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Moment-based approximation for variance of semi-Markovian random walk with gamma distributed interference of chance

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Abstract: This study proposed moment-based approximations for the expected value and variance of the ergodic distribution of the semi-Markovian random walk process $(X(t))$ with gamma distributed interference of chance. Many studies have investigated analogous moment problems by using an asymptotic approach. The key distinguishing aspect of this study from others in the literature is obtaining Kambo's approximations for the moments of $X(t)$ instead of asymptotic expansions. Firstly, the approximation formulas for the moments of boundary functional $S_{N(z)}$ of $X(t)$ were obtained. Then using these results, approximation formulas for the first two moments of the ergodic distributions of $X(t)$ were derived. Finally, the expected value and variance of $X(t)$ were calculated by using the Monte Carlo simulation method for two concrete distributions (Gaussian and Uniform).

Key words: Semi-Markovian random walk process, moment-based approximation, ergodic distribution, boundary functional, interference of chance

1. Introduction

This study investigates the random walk process $X(t)$ with a gamma distributed interference of chance. These processes can be utilized to analyze diverse problems that arise within the domains of reliability, inventory, stock control, queuing, physics, stochastic finance, and mathematical biology applications. The considered process is a special case of a class of stochastic processes known as "Stochastic processes with a discrete interference of chance" as introduced by A.N. Kolmogoroff [16]. Gikhman and Skorohod [7] proved the general ergodic theorem for this class. Since difficulties arise due to the complexity of the mathematical structure of the exact formulas in practice, asymptotic approaches have been employed. Researches have been done in several directions to find simpler asymptotic expansions for random walk processes. Lotov [18] obtained the asymptotic expansions for the first three moments of the ladder height of the Gaussian random walk using Hadamard expansion. Chang and Peres [5] derived asymptotic expansions for the first four moments of the ladder height employing the

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Riemann zeta function. Nagaev [19] advanced the research on this topic by deriving asymptotic expansions for the first five moments of the ladder height. Additionally, Janssen and Leeuwaardeen [11] examined the cumulants associated with the boundary functional.

Further to the foregoing research, random walk processes have been analyzed over recent years with the interference of chances with different types of distributions and asymptotic expansions were derived for the moments of $X(t)$ see [1–3]. For example, in studies [8] and [9] the weak convergence theorem for the distribution of the maximum of a Gaussian random walk and for the distribution of a random walk were investigated, respectively. In studies [4], [13], and [14] a semi-Markovian random walk process is investigated by using gamma, Weibull, and generalized beta distributed interference of chance, respectively. The study [15] examined a semi-Markovian random walk process with a special barrier. As a result asymptotic expansions for the moments of $X(t)$ are acquired. One of the most important recent investigations of random walk processes in the literature is the work of Hanalioglu et al. [10]. The study [10] examined the process $X(t)$ with normal distributed interference of chance. They proposed exact formulas in terms of boundary functional $S_{N(z)}$ and obtained asymptotic results for the first four moments of $X(t)$. However, as the results obtained in the above-mentioned research are asymptotic expansions, the remaining terms following the first two or three terms are usually given with small oh (o) or big oh (O) notations. This poses limitations in offering a comprehensive understanding of the convergence of these terms to zero. The aim of this study is to address this gap identified in prior research, particularly in cases where asymptotic expansions have been suggested. It is desirable that the approximations presented for such systems provide accuracy, simplicity, applicability, and clarity. This article is motivated by an interest in proposing a new approach to derive approximate formulas for some important characteristics of a random walk process $X(t)$ that meets the listed requirements. For this aim, we first propose approximations in the sense of Kambo for the moments of the boundary functional $S_{N(z)}$. We also derive approximate formulas for the first and second initial moments and variance of ergodic distribution of the random walk process $X(t)$.

Note that our research builds on the work of Kambo et. al. [12] that provide moment-based approximations for the renewal process. It is observed that the approximation formula derived for the renewal function in Kambo's work is exact for a number of distributions, including exponential, a mixture of two exponential (Erlang with two phases), and K2. These distributions are applicable in a number of different contexts, including modeling of arrival and service distributions in queuing theory, lifetime distributions in reliability analysis and approximation of probability distributions for the evaluation of multimedia systems. Moreover, the results in the [12] study were obtained using the moment matching method. The moment matching estimation method, which is similar to that used in study [12], and our study could also be used for the performance analyses of reconfigurable intelligent surface (RIS)-assisted stochastic unmanned aerial vehicle (UAV) mmWave relay communication system [17]. The study of Li et al. [17] is a good example that similar methods can also be applied to other models employed in practical engineering applications.

The remainder of the article is organized as follows. In Section 2 we give mathematical construction of the considered process $X(t)$. In Section 3 we present exact expressions for moments of ergodic distribution of the process $X(t)$. In Section 4 we first introduce moment-based approximations for the first three moments of the renewal process $N(t)$ proposed by Kambo et al. [12]. Then, we gave the comparison of approximations for renewal function $U(t)$ in the sense of Kambo and Feller with a specific example. We observed that Kambo's approximate formula $U_K(t)$ is more closely aligned with the exact formula $U(t)$ than Feller's asymptotic expansion $U_F(t)$. At the end of Section 4 we proposed moment-based approximations for moments of boundary functional $S_{N(z)}$ inspired by the method by Kambo et al. [12]. In Section 5 we provide moment-based approximations for the moments of the process $X(t)$ by using the first three moments of $S_{N(z)}$. In Section 6 we consider two examples and we calculate the expected value and variance of $X(t)$ by using the Monte Carlo simulation method for two concrete distributions (Gaussian and Uniform).

2. Mathematical construction of process $X(t)$

Suppose $\{(\xi_n, \eta_n, \zeta_n), n = 1, 2, \}$ is a sequence of independent and identically distributed random triples on the probability space $(\Omega, \mathcal{F}, P)$. $\xi_n, n \geq 1$, take only positive values; $\eta_n, n \geq 1$, take both positive and negative values. Moreover, $\{\zeta_n, n = 1, 2, 3\}$ is a sequence of gamma distributed random variables (r.v.) with parameters (α, λ), $\alpha > 0, \lambda > 0$. For this case, the probability density function (p.d.f) $f_\zeta(x)$ of ζ_n, $(n = 1, 2, 3...)$ is given by

$$f_\zeta(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x},\ \alpha > 0,\ \lambda > 0,\ x > 0. \tag{2.1}$$

It should be noted that, the r.v. ζ_n represents the discrete interference of chance.
The renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ are defined as follows, respectively:

$$T_0 = S_0 = 0;\ T_n \equiv \sum_{i=1}^{n} \xi_i;\ S_n \equiv \sum_{i=1}^{n} \eta_i, n = 1, 2,$$

Now we can define a sequence of integer-valued random variables $\{N_n\}$, $n = 0, 1, 2,$ as

$$N_0 = 0,\ N_1 \equiv N(z) = inf\left\{n \geq 1 : z - S_n < 0\right\},\ z > 0;$$

$$N_{n+1} \equiv inf\left\{k \geq N_n + 1 : \zeta_n - (S_k - S_{N_n}) < 0\right\},\ n = 0, 1, 2, ...$$

and $inf\{\varnothing\} = +\infty$ is stipulated. Let

$$\tau_0 = 0,\ \tau_1(z) \equiv \sum_{i=1}^{N(z)} \xi_i, ..., \tau_n \equiv \sum_{i=1}^{N_n} \xi_i,\ n = 2, 3, ...$$

and let define $\nu(t)$ as $\nu(t) \equiv max\{n \geq 0 : T_n \leq t\}, t > 0$. The stochastic process $X(t)$ can now be constructed as follows:

$$X(t) \equiv \zeta_n - \left(S_{\nu(t)} - S_{N_n}\right),\ t \in [\tau_n, \tau_{n+1}),\ n \geq 0,\ t > 0;\ X(0) = \zeta_0 = z > 0. \tag{2.2}$$

The process $X(t)$ defined by (2.2) is called “A semi-Markovian random walk with gamma distributed interference of chance”. Figure 1 shows a trajectory of $X(t)$.

