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The compositions derived from by changing variable in singular distributions

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Abstract: In this paper we concern further results of [32, 33] by using the method mainly related to the finite part of divergent integral, in fact, we consider the powers for positive integers of the composition of the Dirac delta function and an infinitely differentiable function having any number of distinct multiple roots which will be defined as the neutrix limit of the regular sequence $\delta_n^k(f(x))$. Moreover we show that the powers of the composition $\delta(f(x))$ for negative integers can be also defined via neutrix settings. Some compositions as examples to better understand will be given.

Key words: Divergent integral, regular sequence, Hadamard's finite part, Dirac-delta function, singular distribution, neutrix limit, Fisher's method, Faá di Bruno's formulae, cosmology

1. Introduction

Many scientists have long been using singular functions, even though these cannot be properly defined within the framework of classical function theory. The simplest of the singular function is the Dirac delta function which had been used in physics before the formal work of L. Schwartz. It appears with its derivative and even some powers, in many specific problems of mathematical physics and engineering, especially, of quantum field theory, quantum mechanics, quantum electrodynamics and signal processing. So it is of central importance in many branches of physics and engineering. For instance, the Dirac delta function was used to represent matter in the field equations by A. Einstein. Besides the symbol $\delta^2(x)$ often appears in quantum mechanics which leads to reasonable results and one needs to evaluate its square when calculating the transition rates of certain particle interactions.

Now, let us get started by bringing to the reader's mind some results on the compositions of the Dirac delta function with summable functions. First Koh and Li [26] used the fixed δ-sequence and gave meaning to the symbol δ^k. Later Accardi and Boukas [1] gave meaning to the expression $\delta^n = \sum_{k=1}^{n-1} c_k \delta^{(k)}$ $(n \geq 2)$ in order to establish a fock representaition of the renormalized higher powers of white noise. Borys et al. defined the k-th power of Dirac delta impulse in terms of Neutrix notation while the linear system was extended to nonlinear Volterra system described by Volterra and Taylor series [4]. Lately Chenkuan Li and Changpin Li [30] used the Caputo fractional derivatives to redefine $\delta^k(x)$ for all $k \in \mathbb{R}$. The symbol $\delta^k(f(x))$ for the infinitely differentiable function f having single simple root or multiple root at x_0 is meaningfully defined in [34] and has been shown that $\delta^k(f(x)) = 0$ for even k. Further the author proved in [33] that if f has any number of simple

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root, then the composition $\delta^k(f(x))$ exists on the interval (a,b) and also equals to zero for even k. Recently the composition of Dirac delta function and an infinitely differentiable function f having distinct multiple roots is meaningfully defined in [32].

Now we assume that $f(x)$ is an infinitely differentiable real valued function having multiple root at $x_0 \in \mathbb{R}$ with multiplicity n. Then $f(x)$ can be factorized as

$$f(x) = (x-x_0)^n h(x)$$

where $h(x)$ is continuous and infinitely differentiable on the real line defined by

$$h(x) = \begin{cases} \frac{f(x)}{(x-x_0)^n}, & x \neq x_0, \\ \frac{f^{(n)}(x_0)}{n!}, & x = x_0, \end{cases} \tag{1.1}$$

and the sth derivative of t at x_0 is equal to $h^{(s)}(x_0) = \frac{f^{(s+n)}(x_0)}{(s+n)!} \quad (s \in \mathbb{Z}^+)$, and clearly it is irreducible [32]. Further similarly if the function $f(x)$ has distinct multiple roots at the points $x_1, x_2, \ldots, x_n$ of $\mathbb{R}$ with the multiplicities $r_1, r_2, \ldots, r_n$ respectively, then we can this time factorize $f(x)$ as

$$f(x) = (x-x_1)^{r_1}(x-x_2)^{r_2}\ldots(x-x_n)^{r_n} t(x)$$

where $r_i \in \mathbb{Z}^+$ and $t(x)$ is infinitely differentiable function on the real line defined by the equation [32]

$$t(x) = \begin{cases} \frac{f(x)}{(x-x_1)^{r_1}(x-x_2)^{r_2}\ldots(x-x_n)^{r_n}}\ , & x \neq x_i,\ i=1,2,\ldots,n \\ \frac{f^{(r_i)}(x_i)}{r_i!\prod_{i\neq k}(x_i-x_k)^{r_k}}\ , & x = x_i,\ i=1,2,\ldots,n. \end{cases} \tag{1.2}$$

As to get the main results we will need the Fa$\acute{a}$ di Bruno's formulae which defines an equation of the kth derivative of a smooth composite function $\varphi \circ f$ stated by Bruno [7] as

$$\Big[\varphi(f(x))\Big]^{(k)} = \sum_{r=1}^{k} \varphi^{(r)}(f) B_{k,r}(f'(x), f''(x), \ldots, f^{(k-r+1)}(x)) \tag{1.3}$$

where $B_{k,r}(f'(x), f''(x), \ldots, f^{(k-r+1)}(x))$ are the Bell polynomials [5] defined by the equation

$$\begin{aligned} B_{k,r}&\Big(f'(x), f''(x), \ldots, f^{(k-r+1)}(x)\Big) = \\ &= \sum \frac{k!}{b_1!b_2!\ldots b_{k-r+1}!}\Big(\frac{f'(x)}{1!}\Big)^{b_1}\Big(\frac{f''(x)}{2!}\Big)^{b_2}\ldots\Big(\frac{f^{(k-r+1)}(x)}{(k-r+1)!}\Big)^{b_{k-r+1}} \end{aligned} \tag{1.4}$$

where the sum is over all possible combinations of nonnegative integers $b_1, b_2, \ldots, b_{k-r+1}$ satisfying two conditions $b_1 + b_2 + \ldots + b_{k-r+1} = r$ and $b_1 + 2b_2 + 3b - 3 \ldots + (k-r+1)b_{k-r+1} = k$.

Now $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$. Let $f(x)$ be locally summable function that means absolutely integrable in every bounded region R_n, then we can associate every $\varphi \in \mathcal{D}$ with

$$\langle f, \varphi(x)\rangle = \int_{R_n} f(x)\varphi(x)\,dx \tag{1.5}$$

where the integral is actually taken over the bounded region in which $\varphi(x)$ fails to vanish. Equation 1.5 represents a very special kind of continuous linear functional on $\mathcal{D}$ called regular, all other not giving by equation 1.5, (including the delta function) will be called singular. One important property of the space $\mathcal{D}'$ is that every distribution is the limit of a regular sequence, will be defined as below, of infinitely differentiable functions with compact support [19].

To construct such a sequence of regular functions which converges to $\delta(x)$ we assume that ρ is a fixed infinitely differentiable function having the properties:

$$\text{(i)}\quad \rho(x)=0 \text{ for } |x|\geq 1\,,\quad \text{(ii)}\quad \rho(x)\geq 0,\quad \text{(iii)}\quad \rho(x)=\rho(-x)\,,\quad \text{(iv)}\quad \int_{-1}^{1}\rho(x)\,dx=1.$$

Then putting $\delta_n(x)=n\rho(nx)$ for $n=1,2,\ldots,$ we have that $\delta_n(x)$ is a regular sequence [38] of infinitely differentiable functions converging to Dirac delta-function $\delta(x)$ and this sequence is called δ-sequence.

It should be emphasized that the distributions play crucial role in the theory of Partial differential and integral equations theory [28].The reader may refer to [8] as to see a description of distributional point values via delta sequence.

Further if f is a distribution in $\mathcal{D}'$ and $f_n(x)=\langle f(t),\delta_n(x-t)\rangle$, then $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions and converges to $f(x)$.

