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Computing exterior isoclinism of crossed modules

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Abstract: In this paper, we introduce exterior isoclinism for crossed modules, extending the concept of isoclinism from groups to these 2-dimensional algebraic structures. Motivated by work on tensor and exterior products of nonabelian groups, we utilize nonabelian exterior products, exterior squares, and exterior centers to define this new equivalence relation. Additionally, we provide computational tools implemented in the GAPsystem to identify exterior isoclinism families of small groups and crossed modules.

Key words: Exterior isoclinism, crossed module, GAP

1. Introduction

Isoclinism was introduced by [16] as a classification of prime power groups, which is an equivalence relation and weaker than isomorphism. The idea is based on the isomorphism of central quotients and commutators rather than the isomorphism of whole groups. This notion was studied in detail in [15]. Additionally, the relationship between isoclinism families and stem groups was studied in [5]. For some related works, we refer to [18, 24].

The notion of crossed module, generalizing the notion of a G-module, was introduced by Whitehead [26] during his studies on the algebraic structure of the second relative homotopy group. Crossed modules are algebraic objects with rich structure and are considered as ”2-dimensional groups” [8]. They provide a simultaneous generalization of the concepts of normal subgroups and modules over a group. Furthermore, we may regard any group as a crossed module. It is, therefore, of interest to seek generalizations of group theoretic concepts and structures to crossed modules.

A share package, XMod [1], for the GAP [25] computational discrete algebra system was described by C.D. Wensley et al., which contains functions for computing crossed modules of groups and cat^1-groups and their morphisms. Later, the algebraic version of a GAP package, XModAlg [2], was given by Arvasi and Odabas (see [3]). The notions of isoclinism and stem group were generalized to crossed modules as 2-dimensional groups, and computer implementations of these notions for GAP were given in [22]. Additionally, n-isoclinism classes of crossed modules were defined in [23].

In considering the tensor and exterior products of (nonabelian) groups by Brown and Loday, we see that their definitions have close connections with universal commutator relations in a group. In fact, a special case of the exterior product of two groups was discovered by Miller [19] in her group theoretic interpretation of the

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second homology of a group with integer coefficients. The notion of exterior isoclinism of groups, which yields a new classification on the class of all groups, was introduced in [14].

The motivation of this paper lies in [10] and [22], where the exterior square, exterior center, and isoclinism of crossed modules are studied. In this paper, we describe a new equivalence relation named "exterior isoclinism" for crossed modules, similar to isoclinism, using the nonabelian exterior product, exterior square, and exterior center defined in [7, 10]. Additionally, we provide computer implementations to determine exterior isoclinism families of small groups and crossed modules for GAP.

The work is organized as follows:
In Section 2, we will recall basic notions such as isoclinism of groups and crossed modules.
In Section 3, we will recall the notion of exterior isoclinism of groups.
In Section 4, we will introduce the notion of exterior isoclinism of crossed modules.
In Section 5, we will provide functions for computing with these structures that have been developed using the GAP computational discrete algebra programming language.

2. Preliminaries

In this section, we will recall the notions of isoclinism with basic properties (see [16, 22]). In the sequel of the work, we assume all groups to be finite.

2.1. Isoclinism of groups

In general, isoclinism is used for the classification of finite groups, and there are many works concerning the enumeration of groups with finite order related to isoclinism (see [17, 24]).

Definition 2.1 *[16] Let M and N be groups; a pair (μ, ζ) is termed an isoclinism from M to N if:*

1. *$\mu : M/Z(M) \longrightarrow N/Z(N)$ is an isomorphism between central quotients;*
2. *$\zeta : [M, M] \longrightarrow [N, N]$ is an isomorphism between derived subgroups;*
3. *the diagram*

$$\begin{array}{ccc} M/Z(M) \times M/Z(M) & \xrightarrow{c_M} & [M,M] \\ \downarrow{\scriptstyle \mu\times\mu} & & \downarrow{\scriptstyle \zeta} \\ N/Z(N) \times N/Z(N) & \xrightarrow[c_N]{} & [N,N] \end{array}$$

commutes, where c_M, c_N are commutator maps.

If there is an isoclinism from M to N, we shall say that M and N are isoclinic groups and denote this by $M \sim N$. It is well-known that isoclinism is an equivalence relation.

