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Free weakly k-nilpotent n-tuple semigroups

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Abstract: In this paper, we construct a free object in the variety of weakly k-nilpotent n-tuple semigroups and characterize the least weakly k-nilpotent congruence on a free n-tuple semigroup. Moreover, we present separately free weakly k-nilpotent n-tuple semigroups of rank 1, calculate the cardinality of the free weakly k-nilpotent n-tuple semigroup for the finite case, emphasize that the semigroups of the free weakly k-nilpotent n-tuple semigroup are isomorphic and the automorphism group of the free weakly k-nilpotent n-tuple semigroup is isomorphic to the symmetric group. We also describe all maximal n-tuple subsemigroups of the free weakly k-nilpotent n-tuple semigroup and all regular elements of the endomorphism semigroup of the free weakly k-nilpotent n-tuple semigroup, and provide a criterion for an isomorphism of endomorphism semigroups of free weakly k-nilpotent n-tuple semigroups.

Key words: n-Tuple semigroup, free weakly k-nilpotent n-tuple semigroup, free n-tuple semigroup, semigroup, congruence

1. Introduction and preliminaries

The notion of an n-tuple semigroup was considered by Koreshkov [7] in the context of the theory of n-tuple algebras of associative type. There are sufficiently convincing motivations for studying these algebraic structures since n-tuple semigroups are closely related to interassociative semigroups [1, 4, 5], dimonoids [8, 13, 19], trioids [6, 9, 28], doppelsemigroups [21], duplexes [14], g-dimonoids [10, 29], restrictive bisemigroups [12], and bisemigroups [11]. If $n = 1$, n-tuple semigroups are semigroups. For $n = 2$, n-tuple semigroups become doppelsemigroups. The theory of doppelsemigroups has been developed in [2, 3, 15, 17, 20–23, 26, 30]. The theory of free systems in n-tuple semigroup varieties is still in progress [16, 18, 24, 25, 27]. In particular, in the latter five papers, the problem of the construction of free objects for the varieties of n-tuple semigroups, commutative n-tuple semigroups, rectangular n-tuple semigroups, left (right) k-nilpotent n-tuple semigroups and k-nilpotent n-tuple semigroups was solved; furthermore, certain least congruences on a free n-tuple semigroup were characterized. The present paper continues this trend of research by considering so-called weakly k-nilpotent n-tuple semigroups, i.e. n-tuple semigroups with n k-nilpotent semigroups. The class of such algebras forms a subvariety of the variety of n-tuple semigroups for which the problem of describing a free object remained open.

In this paper, we construct a free object in the variety of weakly k-nilpotent n-tuple semigroups and characterize the least weakly k-nilpotent congruence on a free n-tuple semigroup. We also focus on the

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description of free weakly k-nilpotent n-tuple semigroups of rank 1, calculate the cardinality of the free weakly k-nilpotent n-tuple semigroup for the finite case, emphasize that the semigroups of the free weakly k-nilpotent n-tuple semigroup are isomorphic and the automorphism group of the free weakly k-nilpotent n-tuple semigroup is isomorphic to the symmetric group. In addition, we characterize all maximal n-tuple subsemigroups of the free weakly k-nilpotent n-tuple semigroup and all regular elements of the endomorphism semigroup of the free weakly k-nilpotent n-tuple semigroup, and provide a criterion for an isomorphism of endomorphism semigroups of free weakly k-nilpotent n-tuple semigroups.

As usual, we denote the set of all positive integers by $\mathbb{N}$, and for $n \in \mathbb{N}$, denote the set $\{1, 2, \ldots, n\}$ by $\overline{n}$. Let us recall that a nonempty set G equipped with n binary operations $\boxed{1}$, $\boxed{2}$,..., $\boxed{n}$, satisfying the axioms $(x \boxed{r} y) \boxed{s} z = x \boxed{r} (y \boxed{s} z)$ for all $x, y, z \in G$ and $r, s \in \overline{n}$, is called an n-tuple semigroup [7]. In this section and Section 4, for an n-tuple semigroup, we will write G instead of $(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$.

We need the following lemma.

Lemma 1.1 *([16], Lemma 1) In an n-tuple semigroup G, for any $m \in \mathbb{N}\backslash\{1\}$ and any $x_i \in G$, $1 \leq i \leq m+1$, and any $*_j \in \{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\}$, $1 \leq j \leq m$, any parenthesization of*

$$x_1 *_1 x_2 *_2 \ldots *_m x_{m+1}$$

yields the same element from G.