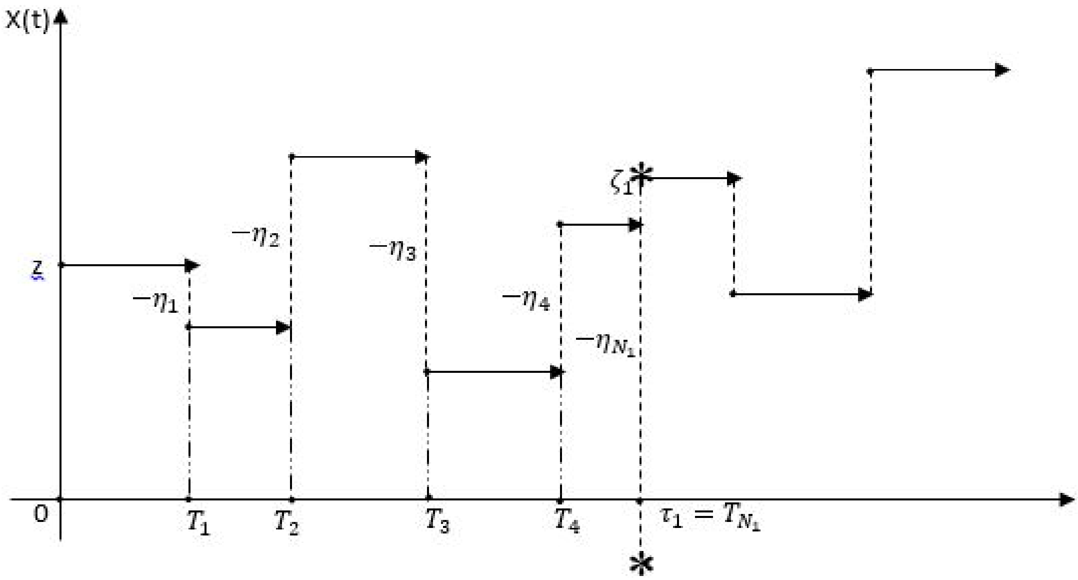

Figure 1. A trajectory of the process $X(t)$.

Before giving the approximate formulas, we need to give the exact formulas for the moments of the ergodic distribution of $X(t)$.

3. Exact expressions for moments of ergodic distribution of process $X(t)$

Now, in order to express the moments of the ergodic distribution $(E(X^k);\ k=1,2)$ of $X(t)$ through moments of $S_{N(z)}$ it is necessary to introduce the following notations:

$$E\left(X^k\right) \equiv lim_{t\to\infty} E\left(X^k(t)\right);\ m_k \equiv E\left(\eta_1^k\right);\ m_{k1} \equiv \frac{m_k}{m_1};\ k=1,2,...,;$$

$$M_k(z) \equiv E\left(S^k_{N(z)}\right);\ E\left(\zeta_1^n M_k(\zeta_1)\right) = \int_0^\infty z^n M_k(z) d\pi(z);\ n=0,1,2,...$$

Here, $\pi(z)$ is the distribution of r.v. ζ_1, i.e. $\pi(z) = P\{\zeta_1 \le z\}$. Moreover the probability density function of ζ_1 is defined by (2.1). In other words $\pi^{'}(z) = f_\zeta(z)$.

Let us now start our examination by first taking into account the exact formulas of the first two moments of $X(t)$ from the study [10].

Lemma 3.1 *([10], Theorem 3.2) Let the initial sequences of the random variables $\{\xi_n\}$ and $\{\eta_n\}$ satisfy the following supplementary conditions i) $0 < E(\xi_1) < \infty$. ii) $E(\eta_1) > 0$, iii) $E\left(|\eta|_1^3\right) < +\infty$, iv) η_1 is a nonarithmetic random variable. In addition to these conditions let $E(X^2) < +\infty$ be satisfied. Then, the first two moments of the ergodic distribution of process $X(t)$ can be expressed by means of the moments of boundary* *functional* $S_{N(z)}$ *as follows:*

$$E(X)=\frac{1}{E(M_1(\zeta_1))}\left\{E\left((2\zeta_1+m_{21})M_1(\zeta_1)\right)-E\left(M_2(\zeta_1)\right)\right\}, \tag{3.1}$$

$$E\left(X^2\right)=\frac{1}{E(M_1(\zeta_1))}\Bigg\{E\left(M_3(\zeta_1)\right)-3E\left(\left(\frac{1}{2}m_{21}+\zeta_1\right)M_2(\zeta_1)\right)$$
$$+E\left(\left(\frac{(3m_{21}^2-2m_{31})}{2}+3m_{21}\zeta_1+3\zeta_1^2\right)M_1(\zeta_1)\right)\Bigg\}. \tag{3.2}$$

Here, $m_{k1}=m_k/m_1$, $k=1,2,3$.

Now let us define the first ladder epoch (ν_1^+) and the first ladder height (χ_1^+) of the random walk $\{S_n, n=0,1,\}$ that we will use through this section as follows:

$$\nu_1^+\equiv min\{n\geq 1: S_n>0\},\ \chi_1^+\equiv S_{\nu_1^+}\equiv\sum_{i=1}^{\nu_1^+}\eta_i.$$

Let (ν_n^+,χ_n^+), $n=2,3,$ be independent pairs with the same distribution as $\left(\nu_1^+,\chi_1^+\right)$ (see [6], page 392). In this case a stochastic process $H(z)$ generated by the sequence $\{\chi_n^+\}, n=1,2,$ is defined as follows:

$$H(z)\equiv min\left\{n\geq 1:\sum_{i=1}^{n}\chi_i^+>z\right\},\ z>0,\ H(0)=1.$$

The stochastic process $H(z)$ is a renewal process (see [6], page 184). The boundary functionals $N(z)$ and $S_{N(z)}\equiv\sum_{i=1}^{N(z)}\eta_i$ can be represented by the Dynkin principle as follows (see [20]):

$$N(z)\equiv\sum_{i=1}^{H(z)}\nu_i^+;\ S_{N(z)}\equiv\sum_{i=1}^{H(z)}\chi_i^+.$$

Remark 3.1. The calculation of the first and second stationary moments of $X(t)$, as given by the exact formulas in (3.1) and (3.2), is very difficult. To overcome this difficulty, in this study we aim to propose approximation formulas for $E(X)$ and $E(X^2)$ by using Kambo's moment-based method (see [12]). To achieve this goal, we must first examine the moments of $S_{N(z)}$. Lemma 3.2. below provides the Laplace transforms of the first three moments of the boundary functional $S_{N(z)}$. First of all, we should give some notations as follows:

$$E(H(z))\equiv U_+(z)=\sum_{n=0}^{\infty}F_+^{*n}(z),\ F_+(z)\equiv P\left\{\chi_1^+\leq z\right\},\ \varphi(\gamma)=E(exp(-\gamma\chi_1^+)),$$

$$D_n^*(\gamma)=E(\chi_1^{+n}exp(-\gamma\chi_1^+)),\ \mu_n=E(\chi_1^{+n}),\ n=1,2,3,\ \gamma>0.$$

Hereafter and throughout the article, $\widetilde{G}(\gamma)$ and $G^*(\gamma)$ represent the Laplace transform and the Laplace Stieltjes transform of any function $G(z)$. Additionally, the notation $\widehat{G}(z)$ represents a moment-based approximation for any function $G(z)$.

