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AYKUT HAS

BEYHAN YILMAZ

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A non-Newtonian conics in multiplicative analytic geometry

Aykut HAS*, **Beyhan YILMAZ**
Department of Mathematics, Faculty of Science, Kahramanmaraş Sütcü İmam University, Kahramanmaraş, Turkiye

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Abstract: In this study, conics (circle, ellipse, hyperbola) are characterized by taking into account basic multiplication operations in multiplicative space. For this purpose, firstly multiplicative axes and regions are introduced. Additionally, the multiplicative cone definition is given and visualized on the figure. General definitions and theorems of non-Newtonian conics are given. Additionally, examples were given and drawings were made to make the resulting characterizations and theorems more memorable.

Key words: Non-Newtonian calculus, multiplicative calculus, conics, multiplicative analytic geometry

1. Introduction

Arithmetic is traditionally thought of as the process of measuring and calculating using basic arithmetic operations (addition, subtraction, multiplication, division) to explain relationships between numbers and solve problems. However, in a more technical definition, arithmetic represents an ordered field structure over a special subset of the set of real numbers. The operations (addition, multiplication) and ordering relationship that constitute this sequential field structure can be expressed in a different way than traditional definitions. These definitions are achieved by obtaining the subset that will form the basis of arithmetic from the set of real numbers through a special transformation. This transformation is called the 'generator' function. The generator function is used to map the set of real numbers and the subset of this set in a one-to-one and onto way. Now let us examine the definition of α-arithmetic, one of the traditional non-Newtonian calculation methods using generator functions. The set $\{\alpha(x) : x \in \mathbb{R}\}$ is called the set of real numbers according to the non-Newtonian calculation style and is denoted by $\mathbb{R}_\alpha$. The basic operations in the set $\mathbb{R}_\alpha$ are given below.

Table 1. Basic $\alpha-$operations.

$a +_\alpha b$	$\alpha\{\alpha^{-1}(a) + \alpha^{-1}(b)\}$
$a -_\alpha b$	$\alpha\{\alpha^{-1}(a) - \alpha^{-1}(b)\}$
$a \cdot_\alpha b$	$\alpha\{\alpha^{-1}(a)\alpha^{-1}(b)\}$
$a/_\alpha b$	$\alpha\{\alpha^{-1}(a)/\alpha^{-1}(b)\}$

The foundation of the relationship between differential and integral is laid by the famous mathematicians Newton and Leibniz in the 17th century. Derivative and integral are the basis for analysis and calculation. As a result of this, calculus or infinitesimal (Newton) calculus are developed. In this process, most scientists drew logical and conceptual conclusions from infinitely large and infinitely small quantities and applied them

*Correspondence: ahas@ksu.edu.tr
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to the concepts of functions, limits and integrals. Today, this analysis is the most used mathematical theory. Considering that addition is the most basic operation, this analysis is called additive or classical analysis. Non-Newtonian analysis based on multiplication is first examined by Volterra V. and Hostinsky B. [24]. In this context, this analysis is also called *"Multiplicative analysis"*. Following the definition of the Volterra analysis, Grossman M. and Katz R. conducted some new studies between 1972 and 1983 this led to the development of non-Newtonian analysis, which included basic definitions and concepts. As a result, various types of analysis is emerged. Examples of these analyzes are geometric analysis, bigeometric analysis, and anageometric analysis, etc. can be given. [14, 15]. Nowadays, multiplicative analysis has started to be used in almost every field. Multiplicative analysis is in a more advantageous position than classical analysis, especially in fractal dimensions [20, 21].In addition, the biggest advantage of multiplicative analysis is that many problems that cannot be solved by basic methods in traditional analysis can be solved by multiplicative analysis. For example, there is no traditional solution for $\int \cos(x^2)dx$. However, this integral can be solved simply by switching to non-Newtonian operations.

$$\int_* \cos_*(x^2)d_*x = e^{\int \frac{1}{x}\log(e^{cos(\log x^2)})} = e^{\frac{1}{2}\sin(\log x^2)+c}.$$

In this respect, multiplicative analysis has attracted the attention of many researchers on various subjects [2–9, 13, 16, 22, 23]. In addition, in recent studies, both non-Newtonian space and non-Newtonian oprations are used compared to the past. Books published by Georgiev S. serve as the main sources on this subject. This usage has made the use of non-Newtonian analysis both easier and more functional [10–12].

The relationship between multiplicative analysis and geometry is still very new. Books published by Georgiev S. are the main sources in this field. Georgiev's books *"Multiplicative Analytic Geometry"* and *"Multiplicative Differential Geometry"* give basic definitions and theorems for analytical geometry and differential geometry. Additionally, Kaya S. et al. investigated vectors in multiplicative space with their geometric properties [19]. Another Evren M.E. et al. rectifying curves are examined in multiplicative space [1]. Has A. and Yilmaz B. examined magnetic curves with multiplicative arguments [17]. Additionally, Has A. et al. constructed the multiplicative Lorentz-Minkowski space based entirely on multiplicative properties [18].

The primary objective of this study is to establish a novel coordinate system rooted in the concept of multiplicative distance within multiplicative space and to derive multiplicative conic sections utilizing this coordinate framework. In this context, a multiplicative coordinate system is formulated with an emphasis on multiplicative numbers. Within this coordinate system, multiplicative conic shapes such as multiplicative circles, multiplicative ellipses, and multiplicative hyperbolas are positioned, and their characterizations are ascertained. To facilitate a more comprehensive comprehension of the subject matter, visual representations are created using Geogebra.

