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On the connection between $\sigma_\epsilon(A_1 \otimes A_2)$ and $\sigma_\epsilon(A_1)$, $\sigma_\epsilon(A_2)$ for certain special operators

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Abstract: In this paper, the connection between the ϵ-pseudospectrum of the tensor product operator $A_1 \otimes A_2$ and the ϵ-pseudospectrums of operators A_1 and A_2 has been investigated and some results are given about this connection under certain conditions.

Key words: Tensor product, pseudospectrum, outer product

1. Introduction

In this study, it will be used that H is separable Hilbert space and $(\cdot\,,\,\cdot)_H$, $\|\cdot\|_H$ are inner product and norm on H, respectively. Also, all linear bounded operators space and Hilbert-Schmidt operators on H class will be denoted by $\mathfrak{B}(H)$ and $\mathfrak{H}(H)$, respectively.

Given $x_1, x_2 \in H$ and $A_1, A_2 \in \mathfrak{B}(H)$, the definitions of *the single tensor product* $x_1 \otimes x_2 \in H \otimes H$, *the outer product* $x \odot y \in \mathfrak{H}(H)$ and the tensor product of operator $A_1 \otimes A_2$ on the tensor product space $H \otimes H$ are given in [6, 7].

The definition of ϵ-pseudospectrum is given and studied by several authors; (see [1, 3, 8–12]). While the ϵ-pseudospectrum of a normal operator in a Hilbert space coincides with the ϵ-neighborhood of the spectrum, the situation is more involved for nonnormal operators. It is known that the spectral properties of a nonnormal operator (or matrix) not only depend on its spectrum, but are also affected by the resolvent growth. The pseudospectra are a good language to describe this growth. Their applications include the finite section method for Toeplitz matrices, growth bounds for semigroups, numerics for differential operators, matrix iterations, linear models for turbulence, etc. Moreover, pseudospectra are a useful tool for analyzing operators, furnishing a lot of information about the algebraic and geometric properties of operators and matrices. They play quite a natural role in numerical computations, especially in those involving spectral perturbations. The book [11] gives an extensive account of the pseudospectra, as well as investigations and applications in numerous fields.

Let $A \in \mathfrak{B}(H)$ and $\epsilon > 0$ arbitrary. The $\epsilon-$pseudospectrum $\sigma_\epsilon(A)$ of A is the set of $\lambda \in \mathbb{C}$ defined equivalently by any of the conditions

$$\sigma_\epsilon(A) = \{\lambda \in \mathbb{C} : \| (A - \lambda I)^{-1} \| > \epsilon^{-1}\}, \tag{1.1}$$

$$\sigma_\epsilon(A) = \{\lambda \in \mathbb{C} : \| (A - \lambda I)\, x \|_H < \epsilon \;\; for\ some\ unit\ vector\ x \in H\}, \tag{1.2}$$

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Here, it is used the convention that $\| (A-\lambda I)^{-1} \| = \infty$ if $\lambda \in \sigma(A)$. Therefore, it seems that $\sigma(A) \subset \sigma_\epsilon(A)$ where $\sigma(A)$ is the spectrum set of A.

If $\sigma_\epsilon(A) = \{\lambda \in \mathbb{C} : \| (A-\lambda I)\, x \|_H < \epsilon \ \mathit{for\ some\ unit\ vector}\, x \in H\}$ as in (1.2), then λ is an ϵ-pseudoeigenvalue of A and x is a corresponding ϵ-pseudoeigenvector. In other words, the $\epsilon-$pseudospectrum is the set of ϵ-pseudoeigenvalues.

The outer product of a pair of vectors $x, y \in H$ is a rank-one bounded linear transformation on H such that

$$(x \odot y)\, z = (z, x)_H\, y \tag{1.3}$$

for every $z \in H$. It is easily seen that $\| x \odot y \|_{\mathfrak{N}} = \| x \|_H \| y \|_H$ and $(x \odot y)^* = y \odot x$.

Some results of the outer product's $\epsilon-$pseudospectrum and $\epsilon-$ pseudospectral radius were given in the studies [3] by J. Cui, C. Li and Y. Poon and [1] by M. Bendaoud, A. Benyouness and M. Sarih, respectively.