Lemma 3.2 *([21], Theorem 4.2) Let the condition of $\mu_3 \equiv E(\chi_1^{+3}) < \infty$ be satisfied. Then, the Laplace transforms of the first three moments $(M_k(z))$ of the boundary functional $S_{N(z)}$ are as follows:*

$$\widetilde{M}_1(\gamma) \equiv \int_{z=0}^{\infty} e^{-\gamma z} E\left(S_{N(z)}\right) dz = \mu_1 \widetilde{U}_+(\gamma), \tag{3.3}$$

$$\widetilde{M}_2(\gamma) \equiv \int_{z=0}^{\infty} e^{-\gamma z} E\left(S_{N(z)}^2\right) dz = 2\mu_1 \widetilde{U}_+(\gamma) U_+^*(\gamma) D_1^*(\gamma) + \mu_2 \widetilde{U}_+(\gamma), \tag{3.4}$$

$$\begin{aligned} \widetilde{M}_3(\gamma) &\equiv \int_{z=0}^{\infty} e^{-\gamma z} E\left(S_{N(z)}^3\right) dz \\ &= 6\mu_1 \widetilde{U}_+(\gamma) U_+^{*2}(\gamma) D_1^{*2}(\gamma) + 3\mu_1 \widetilde{U}_+(\gamma) U_+^*(\gamma) D_2^*(\gamma) \\ &+ 3\mu_2 \widetilde{U}_+(\gamma) U_+^*(\gamma) D_1^*(\gamma) + \mu_3 \widetilde{U}_+(\gamma). \end{aligned} \tag{3.5}$$

Remark 3.2. Our first objective is to derive approximations for the first three moments of $S_N(z)$ based on the results of [12], which provided an approximation for the first three moments of the renewal process.

4. Moment-based approximations for moments of boundary functional $S_{N(z)}$

Moment-based approximation formulas in the sense of Kambo are given in the work [12] with Proposition 4.1. We need to introduce following notation in order to give Proposition 4.1:

$$N(t) = max\left\{n \geq 0 : \sum_{i=1}^{n} X_i \leq t\right\}, \sum_{i=1}^{0} X_i \equiv 0,\ t > 0,\ N(0) = 0. \tag{4.1}$$

Here the random variables X_n, $n = 1, 2$ are independent and identically distributed and positive valued. Note that here $N(t)$ represents the renewal function in the sense of Smith.

Proposition 4.1 *([12], Theorem 3.1) Suppose $Y_n = X_1 + X_2 + ... + X_n$, $n \geq 1$. Moreover, suppose the first three raw moments of X_i exist and are known. Then the following approximation holds for the first three moments of the renewal process $N(t)$:*

$$E\left(N(t)\right) \approx \frac{t}{\mu_1} + \frac{(\mu_2 - 2\mu_1^2)(1 - e^{s_0 t})}{2\mu_1^2}, \tag{4.2}$$

$$\begin{aligned} Var\left[N(t)\right] \approx \frac{\sigma^2}{\mu_1^3} t + \left[\frac{2\sigma^2}{\mu_1^2} + \frac{3}{4} + \frac{5\sigma^4}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3}\right] + \left[\frac{2\nu}{\mu_1} + \nu s\right] t e^{s_0 t} \\ - \left[\frac{5\sigma^2}{2\mu_1^2} + \frac{1}{2} + \frac{\sigma^4}{\mu_1^4} - \frac{2\mu_3}{3\mu_1^3}\right] e^{s_0 t} - \nu^2 e^{2 s_0 t}, \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
E\left[N^3(t)\right] \approx\ & \frac{t^3}{\mu_1^3} + \frac{3(1+3\nu)}{\mu_1^2}t^2 + \left[\frac{1}{\mu_1} + \frac{12\nu}{\mu_1} + \frac{18\nu}{\mu_1^2 s_0} + \frac{18\nu^2}{\mu_1}\right] t + \nu + \frac{12\nu}{\mu_1 s_0} \\
&+ \frac{18\nu}{\mu_1^2 s_0^2} + \frac{36\nu^2}{\mu_1 s_0} - 6\nu^2\left(\frac{1}{s_0} - \nu\right) + \left[3\nu^2(s_0 - \nu s_0^2)\right] t^2 e^{s_0 t} \\
&+ \left[\frac{18\nu^2}{\mu_1} - 6\nu^2(1-\nu s_0)\right] t e^{s_0 t} \\
&+ \left[-\nu - \frac{12\nu}{\mu_1 s_0} - \frac{18\nu}{\mu_1^2 s_0^2} - \frac{36\nu^2}{\mu_1 s_0} + 6\nu^2\left(\frac{1}{s_0} - \nu\right)\right] e^{s_0 t} \qquad (4.4)
\end{aligned}
$$

for $s_0 = \frac{6\mu_1(\mu_2 - 2\mu_1^2)}{(3\mu_2^2 - 2\mu_1\mu_3)},\ \nu = \frac{\mu_2 - 2\mu_1^2}{2\mu_1^2},\ \sigma^2 = \mu_2 - \mu_1^2,\ \mu_n = E(X_1^n),\ n = 1, 2, 3.$

Remark 4.1. In the following parts of this article, $\widehat{U}_+(z)$ will express the moment-based approximation of the renewal function $U_+(z) \equiv E\left(H(z)\right)$ in the sense of Kambo. As observed in (4.1), in order to write the equation $H(z) \equiv N(z) + 1$ with probability 1, it is sufficient to replace X_i with χ_i^+. Thus the following expression can be written.

$$
\widehat{U}_+(z) = \frac{z}{\mu_1} + c_F + (1 - c_F)e^{s_0 z}. \qquad (4.5)
$$

Note that $c_F = \mu_2/(2\mu_1^2)$ is Feller's coefficient. At this point, we present a specific example to demonstrate that the approximate formulas derived according to Kambo's approach are more efficient compared to the formulas available in the existing literature (commonly referred to as Feller's approximation).

Example : $\{\eta_n,\ n = 1, 2, ...\}$ is a sequence of Erlang distributed random variables with parameters $(3, \lambda), \lambda > 0$. In other words, the probability density function $f_\eta(t)$ of η_n is expressed as follows:

$$
f_\eta(t) = \frac{\lambda^3 t^2}{2} e^{-\lambda t},\ t \geq 0.
$$

The main aim of this example is to show that Kambo's approximate formula $U_K(t)$ is more closely aligned with the exact formula $U(t)$ than Feller's asymptotic expansion $(U_F(t))$ for relatively small values of λt. Here

$$
\begin{aligned}
U(t) &= \frac{\lambda t}{3} + \frac{2}{3} + \frac{1}{3} e^{-\frac{3}{2}\lambda t}\left[cos\left(\frac{\sqrt{3}\lambda t}{2}\right) + \frac{1}{\sqrt{3}} sin\left(\frac{\sqrt{3}\lambda t}{2}\right)\right], \\
U_F(t) &= \frac{\lambda t}{3} + \frac{2}{3} + o(1), \\
U_K(t) &= \frac{\lambda t}{3} + \frac{2}{3} + \frac{1}{3} e^{-\frac{3}{2}\lambda t}.
\end{aligned}
$$

Here $U(t)$ is the renewal function generated by Erlang distribution with parameters $(3, \lambda)$. The following notations are used to measure closeness:

$$
\Delta_F \equiv \left|U(t) - U_F(t)\right|,\ \Delta_K \equiv \left|U(t) - U_K(t)\right|;\ \delta_F \equiv \frac{\Delta_F}{U(t)} 100\%;\ \delta_K \equiv \frac{\Delta_K}{U(t)} 100\%,
$$

$$
AP_F = 100 - \delta_F;\ AP_K = 100 - \delta_K.
$$

Here, Δ_F and Δ_K are absolute errors of approximate formulas in terms of Feller (existing method) and Kambo (proposed method), respectively; δ_F and δ_K are relative errors; AP_F and AP_K are the percentage of accuracy of these two approximate formulas. The table below shows the values of measures of closeness, obtained for $\lambda t = 0.2; 0.3; ; 1.5$:

Table 1. Comparison of the results obtained by the proposed method and the existing method.