2. Multiplicative calculus and multiplicative space

In this section, basic definitions and theorems will be given about the multiplicative space created by choosing the generator exponential function (exp). Generator function α is chosen as (exp) function. Svetlin's G. books will be used for the basic informations given in this section [10–12].

$$\begin{aligned} \alpha \;&:\; \mathbb{R} \to \mathbb{R}^+ \qquad\qquad & \alpha^{-1} : \mathbb{R}^+ \to \mathbb{R} \\ & a \to \alpha(a) = e^a & b \to \alpha^{-1}(b) = \log b \end{aligned}$$

By choosing the generator (exp) function as, a function is defined from real numbers to the positive side of real numbers. Thus, the set of real numbers in the multiplicative space is defined as follows

$$\mathbb{R}_* = \{exp(a) : a \in \mathbb{R}\} = \mathbb{R}^+ - \{0\}.$$

Similarly, positive and negative multiplicative numbers are as follows

$$\mathbb{R}_*^+ = \{exp(a) : a \in \mathbb{R}^+\} = (1, \infty)$$

and

$$\mathbb{R}_*^- = \{exp(a) : a \in \mathbb{R}^-\} = (0, 1).$$

Additionally, with the help of the function exp, the basic operations in the multiplicative space can be seen from Table 1, for all $a, b \in \mathbb{R}_*$, $b \neq 1$

Table 2. Basic multiplicative operations.

$a +_* b$	$e^{\log a + \log b}$	ab
$a -_* b$	$e^{\log a - \log b}$	$\frac{a}{b}$
$a \cdot_* b$	$e^{\log a \log b}$	$a^{\log b}$
$a /_* b$	$e^{\log a / \log b}$	$a^{\frac{1}{\log b}}$

Given by Table 2, a multiplicative structure is formed by the field $(\mathbb{R}_*, +_*, \cdot_*)$. Each element of the space $\mathbb{R}_*$ is referred to as a multiplicative number and is denoted by $a_* \in \mathbb{R}_*$, where $a_* = exp(a)$. For the sake of simplicity in notation, we will denote multiplicative numbers as $a \in \mathbb{R}_*$ instead of a_* in the rest of the study. In addition, the unit elements of multiplicative addition and multiplication operations are $0_* = 1$ and $1_* = e$, respectively.

Now some useful operations on the multiplicative space will be given. Multiplicative space is defined based on the absolute value multiplication operation. Since distance is an additive change in Newtonian (additive) space, the absolute value is defined as additive. However, since distance is a multiplicative change in multiplicative space, the multiplicative absolute value is as follows

$$|a|_* = \begin{cases} a, & a \geq 0_* \quad \text{or} \quad a \in [1, \infty) \\ -_* a, & a < 0_* \quad \text{or} \quad a \in (0, 1). \end{cases}$$

where $-_* a = 1/a$. Also in the multiplicative space we have,

$$a^{k*} = \underbrace{a \cdot_* a \cdot_* a \cdot_* \dots \cdot_* a}_{k-times} = e^{(\log a)^k}$$

for $a \in \mathbb{R}_*$ and $k \in \mathbb{R}$ and

$$a^{\frac{1}{2}*} = e^{(\log a)^{\frac{1}{2}}} = \sqrt[*]{a}.$$

The inverses of multiplicative addition and multiplication in multiplicative space are as follows, respectively

$$-_* a = 1/a, \quad a^{-1_*} = e^{\frac{1}{\log a}}.$$

We have the following formulas for $a, b \in \mathbb{R}_*$

$$(a +_* b)^{2*} = a^{2*} +_* e^2 \cdot_* a \cdot_* b +_* b^{2*}, \tag{2.1}$$

$$a^{2*} -_* b^{2*} = (a +_* b) \cdot_* (a -_* b). \tag{2.2}$$

The vector definition in n-dimensional multiplicative space $\mathbb{R}_*^n$ is given by

$$\mathbb{R}_*^n = \{(x_1, ..., x_n) : x_i \in \mathbb{R}_*, i \in 1, 2, 3, ..., n\}.$$

$\mathbb{R}_*^n$ is a vector space on $\mathbb{R}_*$ with the pair of operations

$$\begin{aligned} \mathbf{x} +_* \mathbf{y} &= (x_1 +_* y_1, ..., x_n +_* y_n) = (x_1 y_1, ..., x_n y_n), \\ a \cdot_* \mathbf{x} &= (a \cdot_* x_1, ..., a \cdot_* x_n) = (e^{\log a \log x_1}, ..., e^{\log a \log x_n}), \quad a \in \mathbb{R}_* \end{aligned}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}_*^n$.

Suppose that $P = (a_1, b_1), Q = (a_2, b_2) \in \mathbb{R}_*^2$. Define the multiplicative distance between the points P and Q as follows

$$d_*(P, Q) = |Q -_* P|_* = e^{\left((\log \frac{a_1}{a_2})^2 + (\log \frac{b_1}{b_2})^2\right)^{\frac{1}{2}}}. \tag{2.3}$$

Also, the multiplicative midpoint $M \in \mathbb{R}_*^2$ of the points $P, Q \in \mathbb{R}_*^2$ in the multiplicative space as

$$M = (1_* /_* e^2) \cdot_* (P +_* Q). \tag{2.4}$$

The symbol $\mathbb{E}_*^2$ will be used to denote the set $\mathbb{R}_*^2$ equipped with the multiplicative distance d_*. Let $\mathbf{x}$ and $\mathbf{y}$ be any two multiplicative vectors in the multiplicative vector space $\mathbb{R}_*^n$. The multiplicative inner product of the $\mathbf{x}$ and $\mathbf{y}$ vectors is

$$\langle \mathbf{x}, \mathbf{y} \rangle_* = e^{\langle \log \mathbf{u}, \log \mathbf{v} \rangle}.$$

Moreover, if the multiplicative vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other, a relation can be given as

$$\langle \mathbf{x}, \mathbf{y} \rangle_* = 0_*.$$

For a vector $\mathbf{x} \in \mathbb{R}_*^n$, the multiplicative norm of $\mathbf{x}$ is defined as follows,

$$\|\mathbf{x}\|_* = e^{\langle \log \mathbf{x}, \log \mathbf{x} \rangle^{\frac{1}{2}*}}.$$