If $f(x)$ is an infinitely differentiable function having simple roots at the points $x_1,x_2,\ldots,$ the composition $\delta(f(x))$ is defined by Gel'fand and Shilov [19] as

$$\delta(f(x))=\sum_n\frac{\delta(x-x_n)}{|f'(x_n)|}.$$

Differentiating this equation, we obtain the distribution $\delta^{(k)}(f(x))$ as

$$\delta^{(k)}(f(x))=\sum_n\frac{1}{|f'(x_n)|}\left(\frac{1}{|f'(x_n)|}\frac{d}{dx}\right)^k\delta(x-x_n).$$

Antosik [2] defines the composition $g(f(x))$ of distributions f and g as the limit of the sequence of composition $\{g_n(f_n)\}$ on $\mathbb{R}$ proving that the limit exists and converges to a distribution $h(x)$. By this definition he obtained

$$(i)\sqrt{\delta}=0,\quad (ii)\sqrt{\delta^2+1}=1+\delta,\quad (iii)\ log(1+\delta)=0,\quad (iv)\sin\delta=0,\quad (v)\tfrac{1}{1+\delta}=1.$$

In addition Antosik introduced the interesting formulas in [2], which may be a matter of interest for physicits,

$$\sqrt[n]{\sum_{i=0}^{n}a_i\delta^i}=\sqrt[n]{a}_n+\delta,\qquad \log\Big(\sum_{i=0}^{n}a_i\delta^i\Big)=0,$$

Where $a_i>0$ for $i=0,\ldots,n$. Notice that the powers of δ^i of δ should be meant as results of the operation of substitution of δ into a certain continuous function, not the operation of product of distributions. In addition Antosik assumed in [3] that $g(x)$ is continuous function from $\mathbb{R}$ in to $\mathbb{R}$ such that $\lim_{x\to-\infty}(g(x)/x)=a$, $\lim_{x\to\infty}(g(x)/x)=b$ and f is a measure with Lebesgue decomposition, then he proved that the composition $g(f)$ existed.

2. The expression $\delta^k(f(x))$ for a function having multiple roots

The method of the discarding of unwanted infinite quantities from divergent integrals resulting finite value is known Hadamard's finite part [8, 9, 39–41], related ideas have been explored by van der Corput [6] who similarly noticed that functions of certain type can be neglected in the study of the asymptotic behaviour integrals, while Fisher has approched the subject via the theory of neutrices [13, 15–18], Raju [37] has tackled the problem by nonstandard analysis. Some of the recent results on the finite part of divergent integrals can be found in [10, 11, 22, 27, 31, 40].

In general, by Antosik's definition above, it is not possible to define the composition for many pairs of distributions. Nevertheless, Fisher [15] gave the following definition enabling the composition defined for larger class of distributions, but before we let N denotes the neutrix having domain $N' = \{1, 2, \ldots, n, \ldots\}$, range the real numbers with negligible functions which are finite linear sums of the functions $n^\lambda \ln^{r-1} n, \ln^r n$ $(\lambda > 0, r = 1, 2, \ldots)$ and all functions which converge to zero in the usual sense as n tends to infinity, [6, 13].

Definition 2.1 *Let F be distributions in $\mathcal{D}'$ and let f be infinitely differentiable function. We say that the distribution $F(f(x))$, the neutrix composition of F and f, exists and is equal to $h(x)$ on the interval (a, b) if the neutrix limit*

$$\mathrm{N}\!-\!\lim_{n\to\infty} \left[\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\, dx\right] = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}$ with support contained in the interval (a, b), where $F_n(x) = (F * \delta_n)(x)$, and N is defined as above [6, 15, 16].

The reader can find some examples of the neutrix limit and some examples of compositions and some applications of the neutrix limit in conjunction with special functions in [15, 17, 23, 36] respectively. It should be pointed out that the essential use of the neutrix limit is to extract an appropriate finite part from a divergent quantity as one has usually done to subtract the divergent terms via rather complicated procedures in the renormalization theory [9, 10, 27].

In order to obtain more results on compositions of distributions we first recall the following proposition given in [34].

Proposition 2.2 *Let $f(x)$ be an infinitely differentiable function on the interval (a, b). Assume that $f(x)$ does not have any root in the interval (a, b). Then the distribution $\delta^k(f(x))$ exists on the interval (a, b) and*

$$\delta^k(f(x)) = 0. \tag{2.1}$$

for all $k \in \mathbb{Z}^+$.

Theorem 2.3 *Let $f(x)$ be an infinitely differentiable function having multiple root at x_0 with multiplicity s on the open interval (a,b). Then the k-th power of the composition $\delta(f(x))$ of Dirac delta function and f exists in the sense of Fisher's definition and for $\varphi \in \mathcal{D}$*

$$\langle \delta^k(f(x)), \varphi(x)\rangle = \sum_{r=0}^{ks-1} \binom{ks-1}{r} \frac{2c_{k,\rho}}{s(ks-1)!|f^{(ks-r)}(x_0)|} \times$$

$$\times \sum_{i=1}^{r} (-1)^i B_{r,i}\Big(\frac{1}{|f'(x_0)|}, \frac{1}{|f''(x_0)|}, \dots, \frac{1}{|f^{(r-i+1)}(x_0)|}\Big)\langle \delta^{(i)}(x-x_0), \varphi(x)\rangle \tag{2.2}$$

for $k,s = 1,3,\dots,$ and otherwise $\delta^k(f(x)) = 0$, where Faá di Bruno's formulae is used in terms of exponential Bell polynomials $B_{r,i}(x_1,x_2,\dots,x_{r-i+1})$ defined as in equation 1.4 and $c_{k,\rho} = \int_0^1 u^{k-1}\rho^k(u)\,du$. In particular for $k = s = 1$, $\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$.

Proof For the sake of completeness, we prove the theorem for the case $x_0 = 0$, the case $x_0 \neq 0$ will then follow by translation. It now follows that we may write $f(x) = x^s h(x)$ where $h(x)$ is defined as in equation 1.1.

Let us write $f_1(x) = xh^{1/s}(x)$ and we assume that the interval (a,b) containing origin is bounded and $f_1'(x) \neq 0$ on this interval.Then the equation $y = f_1(x)$ will have inverse $x = g(y) \in C^\infty$ on the interval (a,b). Now let $\varphi \in \mathcal{D}$ with $\sup(\varphi) \subset (a,b)$. Then

$$\int_{-\infty}^{\infty} \delta_n^k(f(x))\varphi(x)\,dx = \int_0^{\infty} \delta_n^k(f(x))\varphi(x)\,dx + \int_0^{\infty} \delta_n^k(f(-x))\varphi(-x)\,dx. \tag{2.3}$$

On making the substitution $t^{1/s} = f_1(x)$ or $x = g(t^{1/s})$ we have

$$\int_0^{\infty} \delta_n^k(f(x))\varphi(x)\,dx = \frac{1}{s}\int_0^{\infty} \delta_n^k(t)\varphi(g(t^{1/s})|g'(t^{1/s})|t^{1/s-1}\,dt.$$

The function $\Psi(y) = \varphi(g(y))|g'(y)|$ is infinitely differentiable function it follows from Taylor's formulae that

$$\Psi(y) = \sum_{i=0}^{ks-1} \frac{\Psi^{(i)}(0)}{i!} y^i + \frac{\Psi^{(ks)}(\xi y)}{(ks)!} y^{ks} \quad (0<\xi<1).$$

Thus we have on making the substitution $nt = u$

$$s\int_0^{\infty} \delta_n^k(f(x))\varphi(x)\,dx = \sum_{i=0}^{ks-1} \frac{\Psi^{(i)}(0)}{i!}\int_0^{\infty} \delta_n^k(t)t^{\frac{i+1}{s}-1}\,dt + \int_0^{\infty} \frac{\Psi^{(ks)}(\xi t^{1/s}))}{(ks)!}\delta_n(t)t^{k-1+1/s}\,dt$$

$$= \sum_{i=0}^{ks-2} \frac{\Psi^{(i)}(0)}{i!} n^{k-1}\int_0^1 \rho^k(u)(\frac{u}{n})^{\frac{i+1}{s}-1}\,du + \frac{\Psi^{(ks-1)}(0)}{(ks-1)!}\int_0^1 u^{k-1}\rho^k(u)\,du +$$

$$+n^k \int_0^1 \frac{\Psi^{(ks)}(\xi(u/n)^{1/s})}{(ks)!}\rho^k(u)(u/n)^{ks}\,du$$

and it now follows that the neutrix limit of $\int_0^\infty \delta_n^k(f(x))\varphi(x)\,dx$ exists and is equal to

$$\begin{aligned} \underset{n\to\infty}{\text{N}-\lim} \int_0^\infty \delta_n^k(f(x))\varphi(x)\,dx &= \frac{\Psi^{(ks-1)}(0)}{s(ks-1)!}\int_0^1 u^{k-1}\rho^k\,du \\ &= c_{k,\rho}\frac{\Psi^{(ks-1)}(0)}{s(ks-1)!}. \end{aligned} \tag{2.4}$$