λt	$U(t)$	$U_F(t)$	$U_K(t)$	Δ_F	Δ_K	$\delta_F(\%)$	$\delta_K(\%)$	$AP_F(\%)$	$AP_K(\%)$
0.2	1.001	0.733	0.980	0.268	0.021	26.75	2.08	73.25	97.92
0.3	1.004	0.767	0.979	0.237	0.024	23.61	2.43	76.39	97.57
0.4	1.008	0.800	0.983	0.208	0.025	20.63	2.48	79.37	97.52
0.5	1.014	0.833	0.991	0.181	0.024	17.85	2.33	82.15	97.67
0.6	1.023	0.867	1.002	0.157	0.021	15.30	2.05	84.70	97.95
0.7	1.034	0.900	1.017	0.134	0.018	12.98	1.70	87.02	98.30
0.8	1.048	0.933	1.034	0.114	0.014	10.91	1.32	89.09	98.68
0.9	1.063	0.967	1.053	0.097	0.010	9.08	0.95	90.92	99.05
1.0	1.081	1.000	1.074	0.081	0.007	7.48	0.60	92.52	99.40
1.1	1.101	1.033	1.097	0.067	0.003	6.10	0.29	93.90	99.71
1.2	1.122	1.067	1.122	0.055	0.000	4.93	0.02	95.07	99.98
1.3	1.145	1.100	1.147	0.045	0.002	3.94	0.20	96.06	99.80
1.4	1.170	1.133	1.174	0.036	0.004	3.11	0.38	96.89	99.62
1.5	1.196	1.167	1.202	0.029	0.006	2.42	0.51	97.58	99.49

As can be seen from the table, when $\lambda t = 0.2$, the accuracy percentage of the Feller approach is 73.25%; comparing the accuracy percentage of the Kambo approach is 97.92%. Additionally, when $\lambda t = 0.3$, these percentages are 76.39% according to Feller and 97.57% according to Kambo, respectively. It becomes clear that at relatively small λt values, the approximate formula in the Kambo sense is more effective than Feller's approximate formula. As the λt value increases, the accuracy percentage of both approximations approaches each other. Particularly, for $\lambda t \geq 7$, the percentages of accuracy will be close enough to each other. From this it can be concluded that especially for small values of λt it is preferable to use approximate formulas derived according to the Kambo approach.

Now we will examine the moments of the boundary functional $S_{N(z)}$.

Remark 4.2. To distinguish from the exact formula $M_k(z) = E\left(S_{N(z)}^k\right)$, $k = 1, 2, 3$; the notations $\widehat{M}_k(z)$ are used for proposed approximations in the sense of Kambo for first three moments of $S_{N(z)}$. Now we will give approximation formulas for first three moments of $S_{N(z)}$ with Theorem 4.2. Note that the coefficient s_0 that we will use in Theorem 4.2 is defined as follows in [12]:

$$s_0 \equiv \frac{6\mu_1(\mu_2 - 2\mu_1^2)}{3\mu_2^2 - 2\mu_1\mu_2}.$$

Theorem 4.2 *Let the following conditions are satisfied:* $\mu_6 = E\left(\chi_1^{+6}\right) < +\infty$ *and* $s_0 < 0$. *Then, the moment-based approximation formulas in the sense of Kambo for the first three moments for* $S_{N(z)}$ *of the process* $X(t)$

are obtained as follows:

$$\widehat{M}_1(z) = z + a_1 + b_1 e^{s_0 z}, \tag{4.6}$$

$$\widehat{M}_2(z) = z^2 + h_1 z + h_2 + (c_1 z + c_2) e^{s_0 z}, \tag{4.7}$$

$$\widehat{M}_3(z) = z^3 + \frac{f_1}{2} z^2 + f_2 z + f_3 + (d_1 z^2 + d_2 z + d_3) e^{s_0 z}. \tag{4.8}$$

Here

$$a_1 = \frac{\mu_2}{2\mu_1},\ b_1 = \frac{2\mu_1^2 - \mu_2}{2\mu_1};$$

$$h_1 = \frac{\mu_2}{\mu_1},\ h_2 = \frac{\mu_3}{3\mu_1},\ h_3 = \frac{7\mu_2\mu_3}{6\mu_1^2} - \frac{3\mu_2^3}{4\mu_1^3} - \frac{\mu_4}{3\mu_1};$$

$$c_1 = h_3 s_0^2 - h_2 s_0 + \mu_2 s_0,\ c_2 = \mu_2 - h_2,$$

$$f_1 = 18\mu_1 - \frac{6\mu_2}{\mu_1},\ f_2 = 9\mu_2 - \frac{9\mu_2^2}{2\mu_1^2},\ f_3 = \frac{\mu_4}{\mu_1} + \frac{27\mu_2\mu_3}{\mu_1^2} - \frac{9\mu_2^3}{4\mu_1^3},$$

$$f_4 = -\frac{\mu_5}{2\mu_1} + \frac{22\mu_3^2}{3\mu_1^2} + \frac{\mu_2^2\mu_3}{4\mu_1^3} - \frac{9\mu_2^4}{4\mu_1^4},$$

$$f_5 = \frac{\mu_6}{8\mu_1} + \frac{\mu_3\mu_4}{2\mu_1^2} - \frac{3\mu_2\mu_5}{8\mu_1^2} + \frac{5\mu_2\mu_3^2}{3\mu_1^3} + \frac{5\mu_2^2\mu_4}{8\mu_1^3} - \frac{5\mu_2^3\mu_3}{4\mu_1^4} + \frac{15\mu_2^5}{16\mu_1^5},$$

$$d_1 = \frac{\mu_3 - f_3}{2} s_0^2 + f_4 s_0^3 - \frac{f_5}{2} s_0^4,\ d_2 = 2(\mu_3 - f_3)s_0 + 3f_4 s_0^2 - f_5 s_0^4,\ d_3 = \mu_3 - f_3.$$

Proof $\widetilde{U}_+(\gamma)$ and $D_n^*(\gamma)$ are defined in Remark 3.1. Moreover, asymptotic expansions for $\widetilde{U}_+(\gamma)$ and $D_n^*(\gamma)$ can be written when $\gamma \to 0$ as follows:

$$\widetilde{U}_+(\gamma) = \frac{1}{\mu_1\gamma^2} + \frac{c_F}{\gamma} + D + o(1), \tag{4.9}$$

$$D_1^*(\gamma) \equiv E\left(\chi_1^+ e^{-\gamma\chi_1^+}\right) = \mu_1 - \gamma\mu_2 + \frac{\gamma^2}{2}\mu_3 + \frac{\gamma^3}{6}\mu_4 + o(\gamma^3), \tag{4.10}$$

$$D_2^*(\gamma) \equiv E\left(\chi_1^{+2} e^{-\gamma\chi_1^+}\right) = \mu_2 - \gamma\mu_3 + \frac{\gamma^2}{2}\mu_4 + \frac{\gamma^3}{6}\mu_5 + \frac{\gamma^4}{24}\mu_6 + o(\gamma^4). \tag{4.11}$$