The multiplicative cross product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{E}_*^3$ is defined by

$$\mathbf{x} \times_* \mathbf{y} = (e^{\log x_2 \log y_3 - \log x_3 \log y_2}, e^{\log x_3 \log y_1 - \log x_1 \log y_3}, e^{\log x_1 \log y_2 - \log x_2 \log y_1}).$$

Let $\mathbf{x}$ and $\mathbf{y}$ represent two unit direction multiplicative vectors in the multiplicative vector space. Let us denote by θ the multiplicative angle between the multiplicative unit vectors $\mathbf{x}$ and $\mathbf{y}$, so θ as follows

$$\theta = \arccos_*(e^{\langle \log \mathbf{x}, \log \mathbf{y} \rangle}).$$

For $\theta \in \mathbb{R}_*$, the definitions multiplicative sine, multiplicative cosine, multiplicative tangent and multiplicative cotangent are as follows

$$\begin{aligned}\sin_* \theta &= e^{\sin \log \theta},\\ \cos_* \theta &= e^{\cos \log \theta},\\ \tan_* \theta &= e^{\tan \log \theta},\\ \cot_* \theta &= e^{\cot \log \theta}.\end{aligned}$$

In addition, multiplicative trigonometric functions provide multiplicative trigonometric relations in parallel with classical trigonometric relations. For example, there is the equality $\sin_*^{2*} \theta + \cos_*^{2*} \theta = 1_*$. For other relations, see [10].

Let P be any point and $[v]_*$ is a multiplicative vector different than the vector 0_*. Then

$$l = \{X : X -_* P \in [v]_*\}$$

is called multiplicative line through P with multiplicative direction $[v]_*$. Here $[v]_* = \{t \cdot_* v : t \in \mathbb{R} \quad \text{and } v \text{ classical vector}\}$. In Figure 1, it is shown a multiplicatine line.

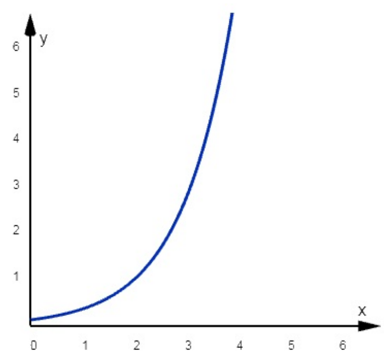

Figure 1. A multiplicative line.

Let $P, Q, R \in \mathbb{E}_*^2$ be multiplicative nonlinear points. The multiplicative triangle PQR is the multiplicative rectilinear figure consisting of the multiplicative segments PQ, PR and RQ. The multiplicative segments are called the multiplicative sides of the multiplicative triangle. In Figure 2, it is shown a multiplicative triangle.

Let $x, y, z \in \mathbb{R}_*$ be the multiplicative side lengths of the multiplicative right triangle. In this case, the multiplicative Pythagorean theorem is satisfied as follows, where the longest side of the multiplicative triangle is x

$$x^{2*} = y^{2*} +_* z^{2*} \tag{2.5}$$

or, equivalently

$$(\log x)^2 = (\log y)^2 + (\log z)^2.$$

3. Multiplicative conics

In this section, conics (circle, ellipse, hyperbola) will be re-characterized by considering basic multiplicative operations in multiplicative space. General definitions and theorems of multiplicative conics in multiplicative

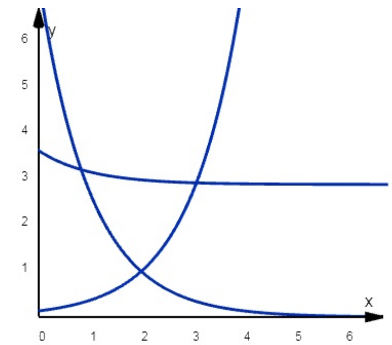

Figure 2. Multiplicative triangle.

space will be given. However, first of all, the multiplicative analytic axes, the multiplicative cone surface and the multiplicative plane, which are important for multiplicative conics, will be defined.

The multiplicative number line is formed by marking the numbers (e.g. e^1, e^2) on the e^x function. In the multiplicative analytic plane, it is represented by the multiplicative cartesian coordinate system. In this system, the plane is defined by two axes as, x_*-axis and y_*-axis. Each axis is represented by two multiplicative lines multiplicative orthogonal to each other at the point of origin $(0_*, 0_*)$, which is where they originally intersect. The line in multiplicative space is given in Figure 1. Therefore, the analytic axes of the multiplicative space are the points produced by the exp function, and we show this in Figure 3.

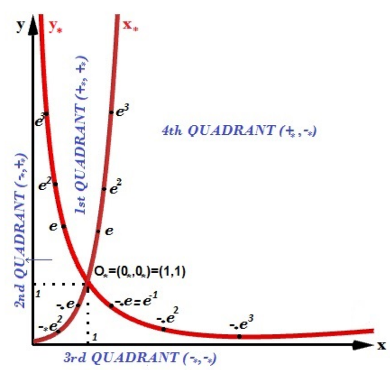

Figure 3. Multiplicative axes and quadrants.

The surface formed by a multiplicative line moving according to a multiplicative curve in multiplicative space and passing through a fixed point that is not in the product plane of the multiplicative curve is called the multiplicative cone surface for $h, k \in \mathbb{R}_*$ and the equation of the multiplicative cone is,

$$e^{(\log x - \log h)^2 + (\log y - \log k)^2}$$

or equally

$$(x -_* h)^{2*} +_* (y -_* k)^{2*} = r^{2*}, \tag{3.1}$$

where (h, k) is the coordinate of the vertex of the cone and r represents the radius of the cone.

Let $P(x,y)$ be a representative point in the multiplicative space. Let us also take a point $P_0(x_0,y_0)$ in the multiplicative space and a multiplicative vector $N(a,b)$ perpendicular to this point. Then the equation of the multiplicative plane is

$$\langle N, \overrightarrow{PP_0} \rangle_* = 0_*.$$

Another equivalent of this, for $a,b,c,d \in \mathbb{R}_*$

$$e^{\log a \log x + \log b \log y + \log c \log z + \log d} = 0_*$$

or

$$a \cdot_* x +_* b \cdot_* y +_* c \cdot_* z +_* d = 0_*. \tag{3.2}$$

In Figure 4 we show the multiplicative cone and the multiplicative plane.