Next consider the integral $\int_0^\infty \delta_n^k(f(-x))\varphi(-x)\,dx$. Similarly we have on making the substitution $-t^{1/s} = f_1(-x)$, where $t^{1/s} \geq 0$, that

$$\begin{aligned} s\int_0^\infty \delta_n^k(f(-x))\varphi(-x)\,dx &= \int_0^\infty \delta_n^k((-1)^s t)\varphi(g(-t^{1/s}))|g'(-t^{1/s})|t^{1/s-1}\,dt \\ &= \int_0^\infty \delta_n^k(t)\Psi(-t^{1/s})t^{1/s-1}\,dt \end{aligned}$$

where Ψ is the function defined above. Thus as similar to equation 2.3, we get by taking the neutrix limit that

$$\begin{aligned} \underset{n\to\infty}{\text{N}-\lim} \int_0^\infty \delta_n^k(f(-x))\varphi(-x)\,dx &= \frac{(-1)^{ks-1}\Psi^{(ks-1)}(0)}{s(ks-1)!}\int_0^1 u^{k-1}\rho(u)\,du \\ &= \frac{(-1)^{ks-1}c_{k,\rho}\Psi^{(ks-1)}(0)}{s(ks-1)!}. \end{aligned} \tag{2.5}$$

It now follows from equations 2.3 - 2.5 that

$$\underset{n\to\infty}{\text{N}-\lim} \int_{-\infty}^\infty \delta_n^k(f(x))\varphi(x)\,dx = \begin{cases} \frac{2c_{k,\rho}\Psi^{(ks-1)}(0)}{s(ks-1)!}, & k,s = 1,3,\dots, \\ 0, & \text{otherwise} \end{cases}$$

proving the existence of $\delta^k(f(x))$ on the interval (a,b) for $s \in \mathbb{N}$. As to evaluate $\Psi^{(ks-1)}(0)$ we use Fa$\dot{a}$ di Brono's formulae in terms of exponential Bell polynomials. It now follows that

$$\begin{aligned} \Psi^{(ks-1)}(0) &= \left\{\Psi^{(ks-1)}(y)\right\}_{y=0} = \{\varphi(g(y))|g'(y)|\}^{(ks-1)}\Big]_{y=0} \\ &= \sum_{r=0}^{ks-1}\binom{ks-1}{r}\{\varphi(g(y))\}^{(r)}\,|g^{(ks-r)}(y)|\Big]_{y=0} \\ &= \sum_{r=0}^{ks-1}\binom{ks-1}{r}\sum_{i=1}^{r}\varphi^{(i)}(x_0)B_{r,i}\Big(g',g'',\dots,g^{(r-i+1)}\Big)|g^{(ks-r)}(y)|\Big]_{y=0} \\ &= \sum_{r=0}^{ks-1}\binom{ks-1}{r}\frac{1}{|f^{(ks-r)}(x_0)|}\times \\ &\qquad \times\sum_{i=1}^{r}B_{r,i}\Big(\frac{1}{|f'(x_0)|},\frac{1}{|f''(x_0)|},\dots,\frac{1}{|f^{(r-i+1)}(x_0)|}\Big)\varphi^{(i)}(x_0) \end{aligned}$$

where $B_{r,i}$ is the incomplete exponential Bell polynomial. What we have proved that the composition $\delta^k(f(x))$

exists and equals to

$$\langle \delta^k(f(x)), \varphi(x)\rangle = \sum_{r=0}^{ks-1} \binom{ks-1}{r} \frac{2c_{k,\rho}}{s(ks-1)!|f^{(ks-r)}(x_0)|} \times$$

$$\times \sum_{i=1}^{r} (-1)^i B_{r,i}\Big(\frac{1}{|f'(x_0)|}, \frac{1}{|f''(x_0)|}, \dots, \frac{1}{|f^{(r-i+1)}(x_0)|}\Big)\langle \delta^{(i)}(x-x_0), \varphi(x)\rangle \tag{2.6}$$

for $k, s = 1, 3, \dots,$ and otherwise $\delta^k(f(x)) = 0$. □

The reader easily notices that if $k \notin \mathbb{Z}^+$, then the neutrix limit of the equations 2.3 and 2.4 equal to zero which results $\delta^k(f(x)) = 0$ and also notices that if $k = 1$ then the theorem 2.3 is in agreement with the theorem 2.3 of [32]. □

Example 2.4 *Let $f(x) = x^r$ and assume that k and r are odd. Then for any $\varphi \in \mathcal{D}$, the function Ψ is identical to φ. Thus*

$$\begin{aligned} \langle \delta^k(f(x)), \varphi\rangle &= \frac{2c_{k_\rho}\varphi^{(kr-1)}(0)}{r(kr-1)!} \\ &= \frac{2(-1)^{kr-1}c_{k_\rho}}{r(kr-1)!}\langle \delta^{(kr-1)}(x), \varphi(x)\rangle \end{aligned}$$

and so

$$\delta^k(x^r) = \frac{2(-1)^{kr-1}c_{k_\rho}}{r(kr-1)!}\delta^{(kr-1)}(x)$$

on the real line for odd k, r. If $k = 1$ then $\delta(x^r) = \frac{(-1)^{r-1}}{r!}\delta^{(r-1)}(x)$ which is in aggrement with the result given in [14]. If $r = 2$, then $\delta(x^2) = 0$. This is in agreement with the result given in [12].

Example 2.5 *Let us consider the function $f(x) = \tanh^3 x$. Then by the notation of the proof of Theorem 2.3, we have $f_1(x) = \tanh x$ which has simple roots at the points $x = 0$ and*

$$\begin{aligned} g(y) &= \tanh^{-1} y = \frac{1}{2}\ln\frac{1+y}{1-y} = \sum_{n=0}^{\infty}\frac{y^{2n+1}}{2n+1} \\ &= y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{1}{7}y^7, \dots, \\ g'(y) &= -\frac{1}{1-y^2} = 1 + y^2 + y^4 + y^6, \dots \end{aligned}$$

on the interval $(-1, 1)$ and so $\Psi(y) = \varphi(\tanh^{-1} y)(1-y^2)^{-1}$ and it can be shown that $\Psi''(0) = 2\varphi(0) + \varphi''(0)$.

It follows from the proof of Theorem 2.3 that

$$\begin{aligned}\langle \delta(\tanh^3 x), \varphi(x)\rangle &= \frac{1}{6}\Psi''(0) = \frac{1}{3}\varphi(0) + \frac{1}{6}\varphi''(0) \\ &= = \frac{1}{3}\langle \delta(x), \varphi(x)\rangle + \frac{1}{6}\langle \delta''(x), \varphi(x)\rangle. \quad \square\end{aligned}$$

Example 2.6 *Let us define the composition* $\delta[\ln(1+|x|)]$. *So we evaluate the neutrix limit of* $\langle \delta_n[\ln(1+|x|)], \varphi(x)\rangle$ *for* $\varphi \in D[-1,1]$. *We may write from Taylor's theorem that* $\varphi(x) = \varphi(0) + x\varphi(\xi x)$ $(0 < \xi < 1)$. *Then*

$$\langle \delta_n[\ln(1+|x|)], \varphi(x)\rangle = \varphi(0)\int_{-1}^{1} \delta_n[\ln(1+|x|)]\,dx + \int_{-1}^{1} \delta_n[\ln(1+|x|)]x\varphi(\xi x)\,dx.$$

For large enough n

$$\int_{-1}^{1} \delta_n[\ln(1+|x|)]\,dx = n\int_{-1}^{1} \rho[\ln(1+|x|)]\,dx = 2n\int_{0}^{1} \rho[\ln(1+|x|)]\,dx$$

and making the substitution $t = n\ln(1+|x|)$ *we have*

$$n\int_{0}^{1} \rho[\ln(1+|x|)]\,dx = \int_{0}^{1} \rho(t)e^{t/n}\,dt = \sum_{k=0}^{\infty}\frac{1}{n^k k!}\int_{0}^{1}\rho(t)t^k\,dt = o(1/n) + \int_{0}^{1}\rho(t)\,dt.$$

It now follows that $\lim_{n\to\infty}\int_{-1}^{1} \delta_n[\ln(1+|x|)]\,dx = 1$, *where* $\int_0^1 \rho(t)\,dt = 1/2$. *Finally since* $x\delta_n[\ln(1+|x|)]$ *is odd function then* $\int_{-1}^{1} \delta_n[\ln(1+|x|)]x\,dx = 0$. *Thus if* ψ *is continuous function then*

$$\lim_{n\to\infty}\int_{-1}^{1} \delta_n[\ln(1+|x|)]\psi(x)\,dx = 0.$$

So what we have proved that for $\varphi \in D[-1,1]$

$$\begin{aligned}\lim_{n\to\infty}\langle \delta_n[\ln(1+|x|)], \varphi(x)\rangle &= \lim_{n\to\infty}\int_{-1}^{1} \delta_n[\ln(1+|x|)]\varphi(x)\,dx = \varphi(0) \\ \langle \delta[\ln(1+|x|)], \varphi(x)\rangle &= \langle \delta(x), \varphi(x)\rangle. \quad \square\end{aligned}$$

The other main contribution of this paper is the following result.

