral identity (3.1) takes the form

$$\int_0^{t_0}\int_\Omega \chi^\varepsilon(\nabla p_0\cdot\boldsymbol{\varphi})dxdt = \int_0^{t_0}\int_\Omega \chi^\varepsilon\big(\mu_1\varepsilon^2\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon) - \pi^\varepsilon\mathbb{I}\big) : \mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})dxdt + \varepsilon\int_0^{t_0}\int_\Omega \chi^\varepsilon\widetilde{\boldsymbol{w}}_f^\varepsilon dxdt. \tag{5.10}$$

Let $\dfrac{\partial\boldsymbol{\varphi}}{\partial t} = \eta(\boldsymbol{x},t)\boldsymbol{\psi}(\dfrac{\boldsymbol{x}}{\varepsilon})$, where $\eta \in W_2^{1,1}(\Omega_T)$, $\eta(\boldsymbol{x},t) = 0$ for $\boldsymbol{x} \in S^0$, $0 < t < T$ and $\boldsymbol{\psi} \in W_2^{1,1}(Y_f)$, *supp*$\boldsymbol{\psi} \subset Y_f$, $\nabla_y\boldsymbol{\psi} = 0$.

Then

$$\mathbb{D}(x,\eta\boldsymbol{\psi}) = \sum_{i,j=1}^3 d_{ij}\big(x,\eta\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big)\boldsymbol{e}_i\otimes\boldsymbol{e}_j, \;\; d_{ij}\big(x,\eta\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big) = \frac{1}{2}\big(\frac{\partial}{\partial x_i}\big(\eta\psi_j(\frac{\boldsymbol{x}}{\varepsilon})\big) + \frac{\partial}{\partial x_j}\big(\eta\psi_i(\frac{\boldsymbol{x}}{\varepsilon})\big) =$$

$$\frac{1}{2\varepsilon}\eta\big(\frac{\partial\psi_j}{\partial y_i}(\frac{\boldsymbol{x}}{\varepsilon})+\frac{\partial\psi_i}{\partial y_j}(\frac{\boldsymbol{x}}{\varepsilon})\big)+\frac{1}{2}\big(\frac{\partial\eta}{\partial x_i}\psi_j(\frac{\boldsymbol{x}}{\varepsilon})+\frac{\partial\eta}{\partial x_j}\psi_i(\frac{\boldsymbol{x}}{\varepsilon})\big),$$

$$\mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})=\frac{\eta}{\varepsilon}\mathbb{D}\big(y,\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big)+\frac{1}{2}(\nabla\eta\otimes\boldsymbol{\psi}+\boldsymbol{\psi}\otimes\nabla\eta)$$

and

$$I_1^\varepsilon(\boldsymbol{\varphi})=-\int_0^{t_0}\int_\Omega\chi^\varepsilon\,\eta\,\big(\varepsilon\mu_1\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon):\mathbb{D}\big(y,\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big)dxdt-\frac{\varepsilon}{2}\mu_1\int_0^{t_0}\int_\Omega\chi^\varepsilon(\nabla\eta\otimes\boldsymbol{\psi}+\boldsymbol{\psi}\otimes\nabla\eta)dxdt.$$

In accordance with Lemma 5.2

$$\lim_{\varepsilon\to0}I_1^\varepsilon(\boldsymbol{\varphi})=-\int_0^{t_0}\int_\Omega\eta\Big(\int_{Y_f}\mu_1\mathbb{D}\big(y,\boldsymbol{W}(\boldsymbol{y};\boldsymbol{x},t):\mathbb{D}(y,\boldsymbol{\psi})dy\big)\Big)dxdt,$$

$$\lim_{\varepsilon\to0}I_2^\varepsilon=-\int_0^{t_0}\int_\Omega\eta\big(\int_{Y_f}(\nabla_x\pi+\nabla_y\Pi)\boldsymbol{\psi}dy\big)dxdt,$$

$$\lim_{\varepsilon\to0}I_3^\varepsilon(\boldsymbol{\varphi})=0. \tag{5.11}$$

Lemma 5.2 and Theorem 4.1 (section 1, chapter 1, [8]) result the equation (5.10). □