Here $c_F = \frac{\mu_2}{2\mu_1^2}$ and $D = \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_2}{6\mu_1^2}$. Using $U_+^*(\gamma) = \gamma\widetilde{U}_+(\gamma)$ and (4.9)-(4.11) in Lemma 3.2 the following expansions are obtained:

$$\widetilde{M}_1(\gamma) = \frac{1}{\gamma^2} + \frac{a_1}{\gamma} + \mu_1 D + o(1), \tag{4.12}$$

$$\widetilde{M}_2(\gamma) = \frac{1}{\gamma^3} + \frac{h_1}{\gamma^2} + \frac{h_2}{\gamma} + h_3 + o(1), \tag{4.13}$$

$$\widetilde{M}_3(\gamma) = \frac{6}{\gamma^4} + \frac{f_1}{\gamma^3} + \frac{f_2}{\gamma^2} + \frac{f_3}{\gamma} + f_4 + f_5\gamma + o(\gamma). \tag{4.14}$$

Using the above expansions and Tauber-Abel's Theorem, the following expansions are obtained, when $z \to \infty$:

$$M_1(z) = z + a_1 + o(1), \tag{4.15}$$

$$M_2(z) = z^2 + h_1 z + h_2 + o(1), \tag{4.16}$$

$$M_3(z) = z^3 + \frac{f_1}{2} z^2 + f_2 z + f_3 + o(1). \tag{4.17}$$

Our aim is to determine the functions $o(1)$ in the above equations in a special way based on Kambo's method. Thus, we can eliminate the hiddenness created by $o(1)$ and obtain concrete expression. Instead of the $o(1)$ functions, let us propose the following functions, respectively, based on Kambo:

$$R_1(z) = b_1 e^{s_0 z}, \tag{4.18}$$

$$R_2(z) = (c_1 z + c_2) e^{s_0 z}, \tag{4.19}$$

$$R_3(z) = (d_1 z^2 + d_2 z + d_3) e^{s_0 z}. \tag{4.20}$$

And so, let us consider the following functions:

$$\widehat{M_1}(z) = z + a_1 + R_1(z), \tag{4.21}$$

$$\widehat{M_2}(z) = z^2 + h_1 z + h_2 + R_2(z), \tag{4.22}$$

$$\widehat{M_3}(z) = z^3 + \frac{f_1}{2} z^2 + f_2 z + f_3 + R_3(z). \tag{4.23}$$

In order to determine the above functions, the coefficients b_i, c_i, d_i must first be determined. It is known that $\lim_{z\to 0} M_1(z) = \mu_1$ and $\lim_{z\to 0} \widehat{M_1}(z) = a_1 + b_1$. Hence there must be $a_1 + b_1 = \mu_1$. Therefore $b_1 = \frac{2\mu_1^2 - \mu_2}{2\mu_1}$ is obtained. Thus, we can replace the function $M_1(z)$ with the function $\widehat{M_1}(z)$ whose coefficients have been determined. So,

$$\widehat{M_1}(z) = z + a_1 + b_1 e^{s_0 z}.$$

In order to determine the other coefficients, we have to consider both the asymptotic expansion of $\widetilde{M_i}(\gamma)$ and the asymptotic expansion of the Laplace transforms $\left(\widetilde{\widehat{M}}_i(\gamma)\right)$ of the functions $\widehat{M_i}(z)$ $(i = 1, 2, 3)$, when $\gamma \to 0$.

For this aim we first obtain the asymptotic expansion for $\widetilde{\widehat{M}}_2(\gamma)$ as follows, when $\gamma \to 0$:

$$\widetilde{\widehat{M_2}}(\gamma) \equiv L_\gamma\left(\widehat{M_2}(z)\right) = \frac{2}{\gamma^3} + \frac{h_1}{\gamma^2} + \frac{h_2}{\gamma} + L_\gamma\left(R_2(z)\right). \tag{4.24}$$

Here

$$L_\gamma\left(R_2(z)\right) = c_1 L_\gamma\left(z e^{s_0 z}\right) + c_2 L_\gamma\left(e^{s_0 z}\right) = c_1 \frac{1}{(s_0 - \gamma)^2} + c_1 \frac{1}{\gamma - s_0} = \frac{c_1}{s_0^2} - \frac{c_2}{s_0} + o(1).$$

The following equation must be ensured when the expansions of the $\widetilde{M_2}(\gamma)$ in (4.13) and the $\widetilde{\widehat{M}}_2(\gamma)$ in (4.24) are matched:

$$h_3 = \frac{c_1}{s_0^2} - \frac{c_2}{s_0}. \tag{4.25}$$

Moreover, it is known that $lim_{z\to 0}M_2(z) = \mu_2$ and $lim_{z\to 0}\widehat{M}_2(z) = h_2 + c_2$. Thus, the following equation is obtained:

$$\mu_2 = h_2 + c_2. \tag{4.26}$$

From the equations (4.25) and (4.26), the following equations are obtained:

$$c_1 = h_3 s_0^2 - h_2 s_0 + \mu_2 s_0,\; c_2 = \mu_2 - h_2.$$

Thus, we can replace the function $M_2(z)$ with the function $\widehat{M}_2(z)$ whose coefficients have been determined. So,

$$\widehat{M}_2(z) = z^2 + h_1 z + h_2 + (c_1 z + c_2)e^{s_0 z}.$$

Similarly, we can determine $\widehat{M}_3(z)$.

$$\widetilde{\widehat{M}}_3(\gamma) \equiv L_\gamma\left(\widehat{M}_3(z)\right) = \frac{6}{\gamma^4} + \frac{f_1}{\gamma^3} + \frac{f_2}{\gamma^2} + \frac{f_3}{\gamma} + L_\gamma\left(R_3(z)\right). \tag{4.27}$$

Here

$$\begin{aligned} L_\gamma\left(R_3(z)\right) &= d_1 L_\gamma\left(z^2 E^{s_0 z}\right) + d_2 L_\gamma\left(z E^{s_0 z}\right) + d_3 L_\gamma\left(E^{s_0 z}\right) \\ &= d_1 \frac{2}{(\gamma - s_0)^3} + d_2 \frac{1}{(s_0 - \gamma)^2} + d_3 \frac{1}{\gamma - s_0} \\ &= \frac{-2d_1}{s_0^3\left(1 - \frac{\gamma}{s_0}\right)^3} + \frac{d_2}{s_0^2\left(1 - \frac{\gamma}{s_0}\right)^2} - \frac{d_3}{s_0\left(1 - \frac{\gamma}{s_0}\right)} \end{aligned}$$

and when $\gamma \to 0$

$$\begin{aligned} L_\gamma\left(R_3(z)\right) &= \frac{-2d_1}{s_0^3}\left[1 + 3\frac{\gamma}{s_0} + o(\gamma)\right] + \frac{d_2}{s_0^2}\left[1 + 2\frac{\gamma}{s_0} + o(\gamma)\right] - \frac{d_3}{s_0}\left[1 + \frac{\gamma}{s_0} + o(\gamma)\right] \\ &= \frac{-2d_1 + d_2 s_0 - d_3 s_0^2}{s_0^3} + \frac{-6d_1 + 2d_2 s_0 - d_3 s_0^2}{s_0^4}\gamma + o(\gamma). \end{aligned}$$

The following equation must be ensured when the expansions of the $\widetilde{M}_3(\gamma)$ in (4.14) and the $\widetilde{\widehat{M}}_3(\gamma)$ in (4.27) are matched:

$$f_4 = \frac{-2d_1 + d_2 s_0 - d_3 s_0^2}{s_0^3}, \tag{4.28}$$

$$f_5 = \frac{-6d_1 + d_2 s_0 - d_3 s_0^2}{s_0^4}. \tag{4.29}$$

Moreover, it is known that $lim_{z\to 0}M_3(z) = \mu_3$ and $lim_{z\to 0}\widehat{M}_3(z) = f_3 + d_3$. Thus, the following equation is obtained:

$$f_3 + d_3 = \mu_3. \tag{4.30}$$

From the equations (4.28), (4.29) and (4.30), the following equations are obtained:

$$d_1 = \frac{\mu_3 - f_3}{2} s_0^2 + f_4 s_0^3 - \frac{f_5}{2} s_0^4;\quad d_2 = 2(\mu_3 - f_3)s_0 + 3f_4 s_0^2 - f_5 s_0^4;\quad d_3 = \mu_3 - f_3.$$