Theorem 2.7 *Let* $f(x)$ *be an infinitely differentiable function having distinct multiple roots at* $x_1, x_2, \ldots, x_n$ *with multiplicities* $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, $(n, \alpha_i \in \mathbb{Z}^+)$ *respectively on the open interval* (a,b). *Then the* k*-th power of the composition* $\delta(f(x))$ *exists for positive integers on the interval* (a,b) *and*

$$\langle \delta^k(f(x)), \varphi(x)\rangle = \sum_{i=1}^{n}\sum_{r=0}^{k\alpha_i-1}\binom{k\alpha_i-1}{r}\frac{2c_{k,\rho}}{\alpha_i(k\alpha_i-1)!|f^{(k\alpha_i-r)}(x_i)|}\times$$

$$\times\sum_{m=1}^{r}(-1)^m B_{r,m}\Big(\frac{1}{|f'(x_i)|}, \frac{1}{|f''(x_i)|}, \ldots, \frac{1}{|f^{(r-m+1)}(x_i)|}\Big)\langle \delta^{(m)}(x-x_i), \varphi(x)\rangle \tag{2.7}$$

for all $k\alpha_i - 1 \in 2\mathbb{Z},\ i = 1, 2, \ldots, n,$ *where* $B_{r,m}(x_1, x_2, \ldots, x_{r-m+1})$ *are again the exponential Bell polynomials. In particular for* $k = \alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n = 1,$ *then*

$$\delta(f(x)) = \sum_n \frac{1}{|f'(x_n)|}\delta(x - x_n).$$

Further if $k\alpha_i \in 2\mathbb{Z}$ *for all* $i = 1, 2, \ldots, n$ *then* $\delta^k(f(x)) = 0.$

Proof We may write $f(x) = (x - x_1)^{\alpha_1}(x - x_2)^{\alpha_2}(x - x_3)^{\alpha_3} \ldots (x - x_n)^{\alpha_n} t(x)$ where $t(x) \in C^\infty$ defined as in equation 1.2.

Let (λ_i, μ_i) be disjoint open subintervals of (a, b) containing x_i such that $\mathcal{A} = \cup_{i=1}^n (\lambda_i, \mu_i)$ for $i = 1, 2, \ldots, n$. Let us write $f_i(x) = \{(x - x_1)^{\alpha_1}(x - x_2)^{\alpha_2}(x - x_3)^{\alpha_3} \ldots (x - x_n)^{\alpha_n} t(x)\}^{1/\alpha_i}$ and assume that the interval (λ_i, μ_i) is bounded and since x_i is simple root we have $f_i'(x) \neq 0$ on (λ_i, μ_i) and also assume that $f_i(x)$ is increasing. Then the equation $y = f_i(x)$ will have inverse $x = g_i(y) \in C^\infty$ on the interval (λ_i, μ_i). Now let $\varphi(x) \in \mathcal{D}$ with $\operatorname{supp}(\varphi) \subset (a, b)$, then we have

$$\begin{aligned}\int_{-\infty}^{\infty} \delta_n^k(f(x))\varphi(x)\,dx &= \int_{\lambda_1}^{\mu_1} \delta_n^k(f(x))\varphi(x)\,dx + \int_{\lambda_2}^{\mu_2} \delta_n^k(f(x))\varphi(x)\,dx + \\ &+ \ldots + \int_{\lambda_n}^{\mu_n} \delta_n^k(f(x))\varphi(x)\,dx + \int_{\mathbb{R}\setminus\mathcal{A}} \delta_n^k(f(x))\varphi(x)\,dx. \end{aligned} \tag{2.8}$$

Now it follows from proposition 2.2 that the last integral on the right hand side of equation 2.8 equals zero.

Next for each i we consider the following integral

$$\int_{\lambda_i}^{\mu_i} \delta_n^k(f(x))\varphi(x)\,dx = \int_{\lambda_i}^{x_i} \delta_n^k(f(x))\varphi(x)\,dx + \int_{x_i}^{\mu_i} \delta_n^k(f(x))\varphi(x)\,dx. \tag{2.9}$$

Making the substitution $t^{1/\alpha_i} = f_i(x)$ or $x = g_i(t^{1/\alpha_i})$ for the second integral on the right-hand side of equation 2.9 then we have

$$\int_{x_i}^{\mu_i} \delta_n^k(f(x))\varphi(x)\,dx = \frac{1}{\alpha_i}\int_0^{\nu_i} \delta_n^k(t)\varphi(g_i(t^{1/\alpha_i}))|g_i'(t^{1/\alpha_i})|t^{1/\alpha_i - 1}\,dt$$

where $\nu_i = f(\mu_i)$.

The function $\Psi(y) = \varphi(g_i(y))|g_i'(y)|$ is infinitely differentiable and it follows from Taylor's theorem that

$$\Psi(y) = \sum_{i=0}^{k\alpha_i - 1} \frac{\Psi^{(i)}(0)}{i!}y^i + \frac{\Psi^{(k\alpha_i)}(\xi y)}{(k\alpha_i)!}y^{k\alpha_i} \quad (0 < \xi < 1).$$

Thus

$$\alpha_i \int_{x_i}^{\mu_i} \delta_n^k(f(x))\varphi(x)\,dx = \sum_{j=0}^{k\alpha_i-1} \frac{\Psi^{(j)}(0)}{j!} \int_0^{\nu_i} \delta_n^k(t) t^{\frac{j+1}{\alpha_i}-1}\,dt +$$

$$+ \int_0^{\nu_i} \frac{\Psi^{(k\alpha_i)}(\xi t^{1/\alpha_i})}{(k\alpha_i)!} \delta_n^k(t) t^{k+1/\alpha_i-1}\,dt$$

$$= \sum_{j=0}^{k\alpha_i-2} \frac{\Psi^{(j)}(0)}{j!} n^{k-1} \int_0^1 \rho(u)(\frac{u}{n})^{\frac{i+1}{\alpha_i}-1}\,du + \frac{\Psi^{(k\alpha_i-1)}(0)}{(k\alpha_i-1)!} \int_0^1 u^{k-1}\rho^k(u)\,du +$$

$$+ n^k \int_0^1 \frac{\Psi^{(k\alpha_i)}(\xi(u/n)^{1/\alpha_i})}{(k\alpha_i)!} \rho^k(u)(u/n)^{k+1/\alpha_i-1}\,du$$

on making the substitution $nt = u$ for $n^{-1} < \alpha_i$. Passing on the neutrix limit of the integral $\int_{x_i}^{\mu_i} \delta_n^k(f(x))\varphi(x)\,dx$ we get

$$\underset{n\to\infty}{\text{N}-\lim} \int_{x_i}^{\mu_i} \delta_n^k(f(x))\varphi(x)\,dx = \frac{c_{k,\rho}\Psi^{(k\alpha_i-1)}(0)}{\alpha_i(k\alpha_i-1)!} \tag{2.10}$$

where $c_{k,\rho} = \int_0^1 u^{k-1} n\rho^k(u)\,du$.

Now consider the integral $\int_{\lambda_i}^{x_i} \delta_n^k(f(x))\varphi(x)\,dx$. Similarly making another substitution $-t^{1/\alpha_i} = f_i(-x)$ or $-x = g_i(-t^{1/\alpha_i})$ where $t^{1/\alpha_i} \geq 0$, that

$$\begin{aligned} \alpha_i \int_{\lambda_i}^{x_i} \delta_n^k(f(x))\varphi(x)\,dx &= \int_0^{\beta_i} \delta_n^k(f(-x))\varphi(-x)\,dx \\ &= \int_0^{\beta_i} \delta_n^k((-1)^{\alpha_i} t)\varphi(g_i(-t^{1/\alpha_i}))|g_i'(-t^{1/\alpha_i})| t^{1/\alpha_i-1}\,dt \\ &= \int_0^{\beta_i} \delta_n^k(t)\Psi(-t^{1/\alpha_i}) t^{1/\alpha_i-1}\,dt \end{aligned}$$

where $\beta_i = -f(\lambda_i)$ and Ψ is defined above.