A. Brown and C. Pearcy showed that $\sigma(A \otimes B) = \sigma(A).\sigma(B)$ in [2]. In this study, it will be aimed to achieve some results regarding the connection between $\sigma_\epsilon(A \otimes B)$ and $\sigma_\epsilon(A), \sigma_\epsilon(B)$ by applying a similar method in [2]. However, it is generally not true that $\sigma_\epsilon(A \otimes B) = \sigma_\epsilon(A)\sigma_\epsilon(B)$, since $\sigma(A \otimes B) \subset \sigma_\epsilon(A \otimes B)$.

For example, suppose that $A_1 = \alpha I$ and $A_2 = \beta I$, for $\alpha, \beta \in \mathbb{C}$. It is seen that $\sigma_\epsilon(A_1) = d(\alpha, \epsilon)$ and $\sigma_\epsilon(A_2) = d(\beta, \epsilon)$ and $\sigma_\epsilon(A \otimes B) = d(\alpha\beta, \epsilon)$ as $d(x, \epsilon) = \{y : |x - y| < \epsilon\}$ [3]. Then, according to the value of α, β and ϵ, the equality $\sigma_\epsilon(A \otimes B) = \sigma_\epsilon(A)\sigma_\epsilon(B)$ does not hold. Also, it cannot be said that $\sigma_\epsilon(A \otimes B) \subset \sigma_\epsilon(A_1)\,\sigma_\epsilon(A_2)$ or $\sigma_\epsilon(A_1)\,\sigma_\epsilon(A_2) \subset \sigma_\epsilon(A \otimes B)$.

The problem of this study is to determine the connection between the $\epsilon-$pseudospectrum of tensor product operator $A_1 \otimes A_2$ and the $\epsilon-$ pseudospectrums of operators A_1 and A_2 in some special cases.

2. Some results For the connection between $\sigma_\epsilon(A_1 \otimes A_2)$ and $\sigma_\epsilon(A_1)$, $\sigma_\epsilon(A_2)$ in some special cases

It is shown the point spectrum set of any operator with $\sigma_p(.)$ notation throughout this study.

Proposition 2.1 *Let $A_i \in \mathfrak{B}(H)$ for $i \in \{1, 2\}$ and $\epsilon > 0$. If $0 \in \sigma_p(A_i)$, then*

$$d(0, \epsilon) \subset \sigma_\epsilon(A_i \otimes A_j)$$

for $i, j \in \{1, 2\}$.

Proof Since other cases can be shown in a similar way, let $i = 1, j = 2$. Since $0 \in \sigma_p(A_1)$, there is a $x \in H, \| x \| = 1$ such that $A_1 x = 0$. Now, let us define $x \otimes x \in \mathfrak{B}(H \otimes H)$. It is clear that $\| x \otimes x \| = 1$. Suppose that $\lambda \in d(0, \epsilon)$. Then,

$$\| (A_1 \otimes A_2 - \lambda)\, x \otimes x \| = \| A_1 x \otimes A_2 x - \lambda x \otimes x \| = |\lambda| < \epsilon.$$

□

Proposition 2.2 *Let $A_i \in \mathfrak{B}(H)$ for $i \in \{1, 2\}$. Then, for $i, j \in \{1, 2\}$ and $0 \neq \lambda \in \sigma_p(A_i)$,*

$$\lambda \sigma_\epsilon(A_j) \subset \sigma_{|\lambda|.\epsilon}(A_i \otimes A_j).$$

Proof Since other cases can be shown in a similar way, let $i = 1, j = 2$. Now, if $\lambda \in \sigma_p(A_1)$, there is a $x_1 \in H, \| x_1 \| = 1$ such that $A_1 x_1 = \lambda x_1$ and if $\lambda_2 \in \sigma_\epsilon(A_2)$, there is a $x_2 \in H, \| x_2 \| = 1$ such that $\| (A_2 - \lambda_2) x_2 \| < \epsilon$. Now, let $x_1 \otimes x_2 \in \mathfrak{B}(H \otimes H)$ be defined. It is easily seen that $\| x_2 \otimes x_1 \| = 1$. Then

$$= \| (A_1 \otimes A_2 - \lambda\lambda_2) x_1 \otimes x_2 \| = \| A_1 x_1 \otimes A_2 x_2 - \lambda\lambda_2 x_1 \otimes x_2 \| = |\lambda| \, \| (A_2 - \lambda_2) x_2 \| < |\lambda| . \epsilon.$$