5.2. Homogenization of the dynamic problem $\mathbb{B}^\varepsilon(r)$ for the solid component

Lemma 5.4 *Under the conditions of the Theorem 1.1 the limiting procedure in the integral identities (3.1) and (3.2) results the following dynamic problem $\mathbb{H}(r)$ displacements and pressure, consisting of the homogenized Lame system*

$$\nabla\cdot\big(\mathfrak{N}^s(r):\mathbb{D}(x,\boldsymbol{w}_s)-p\mathbb{I}\big)=\nabla p_0, \tag{5.12}$$

$$\nabla\cdot\big(\boldsymbol{w}_f+(1-m(r))\boldsymbol{w}_s\big)=0 \tag{5.13}$$

for the liquid displacements $\boldsymbol{w}_f$, solid displacements $\boldsymbol{w}_s$ and presser $p=\dfrac{\partial\pi}{\partial t}$ in the domain Ω_T, completed with boundary conditions

$$\big(\mathfrak{N}^s(r):\mathbb{D}(x,\boldsymbol{w}_s)-p\mathbb{I}\big)<\boldsymbol{n}>=0,\ \ \boldsymbol{x}\in S^1\cup S^1,\ \ 0<t<T, \tag{5.14}$$

$$\boldsymbol{w}_s(\boldsymbol{x},t)=0,\ \ \boldsymbol{x}\in S^0,\ \ 0<t<T, \tag{5.15}$$

In (5.11) $\boldsymbol{n}$ is a unit normal vector to $S=\partial\Omega$, the symmetric strictly positively definite fourth rank tensor $\mathfrak{N}^s(r)$ is given by formula (1.2.38) in [8].

For the proof see the proof of Theorem 1.4 § 1.2.3, section 1.2, chapter I in [8].

Remark 5.5 *Due to to the symmetry of the set Y_f*

$$\mathfrak{N}^s(r)=\big(1-m(r)\big)\mathbb{I}\otimes\mathbb{I}. \tag{5.16}$$

5.3. Homogenization of the diffusion problem $\mathbb{B}^{\varepsilon}(r)$

Lemma 5.6 *Under the conditions of Theorem 1.1 the limiting procedure in the integral identity (3.3) results the following homogenized diffusion problem* $\mathbb{H}(r)$, *consisting of differential equation*

$$\frac{\partial}{\partial t}\big(m(r)c\big) = \nabla \cdot (\alpha_c \mathbb{B}^{(c)}(r) < \nabla c > \tag{5.17}$$

in the domain Ω_T *and boundary and initial conditions*

$$c(\boldsymbol{x},t) = c_0(\boldsymbol{x}), \ \ \boldsymbol{x} \in S^1 \cup S^2, \ t > 0, \tag{5.18}$$

$$\frac{\partial c}{\partial n}(\boldsymbol{x},t) = 0, \ \boldsymbol{x} \in S^0, \ t > 0, \tag{5.19}$$

$$c(\boldsymbol{x},0) = c_0(\boldsymbol{x}), \ \ \boldsymbol{x} \in \Omega. \tag{5.20}$$

The symmetric strictly positively definite matrix $\mathbb{B}^{(c)}(r)$ *is given by formula (10.1.61) in* [8].

See Theorem 10.1, § 10.1, section 1, chapter 10 in [8] for the proof.

Remark 5.7 *Under conditions of the Lemma 5.6*

$$\|c - c_0\|_{2,\Omega_T} + \|\nabla(c - c_0)\|_{2,\Omega_T} \leqslant MT\|\nabla c_0\|_{2,\Omega}. \tag{5.21}$$

5.4. Homogenization of the boundary condition (1.9)

Lemma 5.8 *Let* $r \in \mathfrak{M}_{(0,T)}$ *and*

$$\alpha^{\varepsilon} = \varepsilon\theta, \ \beta^{\varepsilon} = \varepsilon, \tag{5.22}$$

where θ *is given a positive constant.*

Then the homogenization of the boundary condition (1.9) is given by the formula

$$d_n(\boldsymbol{x},t) = \frac{\partial r}{\partial t}(\boldsymbol{x},t) = \theta\, c(\boldsymbol{x},t), \ \ r(\boldsymbol{x},0) = r_0(\boldsymbol{x}). \tag{5.23}$$

For the proof see Lemma 4.2 in [11].