Recall the definition of a k-nilpotent semigroup (see also [21, 22]). A semigroup S is called nilpotent if $S^{n+1} = 0$ for some $n \in \mathbb{N}$. The least such n is called the nilpotency index of S. For $k \in \mathbb{N}$, a nilpotent semigroup of nilpotency index $\leq k$ is called k-nilpotent. An element 0 of an n-tuple semigroup G is called zero [25] if $x \boxed{i} 0 = 0 = 0 \boxed{i} x$ for all $x \in G$ and $i \in \overline{n}$. An n-tuple semigroup G with zero will be called weakly nilpotent if $(G, \boxed{1})$, $(G, \boxed{2}), \ldots, (G, \boxed{n})$ are nilpotent semigroups. A weakly nilpotent n-tuple semigroup G will be called weakly k-nilpotent if $(G, \boxed{1})$, $(G, \boxed{2}), \ldots, (G, \boxed{n})$ are k-nilpotent semigroups. If ρ is a congruence on an n-tuple semigroup G such that G/ρ is a weakly k-nilpotent n-tuple semigroup, we say that ρ is a weakly k-nilpotent congruence.

Note that the operations of any weakly 1-nilpotent n-tuple semigroup coincide and each operation results in a zero semigroup. It is not difficult to verify that the variety of k-nilpotent n-tuple semigroups introduced in [25] is a subvariety of the variety of weakly k-nilpotent n-tuple semigroups. An n-tuple semigroup which is free in the variety of weakly k-nilpotent n-tuple semigroups will be called a free weakly k-nilpotent n-tuple semigroup.

Denote the symmetric group on a set X by $\Im[X]$ and the automorphism group (the endomorphism semigroup) of an n-tuple semigroup G by $Aut\, G$ $(End\, G)$. If $\nu : G_1 \to G_2$ is a homomorphism of n-tuple semigroups, the kernel of ν is denoted by Δ_ν, that is, $\Delta_\nu = \{(x, y) \in G_1 \times G_1 \,|\, x\nu = y\nu\}$.

2. Constructions

In this section, we construct a free weakly k-nilpotent n-tuple semigroup of an arbitrary rank and consider separately free weakly k-nilpotent n-tuple semigroups of rank 1. Note that the free weakly k-nilpotent n-tuple semigroup over a set X is an algebra in the variety of weakly k-nilpotent n-tuple semigroups generated by X and such that every map of X into any other weakly k-nilpotent n-tuple semigroup K can be extended to a homomorphism of the free weakly k-nilpotent n-tuple semigroup over a set X into K. We also calculate the cardinality of the free weakly k-nilpotent n-tuple semigroup for the finite case and establish that the semigroups of the free weakly k-nilpotent n-tuple semigroup are isomorphic.

Let X be an arbitrary nonempty set and let ω be an arbitrary word in the alphabet X. The length of ω is denoted by l_ω. Fix $n \in \mathbb{N}$ and let $Y = \{y_1, y_2, \dots, y_n\}$ be an arbitrary set consisting of n elements. Let further $F[X]$ be the free semigroup on X and $F^\theta[Y]$ the free monoid on Y with the empty word θ. By definition, the length l_θ of θ is equal to 0. For $x \in Y$ and all $u \in F^\theta[Y]$, the number of occurrences of the element x in u is denoted by $d_x(u)$. Obviously, $d_x(\theta) = 0$ for any $x \in Y$. Fix $k \in \mathbb{N}$ and define n binary operations $\boxed{1}, \boxed{2}, \dots, \boxed{n}$ on

$$\Omega_k = \{(w,u) \in F[X] \times F^\theta[Y] \,|\, l_w - l_u = 1,$$

$$d_x(u) + 1 \le k \quad \text{for all} \quad x \in Y\} \cup \{0\}$$

by

$$(w_1,u_1)\boxed{i}(w_2,u_2)$$

$$= \begin{cases} (w_1w_2, u_1y_iu_2), \text{ if } d_x(u_1y_iu_2)+1 \le k \quad \text{for all} \quad x \in Y, \\ \quad 0, \qquad\qquad\quad \text{in all other cases,} \end{cases}$$

$$(w_1,u_1)\boxed{i}0 = 0\boxed{i}(w_1,u_1) = 0\boxed{i}0 = 0$$

for all (w_1,u_1), $(w_2,u_2) \in \Omega_k \backslash \{0\}$ and all $i \in \overline{n}$. The algebra obtained in this manner will be denoted by $FNS_n^k(X)$. The main result of this paper is the following theorem:

Theorem 2.1 *$FNS_n^k(X)$ is a free weakly k-nilpotent n-tuple semigroup.*

Proof We first show that $FNS_n^k(X)$ is an n-tuple semigroup. Let

$$(w_1,u_1),\ (w_2,u_2),\ (w_3,u_3) \in \Omega_k \backslash \{0\} \text{ and } r, s \in \overline{n}.$$