Thus, we can replace the function $M_3(z)$ with the function $\widehat{M}_3(z)$ whose coefficients have been determined. So,

$$\widehat{M}_3(z) = z^3 + \frac{f_1}{2}z^2 + f_2 z + f_3 + (d_1 z^2 + d_2 z + d_3)e^{s_0 z}.$$

The coefficients are as given in the expression of Theorem 4.2. This completes the proof. □

5. Moment-based approximations for moments of process $X(t)$

This section presents a moment-based approximation in the sense of Kambo for the first two moments of the process $X(t)$ by using the first three moments of $S_{N(z)}$. We start with stating following lemma:

Lemma 5.1 *Let the sequence of random variables $\{\zeta_n,\ n = 1, 2, ...\}$ have gamma distribution with parameters α and λ given as in (2.1), and let the following conditions will be satisfied:*

$$\mu_6 = E\left(\chi_1^{+6}\right) < +\infty,\ K \equiv -s_0 > 0.$$

Then the following moment-based approximations in the sense of Kambo are written:

$$E\left(\widehat{M}_1(\zeta_1)\right) = \frac{\alpha}{\lambda} + \frac{\mu_2}{2\mu_1} - \frac{\mu_2 - \mu_1^2}{2\mu_1}\left(\frac{\lambda}{\lambda + K}\right)^{\alpha}, \tag{5.1}$$

$$E\left(\zeta_1\widehat{M}_1(\zeta_1)\right) = \frac{\alpha(\alpha+1)}{\lambda^2} + \frac{\alpha\mu_2}{2\mu_1\lambda} + \frac{\alpha(2\mu_1^2 - \mu_2)\lambda^{\alpha}}{(\lambda + K)^{\alpha+1}}, \tag{5.2}$$

$$E\left(\zeta_1^2\widehat{M}_1(\zeta_1)\right) = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} + \frac{\alpha(\alpha+1)\mu_2}{2\mu_1\lambda^2} + \frac{\alpha(\alpha+1)(2\mu_1^2 - \mu_2)\lambda^{\alpha}}{2\mu_1(\lambda + K)^{\alpha+2}}, \tag{5.3}$$

$$E\left(\widehat{M}_2(\zeta_1)\right) = \frac{\alpha(\alpha+1)}{\lambda^2} + \frac{\alpha\mu_2}{\mu_1\lambda} + \frac{\mu_3}{3\mu_1} + c_1\frac{\alpha\lambda^{\alpha}}{(\lambda + K)^{\alpha+1}} + c_2\left(\frac{\lambda}{\lambda + K}\right)^{\alpha}, \tag{5.4}$$

$$E\left(\zeta_1\widehat{M}_2(\zeta_1)\right) = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} + \frac{\alpha(\alpha+1)\mu_2}{\mu_1\lambda^2} + \frac{\alpha\mu_3}{3\mu_1\lambda} + c_1\frac{\alpha(\alpha+1)\lambda^{\alpha}}{(\lambda + K)^{\alpha+2}} + c_2\frac{\alpha\lambda^{\alpha}}{(\lambda + K)^{\alpha+1}}, \tag{5.5}$$

$$E\left(\widehat{M}_3(\zeta_1)\right) = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} + f_1\frac{\alpha(\alpha+1)}{2\lambda^2} + f_2\frac{\alpha}{\lambda} + f_3 + d_1\frac{\alpha(\alpha+1)\lambda^{\alpha}}{(\lambda + K)^{\alpha+2}} + d_2\frac{\alpha\lambda^{\alpha}}{(\lambda + K)^{\alpha+1}} + d_3\left(\frac{\lambda}{\lambda + K}\right)^{\alpha}. \tag{5.6}$$

Here $B = \alpha + \frac{\lambda\mu_2}{2\mu_1}$.

Proof Moment-based approximations $\widehat{M}_k(z)$ in the sense of Kambo for $M_k(z) = E(S_{N(z)}^k)$ for $k = 1, 2, 3$ are obtained in Theorem 4.2. By using approximate formulas $\widehat{M}_1(z)$, $\widehat{M}_2(z)$ and $\widehat{M}_3(z)$ and making necessary calculations we obtain (5.1)-(5.6). □

The primary objective of this study is outlined in Theorem 5.2. Here we obtained in the sense of Kambo moment-based approximations for the first two moments of the process $X(t)$ by using approximations for the first three moments of $S_{N(z)}$.

Note that $\widehat{E}(X^k)$, $k=1,2$ represents in the sense of Kambo moment-based approximations for the first two moments of the ergodic distribution of $X(t)$.

Theorem 5.2 *Let the conditions of Lemma 3.1 and Lemma 5.1 be satisfied. Then in the sense of Kambo moment-based approximations for the first two moments of the ergodic distribution of the process $X(t)$ are derived as follows, when the parameter λ is sufficiently small:*

$$\widehat{E}(X) = \frac{\alpha(\alpha+1)}{B}\lambda^{-1} + \frac{A_1\alpha}{B} + \frac{3A_1\mu_2 - 2\mu_3}{6\mu_1 B}\lambda - \frac{\mu_1 c(\alpha+1)}{K^\alpha B}\lambda^\alpha,$$

$$\widehat{E}(X^2) = \frac{\alpha(\alpha+1)(\alpha+2)}{B}\lambda^{-2} + \frac{\alpha(\alpha+1)(18\mu_1 c + 3A_1)}{2B}\lambda^{-1} + \frac{\alpha(9\mu_1\mu_3 c + \mu_3 + 3\mu_1 A_2)}{\mu_1 B}$$
$$+ \frac{2\mu_1 f_3 + 3A_2\mu_2 - A_1\mu_3}{2\mu_1 B}\lambda - \frac{(\alpha+1)(\alpha+2)\mu_1 c}{K^\alpha B}\lambda^{\alpha-1} - \frac{(\alpha+1)\mu_1 c(18\mu_1 c + 3A_1)}{2K^\alpha B}\lambda^\alpha.$$

Here $A_1 = m_{21}$, $A_2 = \frac{m_{21}^2}{2} - \frac{m_{31}}{3}$, $m_{k1} = \frac{m_k}{m_1}$, $k=2,3$, $B = \alpha + \frac{\mu_2}{2\mu_1}\lambda$, $c = \frac{2\mu_1^2 - \mu_2}{2\mu_1}$, $K \equiv -s_0 = \frac{6\mu_1(2\mu_1^2-\mu_2)}{(3\mu_2^2 - 2\mu_1\mu_3)}$.

Proof The exact expressions for the first moment $E(X)$ and second moment $E(X^2)$ of ergodic distribution of $X(t)$ are given in Lemma 3.1 with (3.1) and (3.2) respectively in Section 3. Moreover the following in the sense of Kambo moment-based approximation can be given using Lemma 5.1 with (5.7) as follows:

$$\begin{aligned}\frac{1}{E\left(\widehat{M_1}(\zeta_1)\right)} &= \left\{\frac{\alpha}{\lambda} + \frac{\mu_2}{2\mu_1} - \frac{\mu_2 - 2\mu_1^2}{2\mu_1}\left(\frac{\lambda}{\lambda+K}\right)\right\}^{-1} \\ &\approx \left\{\left[\frac{\alpha}{\lambda} + \mu_1 c_F\right]\left[1 + \frac{\mu_1 c}{\alpha K^\alpha}\lambda^{\alpha+1} - \frac{\mu_1 c}{K^\alpha}\left(\frac{\mu_1 c_F}{\alpha^2} + \frac{1}{K}\right)\lambda^{\alpha+1}\right]\right\}^{-1} \\ &\approx \frac{\lambda}{\alpha + \lambda\mu_1 c_F}\left\{1 - \frac{\mu_1 c}{\alpha K^\alpha}\lambda^{\alpha+1} + \frac{\mu_1 c(\mu_1 c_F K + \alpha^2)}{\alpha^2 K^{\alpha+1}}\lambda^{\alpha+2}\right\} \\ &\approx \frac{1}{B}\lambda - \frac{\mu_1 c}{\alpha K^\alpha B}\lambda^{\alpha+2}. \end{aligned} \tag{5.7}$$

Here $c_F = \mu_2/2\mu_1^2$.