Examples 2.2
(1) Isomorphic groups are also isoclinic.
(2) All abelian groups are isoclinic to each other. The pairs (μ, ζ) consist of trivial homomorphisms.
(3) A stem group G is a group satisfying $Z(G) \leq G'$. In every nonabelian isoclinism family, there exists a

stem group, which is the lowest order group in the family. See [16] for details. Consider the isoclinism family of $C_4 \ltimes C_8$; the stem group in the family is Q_8 with order 8, which is the lowest order group in the family.

Gustafson [13] considered "what is the probability that two group elements commute?" The answer is given by what is known as the commutative degree of a group. Let G be a finite group, the commutative degree of G is denoted by $d(G)$ and defined by

$$d(G) = \frac{|\{(x,y) \in G \times G : xy = yx\}|}{|G|^2}$$

Obviously, G is abelian if and only if $d(G) = 1$; furthermore, the following results are given in [18] :

1. If G is abelian, then $d(G) \leq \frac{5}{8}$
2. If $d(G) > \frac{1}{2}$, then G is nilpotent
3. If $d(G) = \frac{1}{2}$ and G is not nilpotent, then the derived subgroup of G is isomorphic to cyclic group of order 3.

Moreover as a important result, Lescot proved that every isoclinic finite groups has same commutative degree (see [18] for details).

2.2. Isoclinism of crossed modules

A *crossed module* consists of a group homomorphism $\partial : S \to R$, endowed with a left action R on S (written by $(r,s) \to {}^r s$ for $r \in R$ and $s \in S$) satisfying the following conditions:

$$\begin{array}{rcll} \partial({}^r s) & = & r(\partial s)r^{-1} & \forall\ s \in S,\ r \in R; \\ {}^{(\partial s_2)}s_1 & = & s_2 s_1 s_2^{-1} & \forall\ s_1, s_2 \in S. \end{array}$$

The first condition is called the *pre-crossed module property* and the second one the *Peiffer identity.* We will denote such a crossed module by $\mathcal{X} = (\partial : S \to R)$. $\mathcal{X}$ is called to be finite if the source and range groups are both finite. In this case, we define the size of $\mathcal{X}$ to be the ordered pair $(|S|, |R|)$.

A *morphism of crossed modules* $(\sigma, \rho) : \mathcal{X}_1 \to \mathcal{X}_2$, where $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$, consists of two group homomorphisms $\sigma : S_1 \to S_2$ and $\rho : R_1 \to R_2$ such that

$$\partial_2 \circ \sigma \ = \ \rho \circ \partial_1, \quad \text{and} \quad \sigma({}^r s) \ = \ {}^{(\rho r)}\sigma s \qquad \forall s \in S,\ r \in R.$$

Standard constructions for crossed modules include the following.

1. Any group G can be regarded as a crossed module in two obvious ways. We can take $1 \xhookrightarrow{inc.} R$ with the inclusion map or $R \xrightarrow{id} R$ with the identity map and action by conjugation.
2. A *conjugation crossed module* is an inclusion of a normal subgroup $N \trianglelefteq R$, where R acts on N by conjugation.
3. If $1 \longrightarrow N \longrightarrow S \xrightarrow{\rho} R \longrightarrow 1$ is a central extension of groups, then $S \xrightarrow{\rho} R$ is a crossed module with the action of R on S via lifting and conjugation.

4. The *direct product* of $\mathcal{X}_1$ and $\mathcal{X}_2$ is $\mathcal{X}_1 \times \mathcal{X}_2 = (\partial_1 \times \partial_2 : S_1 \times S_2 \to R_1 \times R_2)$ is a crossed module with direct product action ${}^{(r_1,r_2)}(s_1, s_2) = ({}^{r_1}s_1, {}^{r_2}s_2)$.

Let $\mathcal{X} = (\partial : S \to R)$ be a crossed module. We get following subgroups of S and R respectively

$$\begin{aligned} S^R &= \{s \in S :{}^r s = s, \forall r \in R\} \\ St_R(S) &= \{r \in R :{}^r s = s, \forall s \in S\}. \end{aligned}$$

Clearly, S^R is in fact a subgroup of $Z(S)$ (the center of S), called the fixed point subgroup of S. Note that $st_R(S)$ is the kernel of the homomorphism from R to $\mathrm{Aut}(S)$ which defines the action of $\mathcal{X}$. $st_R(S)$ is called the stabilizer subgroup of R. If the action of R on S is faithful, that is, if $st_R(S) = 1$, then $\mathcal{X}$ is called a faithful crossed module. Also, $D_R(S) = \left\langle \{{}^r ss^{-1} : s \in S, r \in R\} \right\rangle$ is the subgroup of S, and $D_R(S)$ is called the displacement subgroup.