Lemma 5.9 *Under conditions of Theorem 1.1 there exists an unique solution to the problem* $\mathbb{H}(r)$, *consisting of differential equations*

$$\boldsymbol{w}_f = -\frac{1}{\mu_1}\mathbb{B}^{v}(r) < \nabla(\pi - p_0 t) >, \tag{5.24}$$

$$\nabla \cdot \big(\mathfrak{N}^s(r) : \mathbb{D}(x,\boldsymbol{w}_s) - (p - p_0)\mathbb{I}\big) = \nabla p_0, \tag{5.25}$$

$$\nabla \cdot \big(\boldsymbol{w}_f + (1 - m(r))\boldsymbol{w}_s\big) = 0 \tag{5.26}$$

for liquid displacement $\boldsymbol{w}_f$ *and diffusion equation (5.17), solid displacement* $\boldsymbol{w}_s$, *pressure* $p = \dfrac{\partial \pi}{\partial t}$ *and concentration of the acid c, completed with boundary and initial conditions (5.14), (5.15), (5.18)-(5.20).*

6. Proof of Theorem 1.3: Correctness of the problem $\mathbb{H}$

Let $c(\boldsymbol{x},t)=\mathbb{C}(r)$ be solution to the initial boundary value problem (5.17)–(5.20). As we have mentioned above, the homogenization (5.23) of the boundary condition (1.9) gives us the operator

$$\mathbb{F}(r)\equiv R(\boldsymbol{x},t)=r_0(\boldsymbol{x})-\theta\int_0^t c(\boldsymbol{x},s)ds. \tag{6.1}$$

It is easy to see, that the operator $\mathbb{F}(r)$, defined by formula (6.1) satisfies the Lipschitz condition. Moreover, for some small time interval $(0,T_1)$ it is contraction mapping and maps the set $\mathfrak{M}_{(0T)}$ into itself.

In fact, one has

$$0\leqslant\mathbb{F}(r)\leqslant T_1M\|\nabla c_0\|_{2,\Omega}=T_1M_c,\ |\mathbb{F}(r)|^{(2+\gamma)}_{\Omega_T}\leqslant M(c_0)=M_F;$$
$$|\mathbb{F}(r_1)-\mathbb{F}(r_2)|^{(2+\gamma)}_{\Omega_T}\leqslant T_1M_c|r_1-r_2)|^{(2+\gamma)}_{\Omega_T}. \tag{6.2}$$

That is, on the interval $(0,T_1)$, where $T_1<M_c^{-1}$ the operator $\mathbb{F}(r)$ is compressive and displays the set $\mathfrak{M}_{(0,T)}$ into itself. Fixed point Theorem (Theorem 1, section 8, Chapter 2, [5]) guarantees us the existence of the unique fixed point r^* from the set $\mathfrak{M}_{(0,T_1)}$ and, that is, the validity of the Theorem 1.3 on the time interval $(0,T_1)$.

Next for $t\geqslant T_1$ we put $r_1(\boldsymbol{x},t)=r^*(\boldsymbol{x},T_1)-r(\boldsymbol{x},T_1)+r(\boldsymbol{x},t)$ instead of $r(\boldsymbol{x},t)$ and consider the problem $\mathbb{H}$ on the interval (T_1,T).

Repeating the process for the time intervals $(T_k,T_{k+1}), k=1,2,3....$ we will find function $r^*(\boldsymbol{x},t)$ on the intervals (T_k,T_{k+1}).

If $\lim\limits_{k\to\infty}T_k=\infty$, then the theorem is proved.

If $\lim\limits_{k\to\infty}T_k=T^*<T$ and $r(\boldsymbol{x},T^*)$ is nonzero on some open set in $\Omega_k\subset\Omega$, then by virtue of the obtained estimates of the solutions to the problem $\mathbb{H}$ we can calculate the limits of solutions as $t\to T$ and then will solve the solution to the problem $\mathbb{H}$ on the interval $(T,T+\delta)$ with some small $\delta>0$, which contradicts our assumption. Thus, the process will be terminated if only if $r^*(\boldsymbol{x},T^*)=0$.

7. Conclusion

It is obvious that homogenization of any microscopic mathematical model requires the existence of a solution to this model over any time interval that does not depend on the small parameter of the homogenization. In our case, proving the existence of a solution to model $\mathbb{A}^\varepsilon$ on an arbitrary time interval that does not depend on the small parameter of the homogenization raises insurmountable difficulties. Maybe these difficulties are related to our capabilities, or maybe the problem itself has no solution globally in time. There are examples in both the first case (Hilbert's millennium problem) and the second case, when the question arose about the existence of a classical solution to Stefan's two-phase problem. Back in 1956, the existence of a generalized solution was proven in [4], [14] and only in 1981, the existence of a classical solution locally in time was proven [9], [12]. A little later it was shown that not every generalized solution is classical, even on a fairly small time interval [9].

The suggested method of homogenization the problem A when we know nothing about its classical solvability allows us to prove at least the classical solvability of the homogenized problem H of the problem $\mathbb{A}^\varepsilon$. From the point of view of possible applications in rock mechanics, this is exactly what is needed.

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