Suppose that

$$d_x(u_1y_ru_2y_su_3) + 1 \le k \tag{1}$$

for all $x \in Y$. Then we get

$$\begin{aligned} ((w_1,u_1)\boxed{r}(w_2,u_2))\boxed{s}(w_3,u_3) &= (w_1w_2, u_1y_ru_2)\boxed{s}(w_3,u_3) \\ &= (w_1w_2w_3, u_1y_ru_2y_su_3) \\ &= (w_1,u_1)\boxed{r}(w_2w_3, u_2y_su_3) \\ &= (w_1,u_1)\boxed{r}((w_2,u_2)\boxed{s}(w_3,u_3)) \end{aligned}$$

since (1) implies $d_x(u_1y_ru_2) + 1 \le k$ and $d_x(u_2y_su_3) + 1 \le k$ for all $x \in Y$. If

$$d_x(u_1y_ru_2y_su_3) + 1 > k$$

for some $x \in Y$, we have

$$((w_1,u_1)\boxed{r}(w_2,u_2))\boxed{s}(w_3,u_3) = 0 = (w_1,u_1)\boxed{r}((w_2,u_2)\boxed{s}(w_3,u_3)).$$

The proof of the case when the element 0 appears on both sides of the axioms of an n-tuple semigroup at least once is obvious.

Therefore, $FNS_n^k(X)$ is an n-tuple semigroup.

Take arbitrary elements $(w_i, u_i) \in \Omega_k \backslash \{0\}$, $1 \leq i \leq k+1$. Since

$$d_{y_r}(u_1 y_r u_2 y_r \dots y_r u_{k+1}) + 1 > k$$

for any $r \in \overline{n}$, we conclude that

$$(w_1, u_1)\boxed{r}(w_2, u_2)\boxed{r}\dots\boxed{r}(w_{k+1}, u_{k+1}) = 0.$$

At this point, assuming $x^0 = \theta$ for all $x \in Y$, for any $(x_i, \theta) \in \Omega_k \backslash \{0\}$, where $x_i \in X$, $1 \leq i \leq k$, we get

$$(x_1, \theta)\boxed{r}(x_2, \theta)\boxed{r}\dots\boxed{r}(x_k, \theta) = (x_1 x_2 \dots x_k, y_r^{k-1}) \neq 0.$$

The last arguments guarantee that $(\Omega_k, \boxed{r})$, $r \in \overline{n}$, is a nilpotent semigroup of nilpotency index k. Therefore, by definition, $FNS_n^k(X)$ is a weakly k-nilpotent n-tuple semigroup.

Let us show that $FNS_n^k(X)$ is free in the variety of weakly k-nilpotent n-tuple semigroups.

Obviously, $FNS_n^k(X)$ is generated by the set $X \times \{\theta\}$. Assume that $(K, \boxed{1}, \boxed{2}, \dots, \boxed{n})$ is an arbitrary weakly k-nilpotent n-tuple semigroup. Let $\varphi' : X \times \{\theta\} \to K$ be an arbitrary map. Consider a map $\varphi : X \to K$ such that $x\varphi = (x, \theta)\varphi'$ for all $x \in X$ and define a map

$$\phi : FNS_n^k(X) \to (K, \boxed{1}, \boxed{2}, \dots, \boxed{n}) : \omega \mapsto \omega\phi$$

as

$$\omega\phi = \begin{cases} x_1\varphi\boxed{i_1}x_2\varphi\boxed{i_2}\dots\boxed{i_{s-1}}x_s\varphi, & \text{if } \omega = (x_1 x_2 \dots x_s, y_{i_1} y_{i_2} \dots y_{i_{s-1}}), \\ & \quad x_j \in X, 1 \leq j \leq s, \\ x_1\varphi, & \text{if } \omega = (x_1, \theta),\ x_1 \in X, \\ 0, & \text{if } \omega = 0. \end{cases}$$

According to Lemma 1.1, ϕ is well-defined. We aim to demonstrate that ϕ is a homomorphism. For this, we will use the axioms of an n-tuple semigroup and the identities of a weakly k-nilpotent n-tuple semigroup. If $s = 1$, we will regard a sequence $y_{i_1} y_{i_2} \dots y_{i_{s-1}} \in F^\theta[Y]$ as θ. For arbitrary elements

$$r \in \overline{n}, \quad (w_1, u_1) = (x_1 x_2 \dots x_s, y_{i_1} y_{i_2} \dots y_{i_{s-1}}),$$

$$(w_2, u_2) = (z_1 z_2 \dots z_m, y_{c_1} y_{c_2} \dots y_{c_{m-1}}) \in \Omega_k \backslash \{0\},$$

where $x_d, z_t \in X$, $1 \leq d \leq s \in \mathbb{N}$, $1 \leq t \leq m \in \mathbb{N}$, $y_{i_p}, y_{c_b} \in Y$, $1 \leq p \leq s-1$, $1 \leq b \leq m-1$, we obtain

$$((x_1 x_2 \dots x_s, y_{i_1} y_{i_2} \dots y_{i_{s-1}})\boxed{r}(z_1 z_2 \dots z_m, y_{c_1} y_{c_2} \dots y_{c_{m-1}}))\phi$$

$$= \begin{cases} \left(x_1 x_2 \dots x_s z_1 z_2 \dots z_m, y_{i_1} y_{i_2} \dots y_{i_{s-1}} y_r y_{c_1} y_{c_2} \dots y_{c_{m-1}}\right)\phi, & \text{if} \\ \qquad d_x(u_1 y_r u_2) + 1 \leq k \quad \text{for all} \quad x \in Y, \\ 0\phi, \quad \text{in all other cases.} \end{cases}$$