Definition 2.3 *Let* $\mathcal{X} = (\partial : S \to R)$ *be a crossed module. Then*

$$Z(\mathcal{X}) : S^R \xrightarrow{\partial|} Z(R) \cap st_R(S)$$

is a crossed module. $Z(\mathcal{X})$ *is called center of the crossed module* $\mathcal{X}$.

$\mathcal{X} = (\partial : S \to R)$ *is abelian if and only if* R *is abelian and acts trivially on* S, *which implies that* S *is also abelian. With this in mind,*

$$\mathcal{X}' = D_R(S) \xrightarrow{\partial|} R'$$

is commutator subcrossed module of $\mathcal{X}$ *where* $R' = [R, R]$ *is the commutator subgroup of* R *(see [21]).*

Definition 2.4 *Let* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *be finite crossed modules; two pair* (μ_1, μ_0) *and* (ζ_1, ζ_0) *are termed an isoclinism of crossed modules from* $\mathcal{X}_1$ *to* $\mathcal{X}_2$ *if:*

1. (μ_1, μ_0) *is an isomorphism from* $\mathcal{X}_1/(Z(\mathcal{X}_1))$ *to* $\mathcal{X}_2/(Z(\mathcal{X}_2))$;
2. (ζ_1, ζ_0) *is an isomorphism from* $\mathcal{X}_1'$ *to* $\mathcal{X}_2'$;
3. *the diagram*

$$\begin{array}{ccc} \mathcal{X}_1/(Z(\mathcal{X}_1)) \times \mathcal{X}_1/(Z(\mathcal{X}_1)) & \longrightarrow & \mathcal{X}_1' \\ \Big\downarrow {\scriptstyle (\mu_1,\mu_0)\times(\mu_1,\mu_0)} & & \Big\downarrow {\scriptstyle (\zeta_1,\zeta_0)} \\ \mathcal{X}_2/(Z(\mathcal{X}_2)) \times \mathcal{X}_2/(Z(\mathcal{X}_2)) & \longrightarrow & \mathcal{X}_2' \end{array}$$

is commutative.

If there is an isoclinism of crossed modules from from $\mathcal{X}_1$ to $\mathcal{X}_2$, we shall say that $\mathcal{X}_1$ and $\mathcal{X}_2$ are isoclinic crossed modules.

Examples 2.5

1. *All abelian crossed modules are isoclinic. The pairs* $((\mu_1, \mu_0), (\zeta_1, \zeta_0))$ *consist of trivial homomorphisms.*

2. *Let* (μ, ζ) *be an isoclinism from* M *to* N. *Then* $\mathcal{X}_1 = (\mathrm{id}_M : M \to M)$ *is isoclinic to* $\mathcal{X}_2 = (\mathrm{id}_N : N \to N)$ *where* $(\mu_1, \mu_0) = (\mu, \mu)$ *and* $(\zeta_1, \zeta_0) = (\zeta, \zeta)$.

3. *Let* M *be a group and let* N *be a normal subgroup of* M *with* $NZ(M) = M$. *Then* $N \xhookrightarrow{inc.} M$ *is isoclinic to* $M \xrightarrow{id} M$. *Here* (μ_1, μ_0) *and* (ζ_1, ζ_0) *are defined by* $(inc., inc.)$, (id_{G_1}, id_{G_0}), *respectively.*

3. Exterior isoclinism of groups

Hakima and Jafari [14] introduced the exterior isoclinism of finite groups. We now recall the nonabelian exterior product of groups from [10].

Let M, N be normal subgroups of a group G. The nonabelian exterior product of M and N is the group $M \wedge N$ generated by the elements $m \wedge n$ with $(m, n) \in M \times N$, subject to the relations

$$\begin{array}{c} mm' \wedge n = {}^{m}(m' \wedge n)(m \wedge n), \\ m \wedge nn' = (m \wedge n)^{n}(m \wedge n'), \\ m \wedge n = 1 \text{ whenever } m = n \end{array}$$

where by definition ${}^{x}(y \wedge z) = ({}^{x}y \wedge {}^{x}z) = (xyx^{-1} \wedge xzx^{-1})$ and conjugation ${}^{x}y = xyx^{-1}$ is taken in the group G (see [10]). A more general construction of $M \wedge N$ is given in [6] for arbitrary crossed G-modules M, N. The exterior square of a group G (sometimes also called the nonabelian exterior square) is denoted by $G \wedge G$.