Hence, it is obtained that $\lambda\lambda_2 \in \sigma_{|\lambda|\epsilon}(A_1 \otimes A_2)$. □

Conversely, $\sigma_{\left|\lambda\right|.\epsilon}(A_i \otimes A_j) \subset \lambda\sigma_\epsilon(A_j)$ is not hold in generally.

Example 2.1 *Let* $H = \mathbb{C}^2$, *and* $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $A_2 = 3I$ *on* H. *It is clear that* $\sigma_p(A_1) = \{1, 2\}$ *and* $\sigma_\epsilon(A_2) = d(3, \epsilon)$ *[3]. Also, it is clear that* $A_1 \otimes A_2$ *is a normal operator. So,* $\sigma_\epsilon(A_1 \otimes A_2) = d(3, \epsilon) \cup d(6, \epsilon)$ *[11]. Hence,* $\sigma_{2\epsilon}(A_1 \otimes A_2) \neq 2.\sigma_\epsilon(A_2)$.

Now, some definitions and lemmas will be given since they will be used in the proof of theorems that will be given in the continuation of the study.

Let construct a Hilbert space $H^{'}$ opposed to H, i.e., the space whose elements and law of addition are identical with those of H, but with the reversed inner product $(x, y)_{H'} = (y, x)_H$ and with multiplication of a vector by a scalar redefined as $\lambda \circ x = \overline{\lambda} x$. Note that a one-one correspondence $A \leftrightarrow A^{'}$ can be defined between $\mathfrak{B}(H)$ and $\mathfrak{B}\left(H^{'}\right)$ defined by $A^{'} x = Ax$ for all x. It is clear that

$$\sigma_\epsilon(A^{'}) = \overline{(\sigma_\epsilon(A))} = \{\overline{\lambda} : \lambda \in \sigma_\epsilon(A)\}. \tag{2.1}$$

Lemma 2.1 *[4] The correspondence* $x \otimes y \to x \odot y$ *possesses a unique extension to a Hilbert space isomorphism* φ *of* $H^{'} \otimes H$ *onto the Hilbert space* $\mathfrak{H}(H)$ *of Hilbert-Schmidt operators on* H. *Moreover, if* $A_1, A_2 \in \mathfrak{B}(H)$, *then* $A_1^{'} \otimes A_2$ *is carried by* φ *onto the operator* $T_{A_2 A_1{}^*} \in \mathfrak{B}(\mathfrak{H}(H))$ *defined by*

$$T_{A_2 A_1^*} X = A_2 X A_1^*$$

for all $X \in \mathfrak{H}(H)$.

Let $L_A, R_A \in \mathfrak{B}(\mathfrak{H}(H))$ for $A \in \mathfrak{B}(H)$ be defined by setting for $X \in \mathfrak{H}(H)$, $L_A X = AX$ and $R_A X = XA$, respectively.

It is clear that $R_{A_2} L_{A_1} = L_{A_1} R_{A_2}$ and $T_{A_2 A_1^*} = R_{A_2} L_{A_1^*}$.

A. Brown and C. Pearcy showed that

$$\sigma(L_A) = \sigma(R_A) = \sigma(A)$$

in [2].To see

$$\sigma_\epsilon(L_A) = \sigma_\epsilon(R_A) = \sigma_\epsilon(A),$$

let us define J be an involuntary isometry on $\mathfrak{H}(H)$ such that $J(A) = A^*$ for $A \in \mathfrak{H}(H)$. Then, it is clear that $R_A = J L_{A^*} J$ and since $\| \left(J L_A J - \overline{\lambda}\right) X \|_{\mathfrak{H}(H)} = \| (L_A - \lambda) X^* \|_{\mathfrak{H}(H)}$, for any $X \in \mathfrak{H}(H)$, $\| X \| = 1$, it is obvious that $\sigma_\epsilon(J L_A J) = \overline{(\sigma_\epsilon(L_A))}$.