In the case $d_x(u_1y_ru_2)+1\leq k$ for all $x\in Y$, we have

$$\begin{aligned}
&(x_1x_2\ldots x_sz_1z_2\ldots z_m, y_{i_1}y_{i_2}\ldots y_{i_{s-1}}y_ry_{c_1}y_{c_2}\ldots y_{c_{m-1}})\phi\\
&= x_1\varphi\boxed{i_1}\ldots\boxed{i_{s-1}}x_s\varphi\boxed{r}z_1\varphi\boxed{c_1}\ldots\boxed{c_{m-1}}z_m\varphi\\
&= (x_1\varphi\boxed{i_1}\ldots\boxed{i_{s-1}}x_s\varphi)\boxed{r}(z_1\varphi\boxed{c_1}\ldots\boxed{c_{m-1}}z_m\varphi)\\
&= (x_1x_2\ldots x_s, y_{i_1}y_{i_2}\ldots y_{i_{s-1}})\phi\boxed{r}(z_1z_2\ldots z_m, y_{c_1}y_{c_2}\ldots y_{c_{m-1}})\phi.
\end{aligned}$$

If $d_x(u_1y_ru_2)+1>k$ for some $x\in Y$, we get

$$\begin{aligned}
0\phi = 0 &= x_1\varphi\boxed{i_1}\ldots\boxed{i_{s-1}}x_s\varphi\boxed{r}z_1\varphi\boxed{c_1}\ldots\boxed{c_{m-1}}z_m\varphi\\
&= (x_1\varphi\boxed{i_1}\ldots\boxed{i_{s-1}}x_s\varphi)\boxed{r}(z_1\varphi\boxed{c_1}\ldots\boxed{c_{m-1}}z_m\varphi)\\
&= (x_1x_2\ldots x_s, y_{i_1}y_{i_2}\ldots y_{i_{s-1}})\phi\boxed{r}(z_1z_2\ldots z_m, y_{c_1}y_{c_2}\ldots y_{c_{m-1}})\phi.
\end{aligned}$$

The proofs of the remaining cases are obvious. Therefore, $(a\boxed{r}b)\phi = a\phi\boxed{r}b\phi$ for all $a,b\in FNS_n^k(X)$, all $r\in\overline{n}$, and hence, ϕ is a homomorphism. Clearly, $(x,\theta)\phi=(x,\theta)\varphi'$ for all $(x,\theta)\in X\times\{\theta\}$. Since $X\times\{\theta\}$ generates $FNS_n^k(X)$, the uniqueness of such homomorphism ϕ is obvious. Thus, $FNS_n^k(X)$ is free in the variety of weakly k-nilpotent n-tuple semigroups. $\square$

In order to calculate the cardinality of Ω_k, let $a_0=\alpha_0=\alpha_{-1}=0$ and let $\alpha_p=\sum_{i=1}^{p}a_i$ for $p\in\overline{n}$ and $\alpha=(a_1,\ldots,a_n)\in\{0,\ldots,k-1\}^n$.

Corollary 2.2 *The free weakly k-nilpotent n-tuple semigroup $FNS_n^k(X)$ generated by a finite set $X\times\{\theta\}$ is finite. Specifically, if $|X|=m$, then*

$$|\Omega_k| = 1 + \sum_{\substack{\alpha=(a_1,\ldots,a_n)\in\\ \{0,\ldots,k-1\}^n}} \left(\prod_{l=0}^{n-1}\binom{\alpha_n-\alpha_{l-1}}{a_l} m^{(1+\alpha_n)}\right).$$