The exterior product $M \wedge N$ can also be defined by its universal property. Given a group H and a function $h : M \times N \to H$, we say that h is an exterior pairing if for all $m, m' \in M$, $n, n' \in N$,

$$\begin{array}{c} h(mm', n) = h({}^{m}m', {}^{m}n)h(m, n), \\ h(m, nn') = h(m, n)h({}^{n}m, {}^{n}n'), \\ h(m, n) = 1 \text{ whenever } m = n \end{array}$$

the function $M \times N \to M \wedge N$, $(m, n) \mapsto m \wedge n$ is the universal exterior pairing from $M \times N$. The exterior center of a group defined by [11] as:

$$Z^{\wedge}(G) = \{g \in G : 1 = g \wedge x \in G \wedge G \text{ for all } x \in G\}$$

Definition 3.1 *[14] Let* M *and* N *be groups; a pair* (μ, ζ) *is termed an exterior isoclinism from* M *to* N *if:*

1. μ *is an isomorphism from* $\mu : M/Z^{\wedge}(M)$ *to* $N/Z^{\wedge}(N)$;

2. $\zeta : M \wedge M \longrightarrow N \wedge N$ *is an isomorphism between exterior squares;*

3. *the diagram*

$$\begin{array}{ccc} M/Z^{\wedge}(M) \times M/Z^{\wedge}(M) & \xrightarrow{\quad h_M \quad} & M \wedge M \\ \Big\downarrow{\scriptstyle \mu \times \mu} & & \Big\downarrow{\scriptstyle \zeta} \\ N/Z^{\wedge}(N) \times N/Z^{\wedge}(N) & \xrightarrow[\quad h_N \quad]{} & N \wedge N \end{array}$$

commutative where h_M, h_N *are universal exterior pairings.*

If there is an exterior isoclinism from M to N, we shall say that M and N are exterior isoclinic groups and denoted by $M \hat{\sim} N$.

Proposition 3.2 *Exterior isoclinism is an equivalence relation.*

Examples 3.3

1. *Isomorphic groups are also exterior isoclinic.*
2. *All cyclic groups are exterior isoclinic to each other.*
3. $S_3 \hat{\sim} C_3 \ltimes C_4$

Corollary 3.4 *There is no relation between isoclinism and exterior isoclinism. For example* $D_{16} \sim Q_{16}$ *but they are not exterior isoclinic. Conversely,* $C_4 \ltimes C_4 \hat{\sim} Q_{16}$ *but they are not isoclinic (please see Table 1 and Table 2). However, if they are cyclic then every two groups are both isoclinic and exterior isoclinic to each other.*

Theorem 3.5 *[14] Let* G *be any finite group and* $H \leq G, N \trianglelefteq G$.
(i) If $G = HZ^{\wedge}(G)$, *then* $G \hat{\sim} H$. *The converse of this condition is true when* $H \cap Z^{\wedge}(G) = 1$.
(ii) $G/N \hat{\sim} G$ *if and only if* $N \leq Z^{\wedge}(G)$.

3.1. Exterior degrees of finite groups

Niroomand and Rezaeiv [20] introduced the exterior degree of a finite group G to be the probability for two elements x and y in G such that $x \wedge y = 1$ and wrote a simple algorithm to compute the exterior degree for some small groups. Let G be a finite group, the exterior degree of G is denoted by $d^{\wedge}(G)$ and defined by

$$d^{\wedge}(G) = \frac{|\{(x, y) \in G \times G : x \wedge y = 1\}|}{|G|^2}$$

Proposition 3.6 *Every exterior isoclinic finite groups has same exterior degree.*

Proof If M and N are two exterior isoclinic finite groups, then we will show that $d^{\wedge}(M) = d^{\wedge}(N)$.