Lemma 2.2 *For* $A \in \mathfrak{B}(H)$, $\sigma_\epsilon(L_A) = \sigma_\epsilon(R_A) = \sigma_\epsilon(A)$.

Proof If $\lambda \in \sigma_\epsilon(L_A)$, there is $X \in \mathfrak{H}(H)$, such that $\| X \| = 1$ and it is satisfied that $\| (L_A - \lambda) X \|_{\mathfrak{H}(H)} < \epsilon$. It is known that $\| (L_A - \lambda) X \| \leqslant \| (L_A - \lambda) X \|_{\mathfrak{H}(H)}$ [5]. Now, for $x \in H, x \neq 0$ and $Xx \neq 0$, let us define the vector $x_0 = \dfrac{x}{\| Xx \|}$. It is easily seen that $\| Xx_0 \| = 1$ and

$$\| (A - \lambda) X x_0 \| \leqslant \| (L_A - \lambda) X \| < \epsilon.$$

Therefore, $\sigma_\epsilon(L_A) \subset \sigma_\epsilon(A)$. On the other hand, for $\lambda \in \sigma_\epsilon(A)$, there exists a unit vector, $\| x \| = 1$, such that $\| (A - \lambda) x \| < \epsilon$. Let an operator X be defined as $X = x \odot x \in \mathfrak{H}(H)$. It is clear that $Xx = x$ and $\| X \| = 1$. Since that the operator $(L_A - \lambda) X$ is rank one operator, $\| (L_A - \lambda) X \|_{\mathfrak{N}} = \| (L_A - \lambda) X \|$ by [5], and the following inequality

$$\begin{aligned} \| (L_A - \lambda) X \| \quad &= \sup_{y \in H} \left(\frac{\| (L_A - \lambda) X y \|_H}{\| y \|_H} \right) = \sup_{y \in H} \left(\frac{| (y, x)_H | \, \| (A - \lambda) x \|_H}{\| y \|_H} \right) \\ &< \epsilon \sup_{y \in H} \left(\frac{| (y, x)_H |}{\| y \|_H} \right) \leqslant \epsilon \end{aligned}$$

holds. Hence, $\sigma_\epsilon(L_A) = \sigma_\epsilon(A)$. Now, since $R_A = J L_{A^*} J$, $\sigma_\epsilon(J L_A J) = \overline{(\sigma_\epsilon(L_A))}$, and due to [11] $\left((p.46),\ \overline{(\sigma_\epsilon(A))} = \sigma_\epsilon(A^*)\right)$, it is seen that $\sigma_\epsilon(R_A) = \sigma_\epsilon(A)$. □

Now, let φ be any isomorphism of H onto H'. For each $A_1 \in \mathfrak{B}(H)$, denote by A_1^0 the element of $\mathfrak{B}(H)$ satisfying the equation $A_1^{0'} = \varphi A_1 \varphi^{-1}$. The tensor product of φ with the identity mapping on H is an isomorphism of $H \otimes H$ onto $H' \otimes H$ that carries each operator $A_1 \otimes A_2 \in \mathfrak{B}(H \otimes H)$ onto the operator $A_1^{0'} \otimes A_2 \in \mathfrak{B}\left(H' \otimes H\right)$. Thus by Lemma 2.1 and the equation (2.1), the equality

$$\sigma_\epsilon(A_1 \otimes A_2) = \sigma_\epsilon\left(A_1^{0'} \otimes A_2\right) = \sigma_\epsilon\left(T_{A_2 A_1^{0*}}\right) = \sigma_\epsilon\left(R_{A_2} L_{A_1^{0*}}\right)$$

is obtained. On the other hand, we have the following equation from the equation (2.1),

$$\sigma_\epsilon(A_1) = \sigma_\epsilon(A_1^{0'}) = \overline{(\sigma_\epsilon(A_1^0))} = \sigma_\epsilon(A_1^{0*}).$$

Hence, the problem becomes to find the connection between $\sigma_\epsilon(R_{A_2} L_{A_1})$ and $\sigma_\epsilon(L_{A_1})$, $\sigma_\epsilon(R_{A_2})$.