Proof Let $(w,u)\in\Omega_k\backslash\{0\}$. Then we denote $d_{y_i}(u)$ by a_i for all $i\in\overline{n}$. Clearly, $a_i\in\{0,\ldots,k-1\}$ for all $i\in\overline{n}$, and let $\alpha=(a_1,\ldots,a_n)$. Hence, there are $\prod_{l=0}^{n-1}\binom{\alpha_n-\alpha_{l-1}}{a_l}$ possibilities for the word u. Since $l_u=\alpha_n$ and $l_w=l_u+1$, there are $m^{(1+\alpha_n)}$ possibilities for the word w. Thus, we have

$$\prod_{l=0}^{n-1}\binom{\alpha_n-\alpha_{l-1}}{a_l} m^{(1+\alpha_n)}$$

possibilities for an element $(w,u)\in\Omega_k\backslash\{0\}$ with $d_{y_i}(u)=a_i$ for all $i\in\overline{n}$, where $\alpha=(a_1,\ldots,a_n)\in\{0,\ldots,k-1\}^n$. Let

$$W_{(a_1,\ldots,a_n)} = \{(w,u)\in\Omega_k\backslash\{0\}\,|\,d_{y_i}(u)=a_i \text{ for all } i\in\overline{n}\}.$$

It is easy to verify that

$$\{W_{(a_1,\ldots,a_n)} \mid (a_1,\ldots,a_n) \in \{0,\ldots,k-1\}^n\}$$

is a partition of $\Omega_k \setminus \{0\}$. Hence,

$$|\Omega_k| = 1 + \sum_{\substack{\alpha=(a_1,\ldots,a_n)\in \\ \{0,\ldots,k-1\}^n}} \left(\prod_{l=0}^{n-1} \binom{\alpha_n - \alpha_{l-1}}{a_l} m^{(1+\alpha_n)} \right).$$

□

Now we construct an n-tuple semigroup which is isomorphic to the free weakly k-nilpotent n-tuple semigroup of rank 1.

We put

$$Y(k) = \{u \in F^{\theta}[Y] \mid d_x(u) + 1 \le k \quad \text{for all} \quad x \in Y\} \cup \{0\}.$$

Define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on $Y(k)$ by

$$u_1 \boxed{\text{i}} u_2 = \begin{cases} u_1 y_i u_2, \text{ if } d_x(u_1 y_i u_2) + 1 \le k \quad \text{for all} \quad x \in Y, \\ \qquad 0, \quad \text{in all other cases,} \end{cases}$$

$$u_1 \boxed{\text{i}} 0 = 0 \boxed{\text{i}} u_1 = 0 \boxed{\text{i}} 0 = 0$$

for all $u_1, u_2 \in Y(k) \backslash \{0\}$ and all $i \in \overline{n}$. The obtained algebra will be denoted by $Y(n,k)$.

Corollary 2.3 *If* $|X| = 1$, *then* $Y(n,k) \cong FNS_n^k(X)$.

Proof Let $X = \{c\}$. One can show that the map

$$\alpha : Y(n,k) \to FNS_n^k(X) : u \mapsto u\alpha,$$

defined by the rule

$$u\alpha = \begin{cases} (c^{l_u+1}, u), & \text{if } u \in Y(k) \backslash \{0\}, \\ 0, & \text{if } u = 0, \end{cases}$$

is an isomorphism. □

The following statement establishes a relationship between the semigroups of the free weakly k-nilpotent n-tuple semigroup.

Corollary 2.4 *For any* $r, s \in \overline{n}$, *the semigroups* $(\Omega_k, \boxed{r})$ *and* $(\Omega_k, \boxed{s})$ *of* $FNS_n^k(X)$ *are isomorphic.*

Proof For any $x \in Y$, let

$$\widehat{x} = \begin{cases} s, & \text{if } x = r, \\ r, & \text{if } x = s, \\ x, & \text{otherwise,} \end{cases}$$

and define a map $\tau : (\Omega_k, \boxed{r}) \to (\Omega_k, \boxed{s}) : q \mapsto q\tau$ by putting

$$q\tau = \begin{cases} (w, \widehat{y_1}\widehat{y_2}\ldots\widehat{y_m}), & \text{if } q = (w, y_1y_2\ldots y_m) \in \Omega_k \backslash \{0\}, \\ q, & \text{in all other cases.} \end{cases}$$

An immediate verification shows that τ is an isomorphism.

□

3. The least weakly k-nilpotent congruence on a free n-tuple semigroup

In this section, we present the least weakly k-nilpotent congruence on a free n-tuple semigroup. We will use the notation established in Sections 1 and 2. An n-tuple semigroup which is free in the variety of n-tuple semigroups is called a free n-tuple semigroup. Recall the construction of a free n-tuple semigroup [16].