Let (μ, ζ) be an exterior isoclinism from M to N; one gets

$$\begin{aligned}
|M/Z^{\wedge}(M)|^2 \, d^{\wedge}(M) &= \frac{1}{|Z^{\wedge}(M)|^2} |M|^2 \, d^{\wedge}(M) \\
&= \frac{1}{|Z^{\wedge}(M)|^2} |\{(x, y) \in M \times M : x \wedge y = 1\}| \\
&= \frac{1}{|Z^{\wedge}(M)|^2} |\{(x, y) \in M \times M : h_M(xZ^{\wedge}(M), yZ^{\wedge}(M)) = 1\}| \\
&= |\{(\alpha, \beta) \in M/Z^{\wedge}(M) \times M/Z^{\wedge}(M) : h_M(\alpha, \beta) = 1\}| \\
&= |\{(\alpha, \beta) \in M/Z^{\wedge}(M) \times M/Z^{\wedge}(M) : \zeta(h_M(\alpha, \beta)) = 1\}| \\
&= |\{(\alpha, \beta) \in M/Z^{\wedge}(M) \times M/Z^{\wedge}(M) : h_N(\mu(\alpha), \mu(\beta)) = 1\}| \\
&= |\{(\gamma, \delta) \in N/Z^{\wedge}(N) \times N/Z^{\wedge}(N) : h_N(\gamma, \delta) = 1\}| \\
&= |N/Z^{\wedge}(N)|^2 \, d^{\wedge}(N)
\end{aligned}$$

thus we have $d^{\wedge}(M) = d^{\wedge}(N)$. □

Definition 3.7 *An exterior stem group is a group whose exterior center is contained in its exterior square. In other words a group G is an exterior stem group if $M/Z^{\wedge}(M) \leq M \wedge M$.*

Every group is exterior isoclinic to an exterior stem group, but distinct exterior stem groups may be exterior isoclinic.

4. Exterior isoclinism of crossed modules

In this section, we define the concepts of the exterior center and exterior commutator subcrossed modules, which are essential for introducing the notion of exterior isoclinism for crossed modules.

Definition 4.1 *Let $\mathcal{X} = (\partial : S \to R)$ be a crossed module. Then*

$$Z^{\wedge}(\mathcal{X}) = (\partial| : S^R \to St_R(S) \cap Z^{\wedge}(R))$$

is a crossed module. $Z(\mathcal{X})$ is called exterior center of the crossed module $\mathcal{X}$. Moreover,

$$\mathcal{X} \wedge \mathcal{X} = (\partial| : D_R(S) \to R \ \wedge R)$$

is exterior derived subcrossed module of the crossed module $\mathcal{X}$.

Proposition 4.2 *Given a normal sub–crossed module $\mathcal{X}_1 = (\partial : S_1 \to R_1)$ of a crossed module $\mathcal{X} = (\partial : S \to R)$, there is a quotient crossed module*

$$\mathcal{X}/\mathcal{X}_1 \ = \ (\overline{\partial} : S/S_1 \to R/R_1)$$

where the action is defined by

$${}^{rR_1}(sS_1) \ := \ ({}^{r}s)S_1$$

and the boundary map is given by

$$\overline{\partial}(sS_1) \ := \ (\partial s)R_1.$$

Therefore

$$\mathcal{X}/Z^{\wedge}(\mathcal{X}) \ = \ (\overline{\partial|} : S/S^R \to R/(St_R(S) \cap Z^{\wedge}(R)))$$

is exterior central quotient crossed module.

Proposition 4.3 *Let $\mathcal{X} = (\partial : S \to R)$ be a crossed module. Define the maps*

$$\begin{array}{rcccl} \sigma & : & S/S^R \times R/(St_R(S) \cap Z^{\wedge}(R)) & \longrightarrow & D_R(S) \\ & & (sS^R, r(St_R(S) \cap Z^{\wedge}(R))) & \longmapsto & {}^{r}ss^{-1} \end{array}$$

and

$$\begin{array}{rcccl} \omega & : & R/(St_R(S) \cap Z^{\wedge}(R)) \times R/(St_R(S) \cap Z^{\wedge}(R)) & \longrightarrow & R \ \wedge R \\ & & (r(St_R(S) \cap Z^{\wedge}(R)), r'(St_R(S) \cap Z^{\wedge}(R))) & \longmapsto & r \wedge r', \end{array}$$

for all $s \in S$, $r, r' \in R$. Then the maps σ and ω are well-defined.