Theorem 2.1 *Let* $A_i \in \mathfrak{B}(H)$ *for* $i \in \{1, 2\}$ *and at least one of them is equal* αI. *Then, for* $i, j \in \{1, 2\}$ *and* $\epsilon > 0$,

$$\sigma_{|\alpha|.\epsilon}(A_i \otimes A_j) = \alpha \sigma_\epsilon(A_j).$$

Proof Since other cases can be shown in a similar way, let $i = 1$, $j = 2$ and $A_1 = \alpha I$. Because of Proposition 2.2

$$\alpha \sigma_\epsilon(A_2) \subset \sigma_{|\alpha|\epsilon}(A_1 \otimes A_2).$$

Now, suppose that $\lambda \in \sigma_{|\alpha|\epsilon}(A_i \otimes A_j) = \sigma_{|\alpha|\epsilon}(R_{A_2}L_{A_1})$. Then,

$$\| (R_{A_2}L_{A_1} - \lambda) X \|_{\mathfrak{H}(H)} = |\alpha| \, \| \left(R_{A_2} - \frac{\lambda}{\alpha}\right) X \|_{\mathfrak{H}(H)} < |\alpha|\epsilon.$$

Hence, $\frac{\lambda}{\alpha} \in \sigma_\epsilon(R_{A_2}) = \sigma_\epsilon(A_2)$. □

Corollary 2.1 *Let $A \in \mathfrak{B}(H)$ and $\epsilon > 0$. Then,*

$$\sigma_\epsilon (I \otimes A) = \sigma_\epsilon (A) = \sigma_\epsilon (A \otimes I).$$

Theorem 2.2 *Let $\alpha \in \mathbb{C}$ be nonzero. When $P \in \mathfrak{B}(H)$ is a nontrivial projection,*

$$\sigma_\epsilon (\alpha P \otimes A) = d(0,\epsilon) \cup \alpha\sigma_{\frac{\epsilon}{|\alpha|}} (A) = \sigma_\epsilon (A \otimes \alpha P).$$

Proof It will be proved that $\sigma_\epsilon (\alpha P \otimes A) = d(0,\epsilon) \cup \alpha\sigma_{\frac{\epsilon}{|\alpha|}} (A)$. The proof of the equation $\sigma_\epsilon (A \otimes \alpha P) = d(0,\epsilon) \cup \alpha\sigma_{\frac{\epsilon}{|\alpha|}} (A)$ can be given same way.

Now, because of $\sigma_p(\alpha P) = \{0, \alpha\}$,

$$d(0,\epsilon) \cup \alpha\sigma_{\frac{\epsilon}{|\alpha|}} (A) \subset \sigma_\epsilon (\alpha P \otimes A).$$

by Proposition 2.1 and Proposition 2.2. Now, let $\lambda \in \sigma_\epsilon (\alpha P \otimes A) = \sigma_\epsilon (R_A L_{\alpha P})$. Then there is a $X \in \mathfrak{H}(H)$, $\| X \| = 1$ such that

$$\| (R_A L_{\alpha P} - \lambda) X \|_{\mathfrak{H}(H)} < \epsilon.$$

Since P is a nontrivial projection, $H = KerP \oplus Ker(I-P)$. If $RangeX \subset KerP$, then $|\lambda| < \epsilon$. Now, suppose that $RangeX \not\subset KerP$. Therefore, $Xx = x_1 + x_2, x_1 \in KerP, x_2 \in Ker(I-P)$ for every $x \in H$. Now, for $Xx = x_1 + x_2, x \in H, x_1 \in KerP, x_2 \in Ker(I-P)$, let us define $X_1 : H \to KerP, X_1 x = x_1$ and $X_2 : H \to Ker(I-P), X_2 x = x_2$. It means that $X = X_1 + X_2$. Also, it is easily seen that $PX = PX_2 = X_2$ and $\| X_2 \| = 1$. Now,