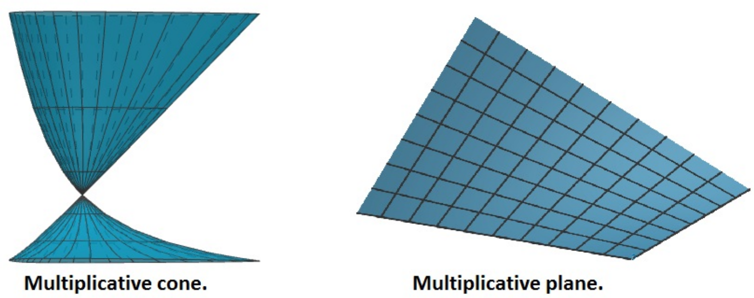

Figure 4. Multiplicative cone and plane, respectively.

Conics are obtained as a result of the intersection of a multiplicative cone and the multiplicative plane at different multiplicative angles. A multiplicative circle is formed when the multiplicative normal of the multiplicative cone and the multiplicative normal of the multiplicative plane intersect parallel. The multiplicative circle is seen in Figure 5.

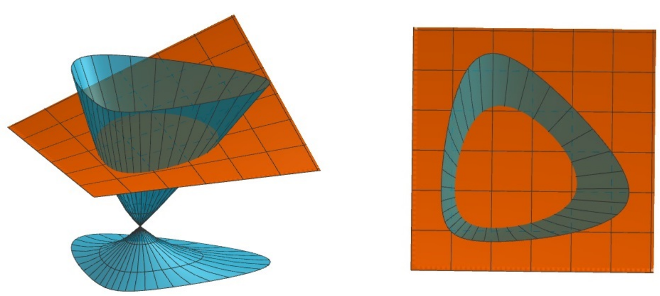

Figure 5. Circle formed by the intersection of the multiplicative cone and the plane.

Basic analytical definitions, theorems and characterizations of multiplicative conics in the multiplicative Cartesian coordinate system will be given in the following sections.

A multiplicative ellipse occurs when the multiplicative direction vector of a multiplicative cone and the multiplicative direction vector of the multiplicative plane intersect with a narrow multiplicative angle. We give the multiplicative ellipse in Figure 6.

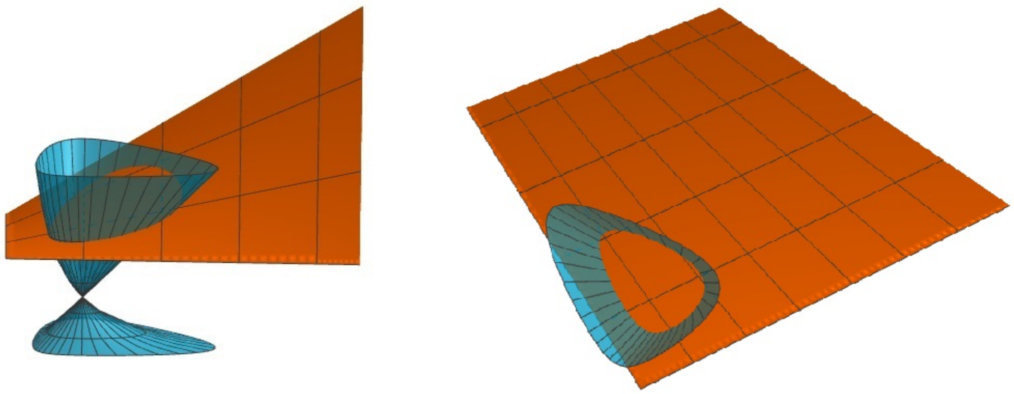

Figure 6. Ellipse formed by the intersection of the multiplicative cone and the plane.

Finally, a multiplicative hyperbola, is obtained by the multiplicative perpendicular intersection of the multiplicative cone and the multiplicative direction vectors of the multiplicative plane. The multiplicative hyperbola is shown in Figure 7.

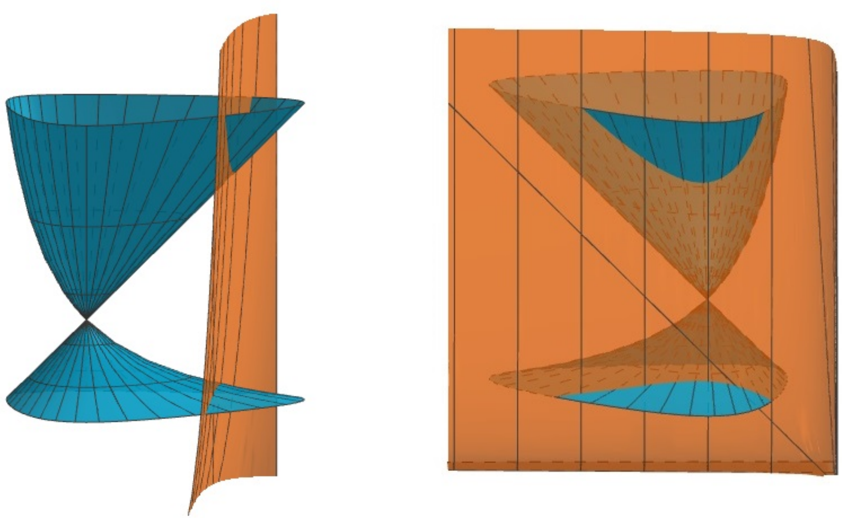

Figure 7. Hyperbola formed by the intersection of the multiplicative cone and the plane.