Proof It can be easily checked by a similar way of Proposition 1 in [22]. □

Definition 4.4 *Let $\mathcal{X}_1 = (\partial_1 : S_1 \to R_1)$ and $\mathcal{X}_2 = (\partial_2 : S_2 \to R_2)$ be crossed modules; two pair (μ_1, μ_0) and (ζ_1, ζ_0) are termed an exterior isoclinism of crossed modules from $\mathcal{X}_1$ to $\mathcal{X}_2$ if:*

1. *(μ_1, μ_0) is an isomorphism from $(\overline{\partial_1|} : S_1/S_1^{R_1} \to R_1/(St_{R_1}(S_1) \cap Z^{\wedge}(R_1)))$ to $(\overline{\partial_2|} : S_2/S_2^{R_2} \to R_2/(St_{R_2}(S_2) \cap Z^{\wedge}(R_2)))$;*
2. *(ζ_1, ζ_0) is an isomorphism from $(\partial_1| : D_{R_1}(S_1) \to R_1 \wedge R_1)$ to $(\partial_2| : D_{R_2}(S_2) \to R_2 \wedge R_2)$;*
3. *the diagrams*

$$\begin{array}{ccc} S_1/S_1^{R_1} \times R_1/(St_{R_1}(S_1) \cap Z^{\wedge}(R_1)) & \xrightarrow{\sigma_1} & D_{R_1}(S_1) \\ \Big\downarrow{\scriptstyle \mu_1 \times \mu_0} & & \Big\downarrow{\scriptstyle \zeta_1} \\ S_2/S_2^{R_2} \times R_2/(St_{R_2}(S_2) \cap Z^{\wedge}(R_2)) & \xrightarrow[\sigma_2]{} & D_{R_2}(S_2) \end{array} \tag{4.1}$$

and

$$\begin{array}{ccc} R_1/(St_{R_1}(S_1) \cap Z^{\wedge}(R_1)) \times R_1/(St_{R_1}(S_1) \cap Z^{\wedge}(R_1)) & \xrightarrow{\omega_1} & R_1 \wedge R_1 \\ \Big\downarrow{\scriptstyle \mu_0 \times \mu_0} & & \Big\downarrow{\scriptstyle \zeta_0} \\ R_2/(St_{R_2}(S_2) \cap Z^{\wedge}(R_2)) \times R_2/(St_{R_2}(S_2) \cap Z^{\wedge}(R_2)) & \xrightarrow[\omega_2]{} & R_2 \wedge R_2 \end{array} \tag{4.2}$$

are commutative.

If there is an exterior isoclinism of crossed modules from $\mathcal{X}_1$ to $\mathcal{X}_2$, we shall say that $\mathcal{X}_1$ and $\mathcal{X}_2$ are exterior isoclinic crossed modules.

Example 4.5 *Let (μ, ζ) be an exterior isoclinism from M to N. Then $\mathcal{X}_1 = (\mathrm{id}_M : M \to M)$ is exterior isoclinic to $\mathcal{X}_2 = (\mathrm{id}_N : N \to N)$ where $(\mu_1, \mu_0) = (\mu, \mu)$ and $(\zeta_1, \zeta_0) = (\zeta, \zeta)$.*

Remark 4.6 *If the crossed modules $\mathcal{X}_1$ and $\mathcal{X}_2$ are simply connected or finite, then the commutativity of diagrams (1) with (2) in Definition 4.4 are equivalent to the commutativity of following diagram.*

$$\begin{array}{ccc} \mathcal{X}_1/(Z^{\wedge}(\mathcal{X}_1)) \times \mathcal{X}_1/(Z^{\wedge}(\mathcal{X}_1)) & \longrightarrow & \mathcal{X}_1 \wedge \mathcal{X}_1 \\ \Big\downarrow{\scriptstyle (\mu_1,\mu_0)\times(\mu_1,\mu_0)} & & \Big\downarrow{\scriptstyle (\zeta_1,\zeta_0)} \\ \mathcal{X}_2/(Z^{\wedge}(\mathcal{X}_2)) \times \mathcal{X}_2/(Z^{\wedge}(\mathcal{X}_2)) & \longrightarrow & \mathcal{X}_2 \wedge \mathcal{X}_2 \end{array}$$

A crossed module $\mathcal{X}$ is a exterior stem crossed module if $Z^{\wedge}(\mathcal{X}) \leq \mathcal{X} \wedge \mathcal{X}$. Every crossed module is exterior isoclinic to an exterior stem crossed module, but distinct exterior stem crossed modules may be exterior isoclinic.

5. Computer implementation

GAP* (Groups, Algorithms, Programming [25]) is the leading symbolic computation system for addressing computational discrete algebra problems. Symbolic computation has driven several key advances in mathematics and computer science, particularly in number theory and coding theory (see [4]). The Small Groups library was instrumental in the landmark "Millennium Project," where Besche, Eick, and O'Brien classified all finite groups of order up to 2000 [9]. The HAP [12] package provides functions for calculating the nonabelian exterior product, exterior square, and exterior center. Additionally, the XMod package offers numerous functions for isoclinism classes of groups and crossed modules, as well as various family invariants.