Taking advantage of Lemma 5.1 the following in the sense of Kambo moment-based approximation can be written as $\lambda \to 0$:

$$\begin{aligned} E\left((2\zeta_1 + m_{21})M_1(\zeta_1)\right) - E\left(M_2(\zeta_1)\right) &= \frac{\alpha(\alpha+1)}{\lambda^2} + A_1\frac{\alpha}{\lambda} + A_1\frac{\mu_2}{2\mu_1} - \frac{\mu_3}{3\mu_1} \\ &+ \left(\frac{2\mu_1^2 - \mu_2}{\mu_1} - c_1\right)\frac{\alpha\lambda^\alpha}{(\lambda+K)^{\alpha+1}} + \left(A_1\frac{2\mu_1^2-\mu_2}{\mu_1} - c_2\right)\frac{\alpha\lambda^\alpha}{(\lambda+K)^\alpha}. \end{aligned} \tag{5.8}$$

Substituting (5.7) and (5.8) into (3.1) following in the sense of Kambo moment-based approximation is derived for $\widehat{E}(X)$:

$$\widehat{E}(X) = \frac{\alpha(\alpha+1)}{B}\lambda^{-1} + \frac{A_1\alpha}{B} + \frac{3A_1\mu_2 - 2\mu_3}{6\mu_1 B}\lambda - \frac{\mu_1 c(\alpha+1)}{K^\alpha B}\lambda^\alpha.$$

Now it aims to find in the sense of Kambo moment-based approximation for the first two moments of ergodic distribution of the process $X(t)$. The exact expression for the second moment of the ergodic distribution of $X(t)$ was given through (3.2) in Section 3 with Lemma 3.1. Using Lemma 5.1 the following in the sense of Kambo moment-based approximation can be written, when $\lambda \to 0$:

$$\begin{aligned}
&E(M_3(\zeta_1)) - 3E\left(\left(\frac{1}{2}m_{21} + \zeta_1\right) M_2(\zeta_1)\right) + E\left(\left(\frac{(3m_{21}^2 - 2m_{31})}{2} + 3m_{21}\zeta_1 + 3\zeta_1^2\right) M_1(\zeta_1)\right) \\
&= \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} + \frac{\alpha(\alpha+1)}{\lambda^2}\left[\frac{f_1}{2} + \frac{3A_1}{2} - \frac{3\mu_2}{2\mu_1}\right] + \frac{\alpha}{\lambda}\left[f_2 - \frac{\mu_3}{\mu_1} + \frac{(3m_{21}^2 - 2m_{31})}{2}\right] \\
&+ f_3 - \frac{A_1\mu_3}{2\mu_1} + \frac{(3m_{21}^2 - 2m_{31})\mu_2}{4\mu_1} + \frac{\alpha(\alpha+1)\lambda^\alpha}{(\lambda+K)^{\alpha+2}}\left[d_1 - 3c_1 + 3\frac{2\mu_1^2 - \mu_2}{2\mu_1}\right] \\
&+ \frac{\alpha\lambda^\alpha}{(\lambda+K)^{\alpha+1}}\left[d_2 - \frac{3A_1}{2} - 3c_2 + 3A_1\frac{2\mu_1^2 - \mu_2}{2\mu_1}\right] \\
&+ \frac{\lambda^\alpha}{(\lambda+K)^\alpha}\left[d_3 + \frac{(3m_{21}^2 - 2m_{31})}{2}\left(\frac{2\mu_1^2 - \mu_2}{2\mu_1}\right)\right].
\end{aligned} \tag{5.9}$$

Substituting (5.7) and (5.9) into (3.2) following in the sense of Kambo moment-based approximation is derived for $\widehat{E}(X^2)$:

$$\begin{aligned}
\widehat{E}(X^2) &= \frac{\alpha(\alpha+1)(\alpha+2)}{B}\lambda^{-2} + \frac{\alpha(\alpha+1)\left(9\mu_1 c + \frac{3A_1}{2}\right)}{B}\lambda^{-1} + \frac{\alpha\left(9\mu_3 c + \frac{\mu_3}{\mu_1}3A_2\right)}{B} \\
&+ \frac{f_3 + \frac{3A_2\mu_2 - A_1\mu_3}{2\mu_1}}{B}\lambda - \frac{(\alpha+1)(\alpha+2)\mu_1 c}{K^\alpha B}\lambda^{\alpha-1} - \frac{(\alpha+1)\mu_1 c\left(9\mu_1 c + \frac{3A_1}{2}\right)}{K^\alpha B}\lambda^\alpha.
\end{aligned}$$

This completes the proof of Theorem 5.2. □

By using the approximate formula of the first two moments of the ergodic distribution of the process $X(t)$, we will obtain the approximation formula for the variance of the ergodic distribution of the process as in Corollary 5.3.

Corollary 5.3 *Under the conditions of Theorem 5.2 moment-based approximation in the sense of Kambo is obtained for the variance of ergodic distribution of $X(t)$ as follows:*

$$\widehat{Var}(X) = \begin{cases} \lambda^{-2}V_1 + \lambda^{-1}V_2 + V_3 + \lambda^{\alpha-1}V_4, & 0 < \alpha \le 2 \\ \lambda^{-2}V_1 + \lambda^{-1}V_2 + V_3 + \lambda V_5, & \alpha > 2. \end{cases}$$

Here

$$V_1 = \frac{\alpha(\alpha+1)(\alpha+2)}{B} - \frac{\alpha^2(\alpha+1)^2}{B^2},$$

$$V_2 = \frac{\alpha(\alpha+1)(18\mu_1 c + 3A_1)}{2B} - \frac{2A_1\alpha^2(\alpha+1)}{B^2},$$

$$V_3 = \frac{\alpha(9\mu_1\mu_3 c - \mu_3 + 3\mu_1 A_2)}{\mu_1 B} - \frac{A_1^2\alpha^2}{B^2} - \frac{\alpha(\alpha+1)(3A_1\mu_2 - 2\mu_3)}{3\mu_1 B^2},$$

$$V_4 = \frac{(\alpha+1)(\alpha+2)\mu_1 c}{K^\alpha B} - \frac{2\alpha(\alpha+1)^2\mu_1 c}{K^\alpha B^2},$$

$$V_5 = \frac{2\mu_1 f_3 + 3A_2\mu_2 - A_1\mu_3}{2\mu_1 B} - \frac{\alpha A_1(3A_1\mu_2 - 2\mu_3)}{3\mu_1 B^2}.$$

Note that here the coefficients A_i and B are given in Theorem 5.2

Remark 5.1 The approximate formulas proposed for expected value $\widehat{E}(X)$ and variance $\widehat{Var}(X)$ of the ergodic distribution of the random walk process $X(t)$ in this study are relatively more precise approaches to those existing in the literature. To demonstrate this, we will now consider two examples.

6. Examples

In this section, the expected value and variance of $X(t)$ were calculated for two concrete distributions (Gaussian and Uniform).