Thus with $n^{-1} < \beta_i$

$$\begin{aligned} \underset{n\to\infty}{\text{N}-\lim} \int_{\lambda_i}^{x_i} \delta_n^k(f(x))\varphi(x)\,dx &= \frac{(-1)^{k\alpha_i-1}\Psi^{(k\alpha_i-1)}(0)}{\alpha_i(k\alpha_i-1)!} \int_0^1 u^{k-1}\rho^k(u)\,du \\ &= \frac{(-1)^{k\alpha_i-1} c_{k,\rho}\Psi^{(k\alpha_i-1)}(0)}{(k\alpha_i-1)!}. \end{aligned} \tag{2.11}$$

It now follows from equations 2.8 - 2.11 that

$$\underset{n\to\infty}{\text{N}-\lim} \int_{\lambda_i}^{\mu_i} \delta_n^k(f(x))\varphi(x)\,dx = \begin{cases} 0, & k\alpha_i - 1 \notin 2\mathbb{Z} \\ \frac{2c_{k,\rho}\Psi^{(k\alpha_i-1)}(0)}{\alpha_i(k\alpha_i-1)!}, & k\alpha_i - 1 \in 2\mathbb{Z} \end{cases} \tag{2.12}$$

proving the existence of the composition $\delta^k(f(x))$ on the interval (λ_i, μ_i) for $i = 1, 2, \ldots, n$. Consequently it follows from equations 2.8 and 2.12 that the composition $\delta^k(f(x))$ exists on the interval (a,b) and equals to

$$\langle \delta^k(f(x)), \varphi(x)\rangle = \sum_{i=1}^{n} \left\{ \frac{2c_{k,\rho}\Psi^{(k\alpha_i-1)}(y)}{\alpha_i(k\alpha_i-1)!} \right\}_{y=0}$$

$$= \sum_{i=1}^{n}\sum_{r=0}^{k\alpha_i-1} \binom{k\alpha_i-1}{r} \frac{2c_{k,\rho}}{\alpha_i(k\alpha_i-1)!} \{\varphi(g_i(y))\}^{(r)} |g_i^{(k\alpha_i-r)}(y)| \Big|_{y=0}$$

$$= \sum_{i=1}^{n}\sum_{r=0}^{k\alpha_i-1} \binom{k\alpha_i-1}{r} \frac{2c_{k,\rho}}{\alpha_i(k\alpha_i-1)!} \times$$

$$\times \sum_{m=1}^{r} \varphi^{(m)}(g_i(y)) B_{r,m}\Big(g_i', g_i'', \ldots, g_i^{(r-m+1)}\Big) |g_i^{(k\alpha_i-r)}(y)| \Big|_{y=0}$$

$$= \sum_{i=1}^{n}\sum_{r=0}^{k\alpha_i-1} \binom{k\alpha_i-1}{r} \frac{2c_{k,\rho}}{\alpha_i(k\alpha_i-1)!} |f^{(k\alpha_i-r)}(x_i)| \times$$

$$\times \sum_{m=1}^{r} B_{r,m}\Big(\frac{1}{|f'(x_i)|}, \frac{1}{|f''(x_i)|}, \ldots, \frac{1}{|f^{(r-m+1)}(x_i)|}\Big) \varphi^{(m)}(x_i)$$

$$= \sum_{i=1}^{n}\sum_{r=0}^{k\alpha_i-1} \binom{k\alpha_i-1}{r} \frac{2c_{k,\rho}}{\alpha_i(k\alpha_i-1)!} |f^{(k\alpha_i-r)}(x_i)| \times$$

$$\times \sum_{m=1}^{r} (-1)^m B_{r,m}\Big(\frac{1}{|f'(x_i)|}, \frac{1}{|f''(x_i)|}, \ldots, \frac{1}{|f^{(r-m+1)}(x_i)|}\Big) \langle \delta^{(m)}(x-x_i), \varphi(x)\rangle$$

for $i=1,2,\ldots,n,\ k\alpha_i - 1 \in 2\mathbb{Z}^+$ where $B_{r,m}(x_1, x_2, \ldots, x_{r-m+1})$ are again the exponential Bell polynomials. If $k\alpha_i \in 2\mathbb{Z}$ for all $i=1,2,\ldots,n$ then the second sum of equation 2.12 is equal to zero, so we have $\delta^k(f(x)) = 0$. This completes the proof. □

Example 2.8 *Let us consider the function $f(x) = \cot^3 x$. Using the notation of the proof of Theorem 2.3 so that $f_1(x) = \cot x$ which has simple roots at the points $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$ and*

$$g(y) = \cot^{-1} y = \frac{\pi}{2} - y + \frac{1}{3}y^3 - \frac{1}{5}y^5 + \frac{1}{7}y^7 - \ldots,$$

$$g'(y) = -\frac{1}{1+y^2} = -1 + y^2 - y^4 + y^6 - \ldots$$

on the interval $(-\infty, \infty)$. Thus $\Psi(y) = -\varphi(\cot^{-1} y)(1+y^2)^{-1}$ and it can be shown that $\Psi''(0) = 2\varphi(\frac{\pi}{2}) - \varphi''(\frac{\pi}{2})$ on the open interval $(0,\pi)$. It follows from the proof of Theorem 2.3 that

$$\langle \delta(\cot^3 x), \varphi(x)\rangle = \frac{1}{6}\Psi''(0) = \frac{1}{3}\varphi(\frac{\pi}{2}) - \frac{1}{6}\varphi''(\frac{\pi}{2})$$

and so

$$\delta(\cot^3 x) = \frac{1}{3}\delta(x - \frac{\pi}{2}) - \frac{1}{6}\delta''(x - \frac{\pi}{2}).\ \square$$

We now extend definition 2.1 with the following definition given in [15].

Definition 2.9 *Let F be distribution in $\mathcal{D}'$ and let f be locally summable function. We say that the distribution $F(f(x))$, the neutrix composition of F and f, exists and is equal to $G(x)$ on the interval (a,b) if the neutrix limit*

$$\underset{n\to\infty}{\mathrm{N-lim}}\left[\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)\,dx\right] = \langle G(x),\varphi(x)\rangle$$

for all φ in $\mathcal{D}$ with support contained in the interval (a,b), where again $F_n(x) = (F*\delta_n)(x)$, and N is the same neutrix as in definition 2.1.

For a summable function $f(x)$ the functions f_+ and f_- are defined by

$$f_+(x) = \begin{cases} f(x), & x \geq 0, \\ 0, & x < 0, \end{cases} \quad \text{and} \quad f_-(x) = \begin{cases} f(x), & x \leq 0, \\ 0, & x > 0. \end{cases}$$

In accordance with the usual practice, we define the summable functions x_+^r and x_-^r by

$$x_+^r = \begin{cases} x^r, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad \text{and} \quad x_-^r = \begin{cases} |x|^r, & x \leq 0, \\ 0, & x > 0. \end{cases}$$

If the term infinitely differentiable function is replaced by a summable function in proposition 2.2, then it becomes as;

Proposition 2.10 *Let $f(x)$ be a summable function and suppose that f is continuous on $[a,b]$ and $f(x) \neq 0$ on this interval, where $a < 0 < b$. Then the composition $\delta^k(f_+(x))$ exists and*

$$\delta^k(f_+(x)) = 0$$

on the interval $(-\infty,b)$ for all $k \in \mathbb{Z}^+$, in particular $\delta^k(H(x)) = 0$ on the interval $(-\infty,\infty)$, where H denotes Heaviside's function.

Proof Let $\varphi \in \mathcal{D}'$ with compact support contained in the interval $(-\infty,b)$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} \delta_n^k(f_+(x))\varphi(x)\,dx &= \int_{-\infty}^{0} \delta_n^k(0)\varphi(x)\,dx + \int_{0}^{\infty} \delta_n^k(f(x))\varphi(x)\,dx \\ &= n^k\rho^k(0)\int_{-\infty}^{0}\varphi(x)\,dx + n^k\int_0^b \rho^k(nf(x))\varphi(x)\,dx\end{aligned}$$

where $n^k\rho^k(0)\int_{-\infty}^{0}\varphi(x)\,dx$ is either negligible or zero. Further, since f is continuous and nonzero on $[0,b]$, we can find an integer N such that $|nf(x)| \geq 1$ for $n > N$. It follows that we have $\rho^k(nf(x)) = 0$ for $n > N$. Thus

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n^k(f_+(x))\varphi(x)\,dx =$$

$$= \mathrm{N}\!-\!\lim_{n\to\infty} n^k\rho^k(0)\int_{-\infty}^{0}\varphi(x)\,dx + \lim_{n\to\infty} n^k\int_0^b \rho^k(ng(x))\varphi(x)\,dx = \langle 0,\varphi(x)\rangle$$

and so $\delta^k(f_+(x)) = 0$. □

Let $f(x)$ be as in the proposition 2.10 and by writing $g(x) = f(-x)$, then $g(x)$ is continuous and nonzero for $-b \le x \le -a$ and so by the proposition 2.9, we have $\delta^k(g_+(x)) = 0$ on the interval $(-\infty,-a)$. Now replacing x by $-x$ we see that $\delta^k(g_+(-x)) = \delta^k(f_-(x)) = 0$. Thus we arrive the following result. □

Corollary 2.11 *Let $f(x)$ be a summable function and suppose that f is continuous and nonzero on the interval $[a,b]$, where $a<0<b$. Then $\delta^k(f_-(x))$ exists and*

$$\delta^k(f_-(x)) = 0$$

on the interval (a,∞) for all $k\in\mathbb{Z}^+$.