We define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on

$$XY_n = \{(w, u) \in F[X] \times F^{\theta}[Y] \mid l_w - l_u = 1\}$$

by

$$(w_1, u_1)\boxed{i}(w_2, u_2) = (w_1w_2, u_1y_iu_2)$$

for all $(w_1, u_1), (w_2, u_2) \in XY_n$ and all $i \in \overline{n}$. The algebra

$$(XY_n, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$$

is denoted by $F_nTS(X)$. By Theorem 2 from [16], $F_nTS(X)$ is a free n-tuple semigroup. For $k \in \mathbb{N}$, let $\beta_{(n,k)}$ be the binary relation on $F_nTS(X)$ defined by

$$(w_1, u_1)\beta_{(n,k)}(w_2, u_2) \text{ if and only if } (w_1, u_1) = (w_2, u_2) \text{ or}$$

$$\begin{cases} d_x(u_1) + 1 > k \text{ for some } x \in Y, \\ d_y(u_2) + 1 > k \text{ for some } y \in Y. \end{cases}$$

Theorem 3.1 *The relation $\beta_{(n,k)}$ on the free n-tuple semigroup $F_nTS(X)$ is the least weakly k-nilpotent congruence.*

Proof We define a map $\delta : F_nTS(X) \to FNS_n^k(X)$ by

$$(w, u)\,\delta = \begin{cases} (w, u), & \text{if } d_x(u) + 1 \leq k \text{ for all } x \in Y, \\ 0, & \text{in all other cases.} \end{cases}$$

We need to show that δ is a homomorphism. Let $(w_1, u_1), (w_2, u_2) \in F_nTS(X)$ and $r \in \overline{n}$. Suppose that $d_x(u_1y_ru_2) + 1 \leq k$ for all $x \in Y$. The latter inequality implies $d_x(u_1) + 1 \leq k$ and $d_x(u_2) + 1 \leq k$ for all $x \in Y$. Then

$$((w_1, u_1)\boxed{r}(w_2, u_2))\delta = (w_1w_2, u_1y_ru_2)\delta = (w_1w_2, u_1y_ru_2)$$

$$= (w_1, u_1)\boxed{r}(w_2, u_2) = (w_1, u_1)\delta\boxed{r}(w_2, u_2)\delta.$$

In the case $d_x(u_1y_ru_2)+1>k$ for some $x\in Y$, we conclude that

$$((w_1,u_1)\boxed{r}(w_2,u_2))\delta=(w_1w_2,u_1y_ru_2)\delta=0=(w_1,u_1)\delta\boxed{r}(w_2,u_2)\delta.$$

Thus, δ is a homomorphism which is surjective since

$$\Omega_k=\{(w,u)\in XY_n\,|\,d_x(u)+1\le k\quad\text{for all}\quad x\in Y\}\cup\{0\}.$$

By Theorem 2.1, $FNS_n^k(X)$ is a free weakly k-nilpotent n-tuple semigroup. Therefore, Δ_δ is the least weakly k-nilpotent congruence on $F_nTS(X)$. It follows from the definition of δ that $\Delta_\delta=\beta_{(n,k)}$.

□

4. Some properties

In this section, we describe some properties of free weakly k-nilpotent n-tuple semigroups. More precisely, we characterize all maximal n-tuple subsemigroups of the free weakly k-nilpotent n-tuple semigroup and all regular elements of the endomorphism semigroup of the free weakly k-nilpotent n-tuple semigroup. We also demonstrate that the automorphism group of the free weakly k-nilpotent n-tuple semigroup is isomorphic to the symmetric group, and that $End(FNS_n^k(X))$ is isomorphic to $End(FNS_n^k(Z))$ if and only if $|X|=|Z|$.

Now, we describe all maximal n-tuple subsemigroups of the free weakly k-nilpotent n-tuple semigroup $FNS_n^k(X)$.

An n-tuple subsemigroup of an n-tuple semigroup G is called proper if it is not equal to G. An n-tuple subsemigroup T of an n-tuple semigroup G is called maximal provided that $T\neq G$ and, for any n-tuple subsemigroup $M\le G$, the inclusion $T\le M$ implies $M=T$ or $M=G$. Equivalently, an n-tuple subsemigroup of an n-tuple semigroup G is maximal if it is a proper n-tuple subsemigroup of G which is not contained in any other proper n-tuple subsemigroup of G.

Proposition 4.1 *Let S be an n-tuple subsemigroup of $FNS_n^k(X)$. Then S is maximal if and only if there is an $x\in X$ such that $S=\Omega_k\setminus\{(x,\theta)\}$.*

Proof Let $x\in X$. Further, let $i\in\overline{n}$ and $(w_1,u_1),(w_2,u_2)\in\Omega_k\setminus\{(x,\theta),0\}$. Then $(w_1,u_1)\boxed{i}(w_2,u_2)\in\{(w_1w_2,u_1y_iu_2),0\}$ with $u_1y_1u_2\neq\theta$. Moreover, $(w_1,u_1)\boxed{i}0=0\boxed{i}(w_1,u_1)=0\boxed{i}0=0$. This shows that $\Omega_k\setminus\{(x,\theta)\}$ forms an n-tuple subsemigroup of $FNS_n^k(X)$, which is clearly maximal.