3.1. Multiplicative circle

Definition 3.1 *The geometric location of points with a multiplicative constant distance* $\mathbf{r} \in \mathbb{R}_*$ *to a multiplicative constant point* M *in the multiplicative plane* $\mathbb{E}_*^2$ *is called the multiplicative circle with radius* $\mathbf{r}$ *centered at* M *. Let the set of such points be denoted by* $\mathbb{S}_*^1$ *. Then* $\mathbb{S}_*^1$ *is*

$$\mathbb{S}_*^1 = \left\{ P \in \mathbb{E}_*^2 : \|\overrightarrow{PM}\|_* = \mathbf{r} \right\}. \tag{3.3}$$

In Figure 8, we present the graph of the multiplicative unit circle.

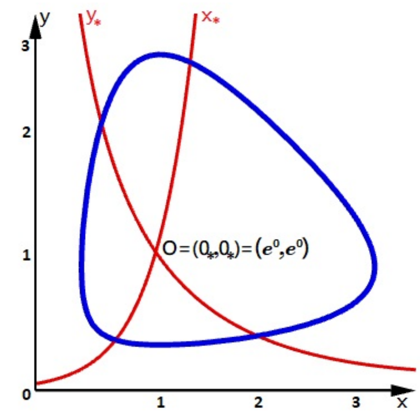

Figure 8. Multiplicative central circle.

All points on the multiplicative circle centered at $M = (a, b)$ for $a, b \in \mathbb{R}_*$ have the same multiplicative distance from the center of the multiplicative circle. Let $P = (x, y)$ be the representation point of points $P_i = (x_i, y_i)$, $(i = 1, 2, 3, 4)$. Then we can see that

$$\|\overrightarrow{P_1M}\|_* = \|\overrightarrow{P_2M}\|_* = \|\overrightarrow{P_3M}\|_* = \|\overrightarrow{P_4M}\|_* = r$$

or

$$\|\overrightarrow{PM}\|_* = r, \tag{3.4}$$

where $r \in \mathbb{R}_*$.

In Figure 9, we present the graph of the multiplicative circle with center $M = (a, b)$ for $a, b \in \mathbb{R}_*$ and radius $r \in \mathbb{R}_*$. Although the distances of the P points to the center $M = (a, b)$ seem to be different in Figure 9, it should not be ignored that the graph is drawn with the traditional distance-sensitive Mathematica program.

Let us consider Eqs. (2.3) and (3.4) to obtain the general equation of the multiplicative circle with center $M = (a, b)$ for $a, b \in \mathbb{R}_*$ and radius $r \in \mathbb{R}_*$, we get

$$\begin{aligned} \|\overrightarrow{PM}\|_* &= (\langle \overrightarrow{PM}, \overrightarrow{PM} \rangle_*)^{\frac{1}{2}*} \\ &= e^{\left((\log \frac{x}{a})^2 + (\log \frac{y}{b})^2\right)^{\frac{1}{2}}}. \end{aligned}$$

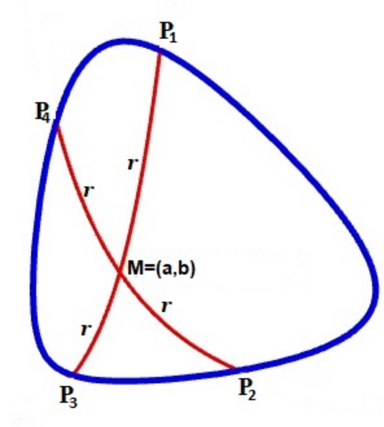

Figure 9. Multiplicative circle with center $M = (a, b)$ and radius r.

Considering that $\|\overrightarrow{PM}\|_*^{2*} = r^{2*}$, it like this

$$\begin{aligned}\|\overrightarrow{PM}\|_*^{2*} &= e^{(\log \frac{x}{a})^2 + (\log \frac{y}{b})^2} \\ &= e^{(\log \frac{x}{a})^2} +_* e^{(\log \frac{y}{b})^2} \\ &= \left(\frac{x}{a}\right)^{2*} +_* \left(\frac{y}{b}\right)^{2*} \\ &= (x -_* a)^{2*} +_* (y -_* b)^{2*}\end{aligned}$$

and finally, we get

$$(x -_* a)^{2*} +_* (y -_* b)^{2*} = r^{2*}. \tag{3.5}$$

Additionally, if Eq. (3.5) is arranged, another expression of the multiplicative circle is as follows

$$x^{2*} -_* e^2 \cdot_* a \cdot_* x +_* a^{2*} +_* y^{2*} -_* e^2 \cdot_* b \cdot_* y +_* b^{2*} - r^{2*} = 0_*.$$

If we take $-_* e^2 \cdot_* a = D$, $-_* e^2 \cdot_* b = E$ and $a^{2*} +_* b^{2*} - r^{2*} = F$, general multiplicative circle equation, is

$$x^{2*} +_* y^{2*} +_* D \cdot_* x +_* E \cdot_* y +_* F = 0_*. \tag{3.6}$$

Theorem 3.2 *Let the multiplicative circle with center $M = (a, b)$ for $a, b \in \mathbb{R}_*$ and radius $r \in \mathbb{R}_*$ be given. When considered together with Eq. (3.6), the center M and the radius r of the multiplicative circle are as follows*

$$M = (-_* D /_* e^2, -_* E /_* e^2), \tag{3.7}$$

$$r = (D^{2*} +_* E^{2*} -_* e^4 \cdot_* F)^{\frac{1}{2}*} /_* e^2. \tag{3.8}$$

Proof Let us consider the general multiplicative circle equation in Eq. (3.6) and for $a \in \mathbb{R}_*$, $a \cdot_* a^{-1*} = e$, the following can be easily seen

$$\begin{aligned} a &= -_* D /_* e^2, \\ b &= -_* E /_* e^2. \end{aligned}$$

From here too, we have

$$M = (-_* D /_* e^2, -_* E /_* e^2).$$

Similarly consider the general multiplicative circle equation in Eq. (3.6) writes as

$$r^{2*} = a^{2*} +_* b^{2*} -_* F$$

or

$$e^{(\log r)^2} = e^{(\log a)^2} +_* e^{(\log b)^2} -_* F.$$

After that, as follow

$$\begin{aligned} e^{(\log r)^2} &= e^{(\log(e^{\log D / \log e^2}))^2} +_* e^{(\log(e^{\log E / \log e^2}))^2} -_* F \\ &= e^{\log e^{(\log(e^{\log D / \log e^2}))^2} + \log e^{(\log(e^{\log E / \log e^2}))^2} - \log F} \\ &= e^{(\log D)^2 -_* (\log e^2)^2 + (\log E)^2 -_* (\log e^2)^2 - \log F} \\ &= D^{2*} /_* e^4 +_* E^{2*} /_* e^4 -_* F. \end{aligned}$$