Proof It is evident. □

Corollary 2.12 *Let $f(x)$ be a summable function and suppose that f is continuous and nonzero on the interval $[a,b]$, where $a<0<b$. Then the composition $\delta^k(f_+(x)-f_-(x))$ exists and*

$$\delta^k(f_+(x)) = \delta^k(f_+(x)-f_-(x)) = 0$$

on the interval (a,b) for all $k\in\mathbb{Z}^+$.

Proof It is evident. □

Theorem 2.13 *let F be a summable function which is $s+1$ times continuously differentiable on the interval $[a,b]$, where $a<0<b$. Suppose that the equation $F(x)=0$ has a single simple root at the point $x_0=0$ in the interval $[a,b]$. If $f=F^s$, then the composition $\delta^k(f_+(x))$ exists on the interval $(-\infty,b)$*

$$\langle\delta^k(f_+(x)),\varphi(x)\rangle = \sum_{r=0}^{ks-1}\binom{ks-1}{r}\frac{c_{k,\rho}}{s(ks-1)!|f^{(ks-r)}(x_0)|}\times$$

$$\times\sum_{i=1}^{r}(-1)^i B_{r,i}\Big(\frac{1}{|f'(x_0)|},\frac{1}{|f''(x_0)|},\ldots,\frac{1}{|f^{(r-i+1)}(x_0)|}\Big)\langle\delta^{(i)}(x-x_0),\varphi(x)\rangle \tag{2.13}$$

and the composition $\delta^k(f_-(x))$ *exists on the interval* (a,∞)

$$\langle \delta^k(f_-(x)),\varphi(x)\rangle = \sum_{r=0}^{ks-1}\binom{ks-1}{r}\frac{c_{k,\rho}}{s(ks-1)!|f^{(ks-r)}(x_0)|}\times$$
$$\times\sum_{i=1}^{r}B_{r,i}\Big(\frac{1}{|f'(x_0)|},\frac{1}{|f''(x_0)|},\dots,\frac{1}{|f^{(r-i+1)}(x_0)|}\Big)\langle \delta^{(i)}(x-x_0),\varphi(x)\rangle \tag{2.14}$$

for $s,k=1,2,\dots,$ *where* $B_{r,i}(x_1,x_2,\dots,x_{r-i+1})$ *was defined as in theorem 2.3 and* $c_{k,\rho}=\int_0^1 u^{k-1}\rho^k(u)\,du$*. In particular*

$$\delta(x_+^s)=\frac{(-1)^{s-1}}{2s!}\delta^{(s-1)}(x) \quad \textit{and} \quad \delta(x_-^s)=\frac{1}{2s!}\delta^{(s-1)}(x) \tag{2.15}$$

on the interval $(-\infty,\infty)$ *for* $s=1,2,\dots$*.*

Proof The proof is as similar as the proof of theorem 2.3. Since $x=0$ is a simple root of the equation $F(x)=0$, this implies that $F'(x)\neq 0$ on the interval $[o,c]$, where $0<c\leq b$. The equation $F(x)=y$ will therefore have inverse $x=g(y)$ on the interval $[0,c]$ and the function g will be $s+1$ times continuously differentiable. Let $\varphi\in\mathcal{D}'$ with $\mathrm{supp}(\varphi)\subset(-\infty,c)$. Then

$$\begin{aligned}\int_{-\infty}^{\infty}\delta_n^k(f_+(x))\varphi(x)\,dx &= \int_{-\infty}^{0}\delta_n^k(0)\varphi(x)\,dx+\int_0^{\infty}\delta_n^k(f(x))\varphi(x)\,dx=\\ &= n^k\rho^k(0)\int_{-\infty}^{0}\varphi(x)\,dx+\int_0^{\infty}\delta_n^k(f(x))\varphi(x)\,dx \end{aligned}\tag{2.16}$$

where again $n^k\rho^k(0)\int_{-\infty}^0\varphi(x)\,dx$ is either negligible or zero. For the second integral in the right hand side of equation 2.16, we have exactly have from equation 2.4 that the neutrix limit of $\int_0^\infty\delta_n^k(f(x))\varphi(x)\,dx$ exists and equal to

$$\begin{aligned}\mathrm{N}\!-\!\lim_{n\to\infty}\int_0^{\infty}\delta_n^k(f(x))\varphi(x)\,dx &= \frac{\Psi^{(ks-1)}(0)}{s(ks-1)!}\int_0^1 u^{k-1}\rho^k(u)\,du\\ &= c_{k,\rho}\frac{\Psi^{(ks-1)}(0)}{s(ks-1)!}.\end{aligned}\tag{2.17}$$

It now follows from equations 2.16 and 2.17 that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n^k(f(x))\varphi(x)\,dx=\frac{c_{k,\rho}\Psi^{(ks-1)}(0)}{s(ks-1)!}$$

proving the existence of $\delta^k(f(x))$ on the interval $(-\infty,b)$ for $k,s\in\mathbb{Z}^+$.

The expression $\Psi^{(ks-1)}(0)$ has been already evaluated in theorem 2.3 as

$$\begin{aligned}\Psi^{(ks-1)}(0) &= \Big\{\Psi^{(ks-1)}(y)\Big\}_{y=0}\\ &= \sum_{r=0}^{ks-1}\binom{ks-1}{r}\sum_{i=1}^{r}\varphi^{(i)}(x_0)B_{r,i}\Big(g',g'',\dots,g^{(r-i+1)}\Big)|g^{(ks-r)}(y)|\Big|_{y=0}\\ &= \sum_{r=0}^{ks-1}\binom{ks-1}{r}\frac{1}{|f^{(ks-r)}(x_0)|}\times\\ &\qquad\times\sum_{i=1}^{r}B_{r,i}\Big(\frac{1}{|f'(x_0)|},\frac{1}{|f''(x_0)|},\dots,\frac{1}{|f^{(r-i+1)}(x_0)|}\Big)\varphi^{(i)}(x_0)\end{aligned}$$

where $B_{r,i}$ is again the incomplete exponential Bell polynomial. Thus we have proved that the composition $\delta^k(f_+(x))$ exists and equals to

$$\begin{aligned}\langle\delta^k(f_+(x)),\varphi(x)\rangle &= \sum_{r=0}^{ks-1}\binom{ks-1}{r}\frac{c_{k,\rho}}{s(ks-1)!|f^{(ks-r)}(x_0)|}\times\\ &\times\sum_{i=1}^{r}(-1)^iB_{r,i}\Big(\frac{1}{|f'(x_0)|},\frac{1}{|f''(x_0)|},\dots,\frac{1}{|f^{(r-i+1)}(x_0)|}\Big)\langle\delta^{(i)}(x-x_0),\varphi(x)\rangle.\end{aligned}\tag{2.18}$$

The equation 2.14 follows from the fact that $\delta^k(f_-(x))=\delta^k(f_+(-x))$. If $F(x)=x$ and $f(x)=F^s(x)=x^s$, then for any $\varphi\in\mathcal{D}$, the function Ψ is identical to φ so that equation 2.15 now follows from the fact that $c_{\rho,1}=\frac{1}{2}$. □ □

Equation 2.15 is in agreement with result given in the results given in [15].