$$|\alpha| \, \| \left(R_A - \frac{\lambda}{\alpha}\right) X_2 \|_{\mathfrak{H}(H)} \leq \| (R_A L_{\alpha P} - \lambda) X \|_{\mathfrak{H}(H)} < \epsilon$$

when $P(\alpha PXA - \lambda X) = \alpha(PX_2A - \lambda PX_2) = \alpha(X_2A - \lambda X_2)$.Hence, $\lambda \in \alpha.\sigma_{\frac{\epsilon}{|\alpha|}}(A)$ □

Corollary 2.2 *Let $A \in \mathfrak{B}(H)$ and P be a nontrivial projection. Then,*

$$\sigma_\epsilon (P \otimes A) = d(0,\epsilon) \cup \sigma_\epsilon (A) = \sigma_\epsilon (A \otimes P).$$

Theorem 2.3 *Let $A_1 \in \mathfrak{B}(H)$ satisfies ${A_1}^2 = 0$ and $A_2 \in \mathfrak{B}(H)$, then*

$$\sigma_\epsilon(A_1 \otimes A_2) = d(0, \sqrt{\epsilon^2 + \| A_1 \| \| A_2 \| \epsilon}) = \sigma_\epsilon(A_2 \otimes A_1).$$

Proof By Proposition 2.4 in [3]. □

Corollary 2.3 *Let $x_1, x_2 \in H$ be arbitrary and $(x_1, x_2)_H = 0$. Now, for $A \in \mathfrak{B}(H)$, $\epsilon > 0$,*

$$\sigma_\epsilon((x_1 \odot x_2) \otimes A) = d(0, \sqrt{\epsilon^2 + \| x_1 \| \| x_2 \| \| A_2 \| \epsilon}) = \sigma_\epsilon(A \otimes (x_1 \odot x_2)).$$

References

[1] Bendaoud M, Benyouness A, Sarih M. Preservers of pseudo spectral radius of operator products. Linear Algebra and its Applications 2016; 489: 186-198. https://doi.org/10.1016/j.laa.2015.10.022

[2] Brown A, Percy C. Spectra of tensor products of operators. Proceedings of the American Mathematical Society 1966; 17: 162-166. https://doi.org/10.2307/2035080

[3] Cui J, Li C, Poon Y. Pseudospectra of special operators and pseudospectrum preservers. Journal of Mathematical Analysis and Applications 2014; 419 (2): 1261-1273. https://doi.org/10.1016/j.jmaa.2014.05.041

[4] Dixmier J. Les Algèbras D'opèrateurs Dans L'espace Hilbertien. Paris: Gauthier-Villars, 1957.

[5] Gohberg I, Goldberg S, Kaashoek MA. Classes of Linear Operators vol 1. Basel: Birkhäuser, 1990.

[6] Kubrusly CS. A concise introduction to tensor product. Far East Journal of Mathematical Sciences 2006; 22: 137-174.

[7] Kubrusly CS, Levan N. Preservation of tensor sum and tensor product. Acta Mathematica Universitatis Comenianae 2011; 80: 132-142.

[8] Kumar GK, Kulkarni SH. Linear maps preserving pseudospectrum and condition spectrum. Banach Journal of Mathematical Analysis 2012; 6 (1): 45-60.

[9] Sukumar D. Some comparative results on eigenvalues, pseudospectra and conditionspectra. The Journal of Analysis 2021; 29: 607-617. https://doi.org/10.1007/s41478-019-00201-4

[10] Trefethen LN. Computation of pseudospectra. Acta Numerica 1999; 8: 247-295. https://doi.org/10.1017/S0962492900002932

[11] Trefethen LN, Embree M. Spectra and Psedospectra, The Behavior of Nonormal Matrices and Operators. Princeton: Princeton University Press, 2005. https://doi.org/10.2307/j.ctvzxx9kj.

[12] Trefethen LN. Pseudospectra of linear operators. SIAM Rev. 1997; 39 (3): 383-406.