In this paper, we have developed new functions for GAP to construct exterior isoclinism of groups and exterior stem groups. The function **AreExteriorIsoclinicGroups** is used to check whether or not two groups are exterior isoclinic, while the function **ExteriorIsoclinicGroup** returns a group that is exterior isoclinic to a given group. Two groups of different orders can be exterior isoclinic. Therefore, the function **ExteriorIsoclinicFamily** can be called in two ways: as **ExteriorIsoclinicFamily(**G**)** to construct a list of the positions in the given order of groups that are exterior isoclinic to G; or as **ExteriorIsoclinicFamily(**G**,** *list***)** to construct a list of the positions in a given *list* partitioned according to the exterior isoclinism of G.

In the following GAP session, we compute all isoclinism families and some exterior isoclinism families of groups of order 16 using these functions, demonstrating their application within the scope of this paper.

```
gap> D16 := SmallGroup(16,7);; StructureDescription(D16);
"D16"
gap> Q16 := SmallGroup(16,9);; StructureDescription(Q16);
"Q16"
gap> AreIsoclinicGroups(D16,Q16);
true
gap> AreExteriorIsoclinicGroups(D16,Q16);
false
gap> C4lxC4 := SmallGroup(16,4);; StructureDescription(C4lxC4);
"C4 : C4"
gap> AreExteriorIsoclinicGroups(Q16,C4lxC4);
true
gap> AreIsoclinicGroups(Q16,C4lxC4);
false
gap> IsoclinicFamily(SmallGroup(16,1));
[ 1, 2, 5, 10, 14 ]
gap> IsoclinicFamily(SmallGroup(16,3));
[ 3, 4, 6, 11, 12, 13 ]
gap> IsoclinicFamily(SmallGroup(16,7));
[ 7, 8, 9 ]
gap> ExteriorIsoclinicFamily(SmallGroup(16,4));
[ 4, 8, 9 ]
gap> ExteriorIsoclinicFamily(SmallGroup(16,5));
[ 5, 6 ]
gap> ExteriorIsoclinicFamily(SmallGroup(16,10));
[ 10, 12, 13 ]
```

On the other hand, we have implemented the functions **CommutativeDegreeOfGroup** and **ExteriorDegreeOfGroup**, which are used to compute the commutative degree and exterior degree of a group.

In the following GAP session, we obtain the commutative degrees and exterior degrees of several groups.

*https://www.gap-system.org.

```
gap> CommutativeDegreeOfGroup(AlternatingGroup(4));
1/3
gap> ExteriorDegreeOfGroup(AlternatingGroup(4));
7/24
gap> CommutativeDegreeOfGroup(QuaternionGroup(40));
13/40
gap> ExteriorDegreeOfGroup(QuaternionGroup(40));
13/40
```

For groups of order 16, there are 14 isomorphism classes and 3 isoclinism families represented by C_{16}, $(C_4 \times C_2) \ltimes C_2$, and D_{16}. The commutative degree of groups within the same isoclinism family is equal, as expected. See the table below for details:

Table 1
Isoclinism Families of Order 16.

No	Number	Representator	Members	$d(G)$
1	5	[16,1]	$C_{16}, C_4 \times C_4, C_8 \times C_2, C_4 \times C_2^2, C_2^4$	1
2	6	[16,3]	$(C_4 \times C_2) \ltimes C_2, C_4 \ltimes C_4, C_8 \ltimes C_2,$ $C_2 \times D_8, C_2 \times Q_8, (C_4 \times C_2) \rtimes C_2$	5/8
3	3	[16,7]	D_{16}, QD_{16}, Q_{16}	7/16

Moreover, there are 9 exterior isoclinism families for groups of order 16. The exterior degree of groups within the same exterior isoclinism family is equal, as expected. See the table below for details:

Table 2
Exterior Isoclinism Families of Order 16.