Theorem 2.14 *Let F be a summable function which is $s+1$ times continuously differentiable on the interval $[a,b]$, where $a<0<b$. Suppose that the equation $F(x)=0$ has a single simple root at the point $x_0=0$ in the interval $[a,b]$. Then if $f=F^s$, the compositions $\delta^k(f(x))$ and $\delta^k(f_+(x)-f_-(x))$ exist on the interval (a,b) for $k=1,2,\dots$. In particular*

$$\delta^k(\mathrm{sgn}(x)|x|^s)=0\tag{2.19}$$

on the interval $(-\infty,\infty)$ for $k=0,2,4,6\dots,$ and and $s=2,4,\dots$

$$\delta^k(|x|^s)=0\tag{2.20}$$

on the interval $(-\infty,\infty)$ for $k,s=1,3,\dots$

$$\delta(\mathrm{sgn}(x)|x|^s)=\frac{(-1)^s}{s!}\delta^{(s-1)}(x)\tag{2.21}$$

on the interval $(-\infty,\infty)$ for for $k=1,3,\dots,$ and $s=2,4,\dots$

$$\delta(|x|^s)=\frac{(-1)^s}{s!}\delta^{(s-1)}(x)\tag{2.22}$$

on the interval $(-\infty,\infty)$ for $k=0,2,\dots,$ and $s=1,3,\dots$.

Proof We have

$$\int_{-\infty}^{\infty} \delta_n^k(f(x))\varphi(x)\,dx = \int_{-\infty}^{0} \delta_n^k(f(x))\varphi(x)\,dx + \int_{0}^{\infty} \delta_n^k(f(x))\varphi(x)\,dx =$$
$$= \int_{-\infty}^{\infty} \delta_n^k(f_-(x))\varphi(x)\,dx + \int_{-\infty}^{\infty} \delta_n^k(f_+(x))\varphi(x)\,dx - \int_{-\infty}^{\infty} \delta_n^k(0)\varphi(x)\,dx$$

the last term again being either negligible or zero. Thus

$$\begin{aligned} \mathrm{N}\!-\!\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^k(f(x))\varphi(x)\,dx &= \langle \delta^k(f_+(x) + \delta^k(f_-(x)), \varphi(x)\rangle \\ \langle \delta^k(f(x)), \varphi(x)\rangle &= \langle \delta^k(f_+(x) + \delta^k(f_-(x)), \varphi(x)\rangle. \end{aligned}$$

Similarly

$$\begin{aligned} \delta^k(f_+(x) - f_-(x)) &= \delta^k(f_+(x)) + \delta^k(-f_-(x)) \\ &= \delta^k(f_+(x)) + (-1)^k\delta^k(f_-(x)). \end{aligned}$$

Equations 2.19 - 2.22 follow from these results and on using theorems 2.7 and 2.13. □

Letting $F(x) = \sin x$ and $f(x) = \sin^2 x$ Then, by setting $\Phi(y) = \varphi(\sin^{-1}(y)(1-y^2)^{-1/2}$, it was shown in Example 3.8 of [32] that

$$\begin{aligned} \langle \delta(\sin^2 x), \varphi(x)\rangle &= 0 \\ \langle \delta(\sin^2_+ x), \varphi(x)\rangle &= -\frac{1}{4}\delta'(x), \\ \langle \delta(\sin^2_- x), \varphi(x)\rangle &= \frac{1}{4}\delta'(x), \\ \langle \delta'(\sin^2_+ x), \varphi(x)\rangle &= \frac{1}{6}\delta'(x) + \frac{1}{24}\delta'''(x), \\ \langle \delta'(\sin^2_- x), \varphi(x)\rangle &= -\frac{1}{6}\delta'(x) - \frac{1}{24}\delta'''(x), \\ \langle \delta(\operatorname{sgn} x.\sin^2 x), \varphi(x)\rangle &= -\frac{1}{2}\delta'(x), \\ \langle \delta'(\operatorname{sgn} x.\sin^2 x), \varphi(x)\rangle &= \frac{1}{3}\delta'(x) + \frac{1}{24}\delta'''(x) \end{aligned}$$

on the interval $(-\infty, \infty)$. □ □

3. The composition $\delta^k(f(x))$ for negative integers

The meaning given to expression $\delta^{-k}(x)$ for real variable x by equation

$$\delta^{-k}(x) = 0$$

for $k = 1, 2, \dots,$ see [35].

In fact, the symbol δ^{-1} recently appeared in the cosmological models based on the mixture of the anti-Chaplying gas and the paradox of soft singularity crossing, see [24, 25] in which the distributional identities

$$\big[f(\tau)+C\delta(\tau)\big]^{-1}=f^{-1}(\tau),\quad \frac{d}{d\tau}\big[f(\tau)+C\delta(\tau)\big]^{-1}=\frac{d}{d\tau}f^{-1}(\tau)$$

were used. There are several attempts to give meaning to the powers of Dirac delta function, one by using fixed δ-sequnces [26] and fractional and Caputo fractional derivatives [29, 30] respectively and another one by normalization [1]. In [37] Raju evaluated the square of x^{-n} by his definition of the pointwise product of distributions. The author defined the powers of Dirac- delta function for negative integers by $\delta^{-k}=0$ in [35]. Further he proved in [33] that if f is an infinitely differentiable function having n distinct simple roots in the open interval (a,b), then the composition $\delta^{-k}(f(x))$ exists on the interval (a,b) and defined by $\delta^{-k}(f(x))=0$ for $k\in\mathbb{Z}^+$.

In this section by using the alternative definition of composition of distributions [21] also given by Fisher, we give a meaning to the powers of some compositions of singular distributions for negative integers.

Definition 3.1 *Let F and f be distributions in $\mathcal{D}'$. We say that the distribution $F(f(x))$ exists and is equal to $G(x)$ on the interval (a,b) if the double neutrix limit*

$$\mathop{\mathrm{N-lim}}_{n\to\infty}\left[\mathop{\mathrm{N-lim}}_{m\to\infty}\int_{-\infty}^{\infty}F_n(f_m(x))\varphi(x)\,dx\right]=\langle G(x),\varphi(x)\rangle$$

for all φ in $\mathcal{D}$ with support contained in the interval (a,b), where

$$F_n(x)=(F*\delta_n)(x),\quad f_m(x)=(f*\delta_m)(x)$$

for $m,n=1,2,\ldots,$ and N is the neutrix defined in the begining of the section 2.

It is an open question as to whether definition 3.1 is a generalization of definition 2.1 for locally summable functions.

Theorem 3.2 *Let $f_1(x)$ be an infinitely differentiable function and suppose that the equation $f_1(x)=0$ has a single simple root at x_0 in the open interval (a,b). If $f(x)=f_1^s(x)$ $(s\in\mathbb{Z}^+)$ then distribution $\delta^{-k}(f(x))$ exists and*

$$\delta^{-k}(f(x))=0$$

for $k,s=1,2,\ldots.$

Proof As in Theorem 2.3 we prove the theorem for $x_0=0$, then the case $x_0\neq 0$ follows by translation. Since x_0 is simple root, we may assume that $f_1'(x)\neq 0$ on the interval (a,b) containing the origin. So the equation $f_1(x)=t$ then has an inverse $x=g_1(t)$.

Writing $x^{-k} = \frac{(-1)^{k-1}}{(k-1)!}(\ln x)^{(k)}$, we have

$$(x^{-k})_n = x^{-k} * \delta_n(x) = \frac{(-1)^{k-1}}{(k-1)!}\int_{-1/n}^{1/n} \ln|t-x|\delta_n^{(k)}(t)\,dt.$$

Let $\varphi \in \mathcal{D}$ with $supp(\varphi) \subset (a,b)$

$$\Big\langle \left[(\delta_m(f(x))^{-k}\right]_n, \varphi(x)\Big\rangle =$$
$$= \frac{(-1)^{k-1}}{(k-1)!}\int_{|x|\geq g_1(1/m^{1/s})} \varphi(x)\int_{-1/n}^{1/n} \ln|t-\delta_m(f(x))|\delta_n^{(k)}(t)\,dt\,dx +$$
$$+\frac{(-1)^{k-1}}{(k-1)!}\int_{|x|< g_1(1/m^{1/s})} \varphi(x)\int_{-1/n}^{1/n} \ln|t-\delta_m(f(x))|\delta_n^{(k)}(t)\,dt\,dx. \tag{3.1}$$

The function $\delta_m(f(x))$ is equal to zero on the set $\{x : |x| \geq g_1(1/m^{1/s})\}$. Thus

$$\lim_{m\to\infty}\int_{|x|\geq f_1(1/m^{1/s})} \varphi(x)\int_{-1/n}^{1/n} \ln|t-\delta_m(f(x))|\delta_n^{(k)}(t)\,dt\,dx =$$
$$= \int_{-\infty}^{\infty}\varphi(x)\,dx\int_{-1/n}^{1/n}\ln|t|\delta_n^{(k)}(t)\,dt$$

where $g_1(1/m^{1/s})$ tends to zero as $m \to \infty$ and making the substitution $nt = u$, we have

$$\underset{n\to\infty}{\mathrm{N-lim}}\left[\lim_{m\to\infty}\int_{|x|\geq g_1(1/m^{1/s})} \varphi(x)\int_{-1/n}^{1/n} \ln|t-\delta_m(f(x))|\delta_n^{(k)}(t)\,dt\,dx\right] =$$
$$= \int_{-\infty}^{\infty}\varphi(x)\,dx\left[\underset{n\to\infty}{\mathrm{N-lim}}\, n^k\int_{-1}^{1}[\ln|u| - \ln n]\rho^{(k)}(u)\,du\right] = 0. \tag{3.2}$$