Conversely, let S be a maximal n-tuple subsemigroup of $FNS_n^k(X)$. Since $X\times\{\theta\}$ is the least generating set of $FNS_n^k(X)$, we can conclude that $\{(x,\theta)\,|\,x\in X\}\nsubseteq S$, i.e. there is an $x\in X$ with $(x,\theta)\notin S$. Therefore, $S\subseteq\Omega_k\setminus\{(x,\theta)\}$. Since S is a maximal n-tuple subsemigroup of $FNS_n^k(X)$, we obtain $S=\Omega_k\setminus\{(x,\theta)\}$. □

Due to the fact that the set $X'=X\times\{\theta\}$ is the least generating for $FNS_n^k(X)$, we obtain the following description of the automorphism group of the free weakly k-nilpotent n-tuple semigroup: $Aut\,FNS_n^k(X)\cong\Im[X]$. It is natural to consider endomorphisms of the free weakly k-nilpotent n-tuple semigroup $FNS_n^k(X)$.

Since X' is the least generating set of $FNS_n^k(X)$, each map $\varphi:X'\to\Omega_k$ induces an endomorphism of $FNS_n^k(X)$, and conversely, every endomorphism of $FNS_n^k(X)$ is uniquely determined by a map from X' into $FNS_n^k(X)$. This yields the formula

$$\left|End(FNS_n^k(X))\right|=|\Omega_k|^{|X|},$$

whenever X is finite, with $|\Omega_k|$ calculated by Corollary 2.2.

Let $a \in \Omega_k$. An endomorphism $f_a \in End(FNS_n^k(X))$ we call constant if $(x,\theta)f_a = a$ for all $x \in X$.

Proposition 4.2 *The only constant idempotent endomorphisms of $FNS_n^k(X)$ are f_a for all $a \in X' \cup \{0\}$.*

Proof Let $CI(X) = \{f_a \,|\, a \in X' \cup \{0\}\}$. Obviously, every element of $CI(X)$ is a constant idempotent endomorphism of $FNS_n^k(X)$. Conversely, let $f \in End(FNS_n^k(X))$ be constant and idempotent. Suppose that $(x,\theta)f = a \in \Omega_k$ for all $x \in X$. Then $(x,\theta)f^2 = af = a = (x,\theta)f$. The equality $af = a$ implies that $a \in X' \cup \{0\}$. Thus, $f = f_a \in CI(X)$.

□

The following statement provides the criterion for an isomorphism of the endomorphism semigroups of free weakly k-nilpotent n-tuple semigroups.

Proposition 4.3 *Let X and Z be arbitrary nonempty sets. Then*

$$End(FNS_n^k(X)) \cong End(FNS_n^k(Z)) \text{ if and only if } |X| = |Z|.$$

Proof Suppose that $\psi : End(FNS_n^k(X)) \to End(FNS_n^k(Z))$ is an isomorphism. By Proposition 4.2, $CI(X)$ and $CI(Z)$ are the sets of all constant idempotent endomorphisms of $FNS_n^k(X)$ and $FNS_n^k(Z)$, respectively. Then we can conclude that $CI(X)\psi = CI(Z)$. This yields $|CI(X)| = |CI(Z)|$, and thus, $|X| = |Z|$.

Let $f \in End(FNS_n^k(X))$ and $(y,\theta)f = (e_1, e_2)$ for some $y \in X$. We will denote e_1 and e_2 by $[y]_f$ and $(y]_f$, respectively. Suppose now that $|X| = |Z|$. Then there is a bijection $\sigma : X \to Z$. We define $\psi : End(FNS_n^k(X)) \to End(FNS_n^k(Z))$ in the following way: let $f\psi$ be the endomorphism of $FNS_n^k(Z)$ defined by $(x,\theta)f\psi = ([x\sigma^{-1}]_f\overline{\sigma}, (x\sigma^{-1}]_f)$ for all $(x,\theta) \in Z'$, where $\overline{\sigma}$ is the extension of σ to $F[X]$. Since σ is a bijection, we can conclude that ψ is a bijection from $End(FNS_n^k(X))$ in $End(FNS_n^k(Z))$ and let $a \in Z$. Then we have $(a,\theta)(f \circ g)\psi = ([a\sigma^{-1}]_{f\circ g}\overline{\sigma}, (a\sigma^{-1}]_{f\circ g})$. On the other hand, we get $(a,\theta)f\psi \circ g\psi = ([a\sigma^{-1}]_f\overline{\sigma}, (a\sigma^{-1}]_f)g\psi$. It is easy to verify that $([a\sigma^{-1}]_f\overline{\sigma}, (a\sigma^{-1}]_f)g\psi = ([a\sigma^{-1}]_{f\circ g}\overline{\sigma}, (a\sigma^{-1}]_{f\circ g})$. Hence, $(a,\theta)(f \circ g)\psi = (a,\theta)f\psi \circ g\psi$. Consequently, ψ is an isomorphism, i.e.

$$End(FNS_n^k(X)) \cong End(FNS_n^k(Z)).$$

□

Recall that an element a of a semigroup S is called regular provided that there exists $b \in S$ such that $aba = a$. A semigroup S is called regular provided that every element of S is regular. At the end of this section, we consider the question of regularity in $End(FNS_n^k(X))$.