No	Number	Representator	Members	$d^{\wedge}(G)$
1	1	[16,1]	C_{16}	1
2	1	[16,2]	$C_4 \times C_4$	11/32
3	1	[16,3]	$(C_4 \times C_2) \ltimes C_2$	19/64
4	3	[16,4]	$C_4 \ltimes C_4, QD_{16}, Q_{16}$	7/16
5	2	[16,5]	$C_8 \times C_2, C_8 \ltimes C_2$	5/8
6	1	[16,7]	D_{16}	11/32
7	3	[16,10]	$C_4 \times C_2^2, C_2 \times Q_8, (C_4 \times C_2) \ltimes C_2$	11/32
8	1	[16,11]	$C_2 \times D_8$	1/4
9	1	[16,14]	C_2^4	23/128

The function **IsExteriorStemGroup** is used to verify whether the condition in Definition 3.7 is satisfied. The function **ExteriorIsoclinicExteriorStemGroup** returns an exterior stem group that is exterior isoclinic to a given group. The function **AllExteriorStemGroupIds** returns the IdGroup list of exterior stem groups of a specified size.

The following GAP session illustrates the use of these functions:

```
gap> c3c3 := Group([ (1,2,3), (4,5,6) ]);;
gap> IsStemGroup(c3c3);
false
gap> IsExteriorStemGroup(c3c3);
true
gap> IsExteriorStemGroup(c4c2);
false
gap> AllExteriorStemGroupIds(16);
[ [ 16, 2 ], [ 16, 3 ], [ 16, 7 ], [ 16, 11 ], [ 16, 14 ] ]
gap> StructureDescription(ExteriorIsoclinicExteriorStemGroup(c4c2));
"C2 x C2"
```

We have also developed new functions for GAP to construct exterior isoclinism of crossed modules and exterior stem crossed modules. We have introduced the functions **ExteriorCentreXMod** and **ExteriorDerivedSubXMod** based on the definitions of the exterior center of a crossed module and exterior derived subcrossed module given in Definition 4.1.

The function **AreExteriorIsoclinicXMods** is used to check whether or not two crossed modules are exterior isoclinic, while **ExteriorIsoclinicXModFamily** constructs a list of positions in a given *list* partitioned according to the exterior isoclinism of the specified crossed module.

The following GAP session demonstrates that two crossed modules of different orders can be exterior isoclinic:

```
gap> all4 := AllXMods(4,4);;
gap> iso_all4 := AllXModsUpToIsomorphism(all4);;
gap> iso_all18 := AllXModsUpToIsomorphism(AllXMods([18,18]));;
gap> Size(iso_all18[49]); Size(iso_all4[3]);
[ 18, 18 ]
[ 4, 4 ]
gap> AreExteriorIsoclinicXMods(iso_all18[49],iso_all4[3]);
true
```

The function **IsExteriorStemXMod** is used to determine whether a given crossed module is a stem crossed module or not.

```
gap> iso_all82 := AllXModsUpToIsomorphism(AllXMods([8,2]));
gap> XM := PermByObject(iso_all82[3]);
[Group( [ (1,4,7,2,5,8,3,6), (1,7,5,3)(2,8,6,4), (1,5)(2,6)(3,7)(4,8) ] )
->Group( [ (1,2) ] )]
gap> IsXMod(XM);
true
gap> DXM := ExteriorDerivedSubXMod(XM);
[Group( [ (1,3,5,7)(2,4,6,8), (1,5)(2,6)(3,7)(4,8) ] )->Group( () )]
gap> CXM := ExteriorCentreXMod(XM);
[Group( [ (1,5)(2,6)(3,7)(4,8) ] )->Group( () )]
gap> IsSubXMod(DXM,CXM);
true
gap> IsExteriorStemXMod(XM);
true
```

The following GAP session demonstrates that there are 60 crossed modules of order $[4, 4]$, which correspond to 18 isomorphism classes, 2 isoclinism families, and 5 exterior isoclinism families.

```
gap> Length(all4);
60
gap> Length(iso_all4);
18
gap> IsoclinicXModFamily(iso_all4[1],iso_all4);
[ 1, 3, 4, 6, 8, 10, 12, 14, 16, 18 ]
gap> IsoclinicXModFamily(iso_all4[2],iso_all4);
[ 2, 5, 7, 9, 11, 13, 15, 17 ]
gap> ExteriorIsoclinicXModFamily(iso_all4[1],iso_all4);
[ 1, 3, 4, 10, 12 ]
gap> ExteriorIsoclinicXModFamily(iso_all4[2],iso_all4);
[ 2, 5, 11, 13 ]
gap> ExteriorIsoclinicXModFamily(iso_all4[6],iso_all4);
[ 6, 8, 14, 16, 18 ]
gap> ExteriorIsoclinicXModFamily(iso_all4[7],iso_all4);
[ 7, 15 ]
gap> ExteriorIsoclinicXModFamily(iso_all4[9],iso_all4);
[ 9, 17 ]
```

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