Example 1: Suppose that the summands η_i constituting the random walk process $\{S_n\}$ have normal distribution with parameters $\mu = 1$, $\sigma^2 = 1$ $(\eta_1 \sim N(1,1))$. In this case by using Python (3.11.0) version and Monte Carlo simulation method by producing 10^8 trajectories, we calculated the first six moments of first ladder height χ_1^+. Following results were obtained:

$$\mu_1 \cong 1.25,\ \mu_2 \cong 2.18,\ \mu_3 \cong 4.59,\ \mu_4 \cong 10.97,\ \mu_5 \cong 28.98,\ \mu_6 \cong 83.04.$$

Using the values of μ_i, $i = 1,2,3,4,5,6$; the constants c and K can be calculated as follows:

$$c = \frac{2\mu_1^2 - \mu_2}{2\mu_1} \cong 0.378,\ \ K = \frac{6\mu_1(2\mu_1^2 - \mu_2)}{3\mu_2^2 - 2\mu_1\mu_3} \cong 2.547.$$

Considering these moments and Theorem 5.2, in the sense of Kambo moments-based approximations for the expected value and variance of Gaussian random walk process $X(t)$ are derived as follows:

$$\widehat{E}(X) = \frac{\alpha(\alpha+1)}{B}\lambda^{-1} + \frac{2\alpha}{B} - \frac{0.52}{B}\lambda + \frac{0.38(\alpha+1)}{(2.48)^\alpha B}\lambda^\alpha;$$

$$\widehat{Var}(X) = \begin{cases} \sigma_1^2, & 0 < \alpha \le 2 \\ \sigma_2^2, & \alpha > 2. \end{cases}$$

$$\sigma_1^2 = \left[\frac{\alpha(\alpha+1)[(\alpha+2)B - \alpha(\alpha+1)]}{B^2}\right]\lambda^{-2} + \left[\frac{\alpha(\alpha+1)[6.38B - 4\alpha(\alpha+1)]}{B^2}\right]\lambda^{-1}$$
$$+ \left[\frac{\alpha[10.74B - 5.63\alpha - 1.63]}{B^2}\right] + \left[\frac{(\alpha+1)[0.38(\alpha+2)B - 0.75\alpha(\alpha+1)]}{(2.48)^\alpha B^2}\right]\lambda^{\alpha-1},$$

$$\sigma_2^2 = \left[\frac{\alpha(\alpha+1)[(\alpha+2)B - \alpha(\alpha+1)]}{B^2}\right]\lambda^{-2} + \left[\frac{\alpha(\alpha+1)[6.38B - 4\alpha(\alpha+1)]}{B^2}\right]\lambda^{-1}$$
$$+ \left[\frac{\alpha[10.74B - 5.63\alpha - 1.63]}{B^2}\right] + \left[\frac{168.16B - 3.27\alpha}{B^2}\right]\lambda.$$

Here $B = \alpha + 0.87\lambda$.

Example 2: Now suppose that the summands η_i constituting the random walk process $\{S_n\}$ have uniform distribution with parameters $(-1,3)$ $(\eta_1 \sim U(-1,3))$. In this case by using Python (3.11.0) version and Monte Carlo simulation method by producing 10^8 trajectories, we calculated the first six moments of first ladder height χ_1^+. Following results were obtained:

$$\mu_1 \cong 1.43,\ \mu_2 \cong 2.76,\ \mu_3 \cong 6.04,\ \mu_4 \cong 14.17,\ \mu_5 \cong 34.82,\ \mu_6 \cong 88.27.$$

Using the values of μ_i, $i = 1,2,3,4,5,6$; the constants c and K can be calculated as follows:

$$c = \frac{2\mu_1^2 - \mu_2}{2\mu_1} \cong 0.465,\ \ K = \frac{6\mu_1(2\mu_1^2 - \mu_2)}{3\mu_2^2 - 2\mu_1\mu_3} \cong 2.045.$$

Considering these moments and Theorem 5.2, in the sense of Kambo moments-based approximations for the expected value and variance of random walk process $X(t)$ are derived as follows:

$$\widehat{E}(X) = \frac{\alpha(\alpha+1)}{B}\lambda^{-1} + \frac{2.33\alpha}{B} - \frac{0.84}{B}\lambda + \frac{0.46(\alpha+1)}{(2.05)^\alpha B}\lambda^\alpha,$$

$$\widehat{Var}(X) = \begin{cases} \sigma_1^2, & 0 < \alpha \leq 2 \\ \sigma_2^2, & \alpha > 2. \end{cases}$$

$$\sigma_1^2 = \left[\frac{\alpha(\alpha+1)[(\alpha+2)B - \alpha(\alpha+1)]}{B^2}\right]\lambda^{-2} + \left[\frac{\alpha(\alpha+1)[7.68B - 4.67\alpha(\alpha+1)]}{B^2}\right]\lambda^{-1}$$
$$+ \left[\frac{\alpha[21.1B - 8.89\alpha - 3.45]}{B^2}\right] + \left[\frac{(\alpha+1)[0.46(\alpha+2)B - 0.93\alpha(\alpha+1)]}{(2.05)^\alpha B^2}\right]\lambda^{\alpha-1},$$

$$\sigma_2^2 = \left[\frac{\alpha(\alpha+1)[(\alpha+2)B - \alpha(\alpha+1)]}{B^2}\right]\lambda^{-2} + \left[\frac{\alpha(\alpha+1)[7.68B - 4.67\alpha(\alpha+1)]}{B^2}\right]\lambda^{-1}$$
$$+ \left[\frac{\alpha[21.1B - 8.89\alpha - 3.45]}{B^2}\right] + \left[\frac{216.21B - 8.04\alpha}{B^2}\right]\lambda.$$

Here $B = \alpha + 0.96\lambda$.

7. Conclusions

In this article, we study a random walk process $X(t)$ with gamma distributed interference of chance. The key idea is obtaining approximations for the expected value and variance of the ergodic distribution of the process $X(t)$. Our study addresses a significant gap in the existing literature regarding the convergence behavior of the remaining terms in asymptotic expansions. While previous studies have often relied on big-oh and small-oh notations to characterize these terms, in this study, we have provided an approximation to the ergodic distribution that allows us to visualize the convergence behavior of the remaining terms. Here, a new perspective is proposed to obtain approximate formulas by utilizing Kambo's approach.

In Section 4 in particular, we provided an example by considering the renewal function produced by the Erlang distribution in order to measure the accuracy of the approximation proposed by Kambo. As a result of our observation, we have come to the following conclusion: For large values of $E((\zeta_1))$ moment-based in the sense of Kambo approximation results and asymptotic results are closer to each other. However, at small values of $E((\zeta_1))$ the results obtained from moment-based in the sense of Kambo approximation results are more efficient than the Feller's asymptotic results. Here ζ_1 represents the discrete interference of chance and the probability density function is given with Equation (2.1). In Section 5 we provide moment-based approximations for the moments of the process $X(t)$ by using the first three moments of $S_{N(z)}$. Moreover in Section 6 we consider two examples to apply the formulas that we obtained for the first moment and variance of the ergodic distribution of the process $X(t)$.

In future it is possible to propose approximations for higher order moments of the ergodic distribution of the process $X(t)$ in order to obtain approximations for skewness and kurtosis coefficients. Moreover moment-based approximation formulas could be proposed for semi-Markovian random walk process with different types of interference of chances (for example Weibull, Nakagami distribution, Amoroso distribution, Stacy distribution and etc.). In addition, when the distribution of the random variable η_n is symmetrized Gamma distribution or triangle distribution in the interval $[-a, b]$, $a, b > 0$, obtaining approximations in the sense of Kambo for moments of random walk may also be interesting in application.

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