Upon this, we write as

$$e^{\sqrt{(\log r)^2}} = (D^{2*} +_* E^{2*} -_* e^4 \cdot_* F)^{\frac{1}{2}*} /_* e^2$$

or

$$r = (D^{2*} +_* E^{2*} -_* e^4 \cdot_* F)^{\frac{1}{2}*} /_* e^2. \tag{3.9}$$

□

Corollary 3.3 *The following results are available for the general multiplicative circle equation given by Eq. (3.6)*

1. $D^{2*} +_* E^{2*} -_* e^4 \cdot_* F > 0_* \Rightarrow$ *Eq. (3.6) denotes a multiplicative circle.*
2. $D^{2*} +_* E^{2*} -_* e^4 \cdot_* F = 0_* \Rightarrow$ *Eq. (3.6) denotes a point.*
3. $D^{2*} +_* E^{2*} -_* e^4 \cdot_* F < 0_* \Rightarrow$ *Eq. (3.6) does not have a solution in multiplicative space.*

Proof Let Eqs. (3.7) and (3.8) be used in Eq. (3.5), then, we get

$$\left(x +_* D /_* e^2\right)^{2*} +_* \left(y +_* E /_* e^2\right)^{2*} = (D^{2*} +_* E^{2*} -_* e^4 \cdot_* F) /_* e^4. \tag{3.10}$$

We also know that for multiplicative nonnegative number a, we have

$$(-_* a)^{2*} = e^{(\log \frac{1}{a})^2} = e^{(\log a)^2} \geq 0_*.$$

Considering this result for the right side of Eq. (3.10), it is seen that this is a multiplicative nonnegative number. Then, when $D^{2*} +_* E^{2*} -_* e^4 \cdot_* F < 0_*$, Eq. (3.10) has no solution in multiplicative space. Similarly, if $D^{2*} +_* E^{2*} -_* e^4 \cdot_* F > 0_*$, Eq. (3.10) indicates a multiplicative circle with radius $r \in \mathbb{R}_* - (0,1)$ and center $M = (-_* D /_* e^2, -_* E /_* e^2)$. Finally, if $D^{2*} +_* E^{2*} -_* e^4 \cdot_* F = 0_*$, Eq. (3.10) indicates a point. □

Definition 3.4 *Let the multiplicative circle in multiplicative space be given by Eq. (3.5). Here, if $a = 0_*$ and $b = 0_*$ specifically, Eq. (3.5) becomes as follows and is called the central multiplicative circle as*

$$x^{2*} +_* y^{2*} = r^{2*}. \tag{3.11}$$

Additionally, if $a = 0_*$, $b = 0_*$ and $r = 1_*$, Eq. (3.5) is called the multiplicative unit circle and is as follows

$$x^{2*} +_* y^{2*} = 1_*.$$

Corollary 3.5 *Let us take for multiplicative angle θ, $x = a +_* r \cdot_* \cos_* \theta$ and $y = b +_* r \cdot_* \sin_* \theta$ in the multiplicative circle given by Eq. (3.5), then*

$$\begin{aligned}
(x -_* a)^{2*} +_* (y -_* b)^{2*} &= (a +_* r \cdot_* \cos_* \theta -_* a)^{2*} +_* (b +_* r \cdot_* \sin_* \theta -_* b)^{2*} \\
&= (r \cdot_* \cos_* \theta)^{2*} +_* (r \cdot_* \sin_* \theta)^{2*} \\
&= e^{(\log(e^{\log r \log e^{\cos \log \theta}}))^2} +_* e^{(\log(e^{\log r \log e^{\sin \log \theta}}))^2} \\
&= e^{(\log r)^2 \cos^2 \log \theta + (\log r)^2 \sin^2 \log \theta} \\
&= e^{(\log r)^2} \\
&= r^{2*}.
\end{aligned}$$

So the multiplicative parametric equation of the multiplicative circle is

$$x = a +_* r \cdot_* \cos_* \theta, \tag{3.12}$$

$$y = b +_* r \cdot_* \sin_* \theta. \tag{3.13}$$

Let $a = 0_*$ and $b = 0_*$ be taken in Eqs. (3.12) and (3.13) respectively, then

$$\begin{aligned}
x &= r \cdot_* \cos_* \theta, \\
y &= r \cdot_* \sin_* \theta.
\end{aligned}$$

This equations are called multiplicative polar equations of the multiplicative central circle.

3.2. Multiplicative ellipse

Definition 3.6 *The set of points whose multiplicative sum of their multiplicative distances to two fixed points on the multiplicative plane is constant is called multiplicative ellipse. Both fixed points are called multiplicative focus of the ellipse. In Figure 10 we present the graph of the multiplicative ellipse with end vertices of major axis is $A'(-_* a, 0_*)$ and $A(a, 0_*)$, end vertices of the minor axis is $B'(0_*, -_* b)$ and $B(0_*, b)$, the multiplicative coordinates of its two focus are $F'(-_* c, 0_*)$ and $F(c, 0_*)$, where for all $a, b, c \in \mathbb{R}_*$.*

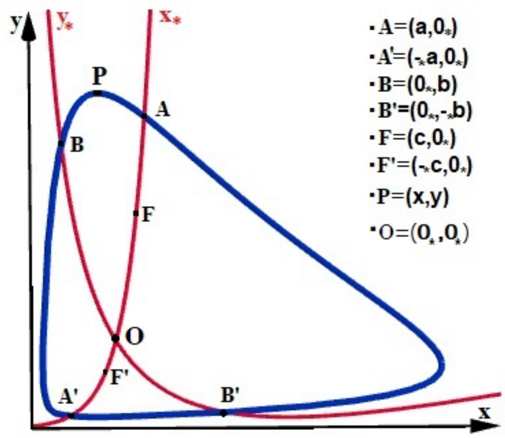

Figure 10. Multiplicative ellipse.