Next

$$\int_{-1/n}^{1/n} |\ln|t-\delta_m(f(x))|\delta_n^{(k)}(t)|\,dt \leq n^{k+1}\sup_t\{|\rho^{(k)}(t)|\}\int_{-1/n}^{1/n}|\ln|t-\delta_m(f(x))||\,dt$$
$$\leq n^{k+1}\sup_t\{|\rho^{(k)}(t)|\}\{|(1/n-\delta_m(f(x)))\ln|1/n-\delta_m(f(x))| +$$
$$-(1/n+\delta_m(f(x)))\ln|1/n+\delta_m(f(x))| + 2/n|\}$$
$$\leq n^k\sup_t\{|\rho^{(k)}(t)|\}(3\ln n + 5) \tag{3.3}$$

for $g_1\Big(\frac{1}{m^{1/s}}(\rho^{-1})^{1/s}(\frac{1}{mn})\Big) \leq |x| < g_1(\frac{1}{m^{1/s}})$.

If $|x| < g_1\Big(\frac{1}{m^{1/s}}(\rho^{-1})^{1/s}(\frac{1}{mn})\Big)$ we have $1/n < \delta_m(f(x)) \leq m\rho(0)$ and

$$\ln(1+\rho(0)m) = \sup_t\left\{\sup_{|x|<g_1\left(\frac{1}{m^{1/s}}(\rho^{-1})^{1/s}(\frac{1}{mn})\right)}\Big|\ln|t-\delta_m(f(x))|\Big|\right\}.$$

Thus

$$\left|\int_{|x|<g_1(\frac{1}{m^{1/s}})} \varphi(x)\int_{-1/n}^{1/n} \ln|t-\delta_m(f(x))|\delta_n^{(k)}(t)\,dt\,dx\right| =$$

$$= \left|\int_{|x|<g_1\left(\frac{1}{m^{1/s}}(\rho^{-1})^{1/s}(\frac{1}{mn})\right)} \varphi(x)\int_{-1/n}^{1/n} \ln|t-\delta_m(f(x))|\delta_n^{(k)}(t)\,dt\,dx + \right.$$

$$\left. + \int_{g_1\left(\frac{1}{m^{1/s}}(\rho^{-1})^{1/s}(\frac{1}{mn})\right)\le|x|<g_1(\frac{1}{m^{1/s}})} \varphi(x)\int_{-1/n}^{1/n} \ln|t-\delta_m(f(x))|\delta_n^{(k)}(t)\,dt\,dx\right|$$

$$\le 4n^k \sup_t\{|\rho^{(k)}(t)|\}\sup|\varphi(x)|\left\{g_1\Big(\frac{1}{m^{1/s}}(\rho^{-1})^{1/s}(\frac{1}{mn})\Big)\ln(1+\rho(0)m) + \right.$$

$$\left. + \Big[g_1(\frac{1}{m^{1/s}}) - g_1\Big(\frac{1}{m^{1/s}}(\rho^{-1})^{1/s}(\frac{1}{mn})\Big)\Big](3\ln n+5)\right\} \to 0 \tag{3.4}$$

as $m\to\infty$.

It now obtain from equations 3.1 - 3.4 that

$$\mathrm{N}\!-\!\lim_{n\to\infty}\left[\mathrm{N}\!-\!\lim_{m\to\infty}\left\langle \left[(\delta_m(f(x)))^{-k}\right]_n, \varphi(x)\right\rangle\right] = 0$$

for all $\varphi\in\mathcal{D}$. □ □

Before giving the generalization of theorem 3.2 we will give the following proposition [33].

Proposition 3.3 *Let $f(x)$ be an infinitely differentiable function on the real line and suppose that f does not have any root. Then the distribution $\delta^{-k}(f(x))$ exists and*

$$\delta^{-k}(f(x)) = 0$$

for $k = 1, 2, \ldots$.

Theorem 3.4 *Let $f(x)$ be an infinitely differentiable function having distinct multiple roots at $x_1, x_2, \ldots, x_n$ with multiplicities $r_1, r_2, r_3, \ldots, r_n$; $n, r_i \in \mathbb{Z}^+$ respectively on the open interval (a,b). Then the distribution $\delta^{-k}(f(x))$ exists on the interval (a,b) and*

$$\delta^{-k}(f(x)) = 0 \tag{3.5}$$

for $k = 1, 2, \ldots$.

Proof We use the same argument of the proof of theorem 2.7 to show the validity of equation 3.5. Let (μ_i, τ_i) be disjoint open subintervals of (a,b) for $i = 1, 2, \ldots, n$ containing x_i such that $A = \cup_{i=1}^n(\mu_i, \tau_i)$.

Now let $\varphi(x) \in \mathcal{D}$ with support contained in the interval (a, b), then we have

$$\begin{aligned}\int_{-\infty}^{\infty} \left[(\delta_m(f(x))^{-k}\right]_n \varphi(x)\,dx &= \int_{\mu_1}^{\tau_1} \left[(\delta_m(f(x))^{-k}\right]_n \varphi(x)\,dx + \\ &+ \int_{\mu_2}^{\tau_2} \left[(\delta_m(f(x))^{-k}\right]_n \varphi(x)\,dx + \ldots + \int_{\mu_n}^{\tau_n} \left[(\delta_m(f(x))^{-k}\right]_n \varphi(x)\,dx + \\ &+ \int_{\mathbb{R}\setminus A} \left[(\delta_m(f(x))^{-k}\right]_n \varphi(x)\,dx \end{aligned} \tag{3.6}$$

where

$$\left[(\delta_m(f(x))^{-k}\right]_n = \frac{(-1)^{k-1}}{(k-1)!} \int_{-1/n}^{1/n} \ln|t - \delta_m(f(x))| \delta_n^{(k)}(t)\,dt.$$

By taking the double neutrix limit of the both sides of equation 3.6 as $m \to \infty$ and $n \to \infty$ respectively, we arrive equation 3.5 by using proposition 3.3 and theorem 2.13. □ □

Example 3.5 *Let us consider the function* $f(x) = (\sinh^{-1} x)^3$. *Using the notation of the proof of theorem 2.3, the equation* $f_1(x) = \sinh x$ *has simple root at the point* $x = 0$ *and will have an inverse*

$$\begin{aligned} g_1(y) &= \sinh y = y + \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \frac{1}{7!}y^7 + \ldots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} y^{2n+1} \\ g_1'(y) &= \cosh y = 1 + \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \frac{1}{6!}y^6 + \ldots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} y^{2n} \end{aligned}$$

on the open interval $(-\infty, \infty)$. *Thus* $\Phi(y) = \varphi(\sinh y)\cosh y$ *and it can be shown that* $\Phi''(0) = \varphi(0) + \varphi''(0)$. *It follows from the proof of theorem 3.4 that*

$$\langle \delta(\sinh^{-1} x)^3, \varphi(x) \rangle = \frac{1}{6}\Phi''(0) = \frac{1}{6}\varphi(0) + \frac{1}{6}\varphi''(0)$$

and so

$$\delta(\sinh^{-1} x)^3 = \frac{1}{6}\delta(x) + \frac{1}{6}\delta''(x)$$

on the interval $(-\infty, \infty)$. *Additionally it follows from theorem 3.4 that*

$$\delta^{-k}(\sinh^{-1} x)^3 = 0$$

for $k \in \mathbb{Z}^+$. □

Conclusion. In the classical theory of distributions, the composition $\delta(f(x))$ is defined for infinitely differentiable function having simple roots at $x_1, x_2, \ldots, x_n$ by Gelfand and Shilov and its k-th power is defined in [33, 34] and the case of f having distinct multiple roots is considered in this study by means of the notion of neutrix introduced by van der Corput and some compositions as an example are evaluated for particular f.

Finally the meaning to the symbol $\delta^{-k}(f(x))$ is given by taking double neutrix limit of the regular sequence $\left[(\delta_m(f(x))^{-k}\right]_n$ for infinitely differentiable function having multiple distinct roots.

Declarations

Conflict of interest

The author declares that there is no conflict of interest.

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