Let ω be an arbitrary word in the alphabet X. The set of all elements $x \in X$ occurring in ω is denoted by $c(\omega)$. Let further Reg be the set of all $f \in End(FNS_n^k(X))$ with $(x,\theta) \in X'f = \{yf \,|\, y \in X'\}$, for all $x \in c(w)$, whenever $(w,u) \in X'f$. Denote the constant transformation of Ω_k with value 0 by f_0^X.

Proposition 4.4 *$Reg \cup \{f_0^X\}$ is the set of all regular elements in $End(FNS_n^k(X))$. $Reg \cup \{f_0^X\}$ is not closed under composition, whenever $|X| \geq 2$.*

Proof Clearly, f_0^X is idempotent and thus regular in $End(FNS_n^k(X))$. Let $f \in Reg$. Then we put

$$X_f = \{(x,\theta) \in X' \,|\, x \in c(w) \text{ and } (w,u) \in X'f\}.$$

Since $f \in Reg$, for each $(x,\theta) \in X_f$, there is $(x,\theta)^* \in X'$ such that $(x,\theta)^* f = (x,\theta)$. Then let $g \in End(FNS_n^k(X))$ such that $(x,\theta)g = (x,\theta)^*$ for all $(x,\theta) \in X_f$ and $(x,\theta)g = 0$ for all $(x,\theta) \in X' \setminus X_f$. Let $(x,\theta) \in X'$ such that $(x,\theta)f \neq 0$. Then it is easy to verify that $(x,\theta)fgf = (x,\theta)f$. If $(x,\theta)f = 0$ then $(x,\theta)fgf = 0gf = 0$ since an endomorphism maps 0 to 0. This shows that f is regular.

Let now $f \in End(FNS_n^k(X)) \setminus \{Reg \cup \{f_0^X\}\}$. Then there is a $(w,u) \in X'f$ and $x \in c(w)$ with $(x,\theta) \notin X'f$. Assume that there is a $g \in End(FNS_n^k(X))$ with $fgf = f$. Let $(a,\theta) \in X'$ such that $(a,\theta)f = (w,u) \in \Omega_k \setminus \{0\}$. Then $(a,\theta)fg = (w,u)g$, and thus, $(w,u)g \neq 0$ since otherwise $(a,\theta)fgf = 0$, a contradiction to $(a,\theta)fgf = (a,\theta)f = (w,u)$. Hence, there is $(w_1,u_1) \in \Omega_k \setminus \{\theta\}$ with $(w,u)g = (w_1,u_1)$. It is clear that $l_w \leq l_{w_1}$. Since $(x,\theta) \notin X'f$ and $(w_1,u_1)f = (w,u)$, we can conclude that $l_{w_1} < l_w$, i.e. $l_w < l_w$, a contradiction. This shows that f is not regular.

Finally, we demonstrate that $Reg \cup \{f_0^X\}$ is not closed under composition, whenever $|X| \geq 2$. For this let $x_1, x_2 \in X$. We put

$$(x_1,\theta)f_1 = (x_2x_2, y_1) \text{ and } (x,\theta)f_1 = (x_2,\theta)$$

for all $(x,\theta) \in X' \setminus \{(x_1,\theta)\}$,

$$(x_2,\theta)f_2 = (x_1x_1, y_1) \text{ and } (x,\theta)f_2 = (x_1,\theta)$$

for all $(x,\emptyset) \in X' \setminus \{(x_2,\theta)\}$. It is easy to verify that f_1 and f_2 define endomorphisms in Reg. We consider

$$(x_1,\theta)(f_1 \circ f_2) = (x_2x_2, y_1)f_2 = (x_1x_1x_1x_1, y_1y_1y_1) \quad \text{and}$$

$$(x,\theta)(f_1 \circ f_2) = (x_2,\theta)f_2 = (x_1x_1, y_1)$$

for all $(x,\theta) \in X' \setminus \{(x_1,\theta)\}$. Since $x_1 \in c(x_1x_1x_1x_1)$, where $(x_1x_1x_1x_1, y_1y_1y_1) \in X(f_1 \circ f_2)$, but $(x_1,\theta) \notin X(f_1 \circ f_2)$ the endomorphism $f_1 \circ f_2$ does not belong to Reg.

□

Corollary 4.5 *Let $|X| = 1$. The regular elements in $End(FNS_n^k(X))$ are exactly the zero f_0^X and the identity map on Ω_k which form a two-element subsemigroup of $End(FNS_n^k(X))$, in particular, a band.*

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