Let us consider the ellipse with focus $F'(-_*c, 0_*)$ and $F(c, 0_*)$ and principal multiplicative axis length $e^2 \cdot_* a$, as shown in Figure 10. Let the representation point of the multiplicative ellipse be $P(x,y)$ for $x, y \in \mathbb{R}_*$. In this case, taking into account Eq. (2.3), the following are given

$$\begin{aligned} d_*(P,F) &= e^{\left((\log \frac{x}{c})^2+(\log y)^2\right)^{\frac{1}{2}}}, \\ d_*(P,F') &= e^{\left((\log xc)^2+(\log y)^2\right)^{\frac{1}{2}}}. \end{aligned}$$

After that, we get

$$\begin{aligned} d_*^{2*}(P,F) &= e^{(\log \frac{x}{c})^2+(\log y)^2} = e^{(\log \frac{x}{c})^2} +_* e^{(\log y)^2} \\ d_*^{2*}(P,F') &= e^{(\log xc)^2+(\log y)^2} = e^{(\log xc)^2} +_* e^{(\log y)^2} \end{aligned}$$

or equivalently written as

$$d_*^{2*}(P,F) = (x -_* c)^{2*} + y^{2*}, \tag{3.14}$$

$$d_*^{2*}(P,F') = (x +_* c)^{2*} + y^{2*}. \tag{3.15}$$

As can be easily seen from Eqs. (2.1), (3.14), and (3.15), we have

$$d_*^{2*}(P,F') -_* d_*^{2*}(P,F) = e^4 \cdot_* c \cdot_* x. \tag{3.16}$$

We also know from the multiplicative ellipse definition that

$$d_*(P,F') +_* d_*(P,F) = e^2 \cdot_* a. \tag{3.17}$$

So, from Eqs. (3.16) and (3.17), as follow

$$(d_*(P,F') +_* d_*(P,F)) \cdot_* (d_*(P,F') -_* d_*(P,F)) = e^4 \cdot_* c \cdot_* x.$$

From this equation we get

$$\begin{aligned} d_*(P,F') -_* d_*(P,F) &= (e^4 \cdot_* c \cdot_* x)/_*(e^2 \cdot_* a) \\ &= e^{(\log e^{\log e^2 \log c \log x})/\log e^{\log a}} \\ &= (e^2 \cdot_* c \cdot_* x)/_* a. \end{aligned} \tag{3.18}$$

Let us consider Eqs. (3.17) and (3.18)

$$\begin{aligned} d_*(P,F') &= e^{\log a + \frac{\log c \log x}{\log a}} \\ &= e^{\log a} +_* e^{\frac{\log c \log x}{\log a}} \\ &= a +_* (c \cdot_* x)/_* a. \end{aligned} \tag{3.19}$$

Similarly

$$\begin{aligned} d_*(P,F) &= e^{\log a - \frac{\log c \log x}{\log a}} \\ &= e^{\log a} -_* e^{\frac{\log c \log x}{\log a}} \\ &= a -_* (c \cdot_* x)/_* a. \end{aligned} \tag{3.20}$$

From Eqs. (3.14) and (3.20), we have

$$(x -_* c)^{2*} + y^{2*} = (a -_* (c \cdot_* x)/_* a)^{2*}.$$

Let this equation can be arranged as follows

$$(a^{2*} -_* c^{2*}) \cdot_* x^{2*} +_* a^{2*} \cdot_* y^{2*} = a^{2*} \cdot_* (a^{2*} -_* c^{2*}). \tag{3.21}$$

Also, in the multiplicative triangle BOF in Figure 10, from the multiplicative Pythagorean theorem

$$a^{2*} = b^{2*} +_* c^{2*}$$

or equivalently written as

$$a^{2*} -_* c^{2*} = b^{2*}. \tag{3.22}$$

Let Eq. (3.22) be used in Eq. (3.21), following equation

$$b^{2*} \cdot_* x^{2*} +_* a^{2*} \cdot_* y^{2*} = a^{2*} \cdot_* b^{2*} \tag{3.23}$$

is obtained. Let us rearrange Eq. (3.23), we get

$$e^{\frac{(\log b)^2 (\log x)^2 + (\log a)^2 (\log y)^2}{(\log a)^2 (\log b)^2}} = e^{\frac{(\log a)^2 (\log b)^2}{(\log a)^2 (\log b)^2}}.$$

From now on as follow

$$e^{(\log x)^2/(\log a)^2} +_* e^{(\log y)^2/(\log b)^2} = e$$

or equivalently written as

$$(x^{2*}/_* a^{2*}) +_* (y^{2*}/_* b^{2*}) = 1_* \tag{3.24}$$

where Eq. (3.24) is called the *general ellipse equation*.

Conclusion 3.7 *Let the multiplicative ellipse equation whose center is the multiplicative origin be given by Eq. (3.24). Then the parametric equation of the multiplicative ellipse is*

$$x = a \cdot_* \cos_* \theta, \tag{3.25}$$

$$y = b \cdot_* \sin_* \theta. \tag{3.26}$$

Proof If we arrange the equation of the ellipse whose center is the multiplicative origin given by Eq. (3.24), we have following

$$e^{(\log x)^2/(\log a)^2} +_* e^{(\log y)^2/(\log b)^2}. \tag{3.27}$$

Let $x = a \cdot_* \cos_* \theta$ and $y = b \cdot_* \sin_* \theta$ be chosen in Eq. (3.27) so, we get

$$\begin{aligned} e^{(\log e^{\log a \log e^{\cos \log \theta} / \log a})^2 + (\log e^{\log b \log e^{\sin \log \theta} / \log b})^2} &= e^{(\cos \log \theta)^2 + (\sin \log \theta)^2} \\ &= e \\ &= 1_*. \end{aligned}$$

□

Definition 3.8 *Let a multiplicative ellipse with multiplicative length $AA' = e^2 \cdot_* a$ of the multiplicative major axis and multiplicative length $BB' = e^2 \cdot_* b$ of the multiplicative minor axis centered at $O = (0_*, 0_*)$ be given in the multiplicative space. The multiplicative circle centered at $O = (0_*, 0_*)$ with multiplicative radius a within the multiplicative ellipse is called the multiplicative major circle of the multiplicative ellipse. Similarly, the multiplicative circle centered at $O = (0_*, 0_*)$ with multiplicative radius length b inside the multiplicative ellipse is called the multiplicative minor circle of the multiplicative ellipse. The multiplicative mayor and minor circle of the multiplicative ellipse can be seen in Figure 11.*

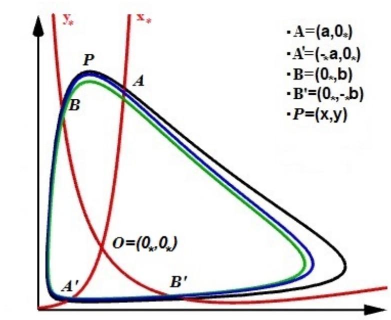

Figure 11. Multiplicative ellipse (Black) and their multiplicative major circle (Blue) and minor circle (Green).

3.3. Multiplicative hyperbola

Definition 3.9 *The geometric location of the points whose distances are constant to the constant points F and F' in the multiplicative plane is called a multiplicative hyperbola. Here, let the constant distance $\|\overrightarrow{AA'}\| = e^2 \cdot_* a$ and the representation point of the hyperbola be $P(x, y)$. Thus, the hyperbola can also be expressed as follows.*

$$H = \{P : d_*(P, F) -_* d_*(P, F') = e^2 \cdot_* a \text{ and } d_*(F, F') = e^2 \cdot_* c,\ a < c\} \tag{3.28}$$

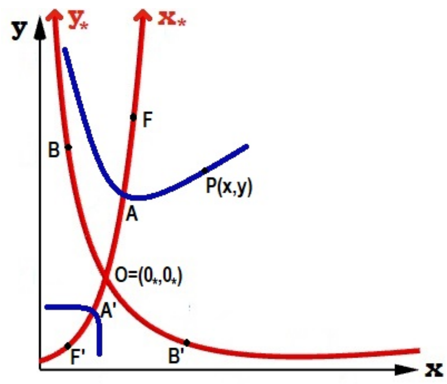

Figure 12. Multiplicative hyperbola.

In Figure 12 we present the multiplicative hyperbola, where A, A', B, B' and F, F' are called the major corners, minor corners and foci of the multiplicative hyperbola, respectively.

$$d_*(P,F) = e^{\left((\log\frac{c}{x})^2+(-\log y)^2\right)^{\frac{1}{2}}},$$
$$d_*(P,F') = e^{\left((\log cx)^2+(\log y)^2\right)^{\frac{1}{2}}},$$

then, we get

$$d_*^{2*}(P,F) = e^{(\log\frac{c}{x})^2+(\log y)^2} = e^{(\log\frac{c}{x})^2} +_* e^{(\log y)^2},$$
$$d_*^{2*}(P,F') = e^{(\log cx)^2+(\log y)^2} = e^{(\log cx)^2} +_* e^{(\log y)^2},$$

or equivalently written as

$$d_*^{2*}(P,F) = (c -_* x)^{2*} + y^{2*}, \quad (3.29)$$
$$d_*^{2*}(P,F') = (c +_* x)^{2*} + y^{2*}. \quad (3.30)$$

From Eqs. (3.29) and (3.30), as follows

$$d_*^{2*}(P,F') -_* d_*^{2*}(P,F) = e^4 \cdot_* c \cdot_* x. \quad (3.31)$$

Consider Eq. (2.2) in Eq. (3.31), so we write

$$(d_*(P,F') -_* d_*(P,F)) \cdot_* (d_*(P,F') +_* d_*(P,F)) = e^4 \cdot_* c \cdot_* x. \quad (3.32)$$

It is known from Eq. (3.28) that $d_*(P,F) -_* d_*(P,F') = e^2 \cdot_* a$. Then it is as follows

$$e^2 \cdot_* a \cdot_* (d_*(P,F') +_* d_*(P,F)) = e^4 \cdot_* c \cdot_* x$$

or

$$\begin{aligned} d_*(P,F') +_* d_*(P,F) &= e^{(\log e^4 \log c \log x)/(\log e^2 \log a)} \\ &= e^{(\log e^2 \log c \log x)/\log a} \\ &= (e^2 \cdot_* c \cdot_* x)/_* a \end{aligned}$$

Then the following results are obtained

$$\begin{aligned} d_*(P,F') +_* d_*(P,F) &= (e^2 \cdot_* c \cdot_* x)/_* a, && (3.33)\\ d_*(P,F) -_* d_*(P,F') &= e^2 \cdot_* a. && (3.34) \end{aligned}$$

As can be easily seen from here, the following equations, are obtained

$$\begin{aligned} e^2 \cdot_* d_*(P,F') &= (e^2 \cdot_* c \cdot_* x)/_* a +_* (e^2 \cdot_* a), \\ e^2 \cdot_* d_*(P,F) &= (e^2 \cdot_* c \cdot_* x)/_* a -_* (e^2 \cdot_* a). \end{aligned}$$

After that

$$\begin{aligned} d_*(P,F') &= a +_* (c \cdot_* x)/_* a, && (3.35)\\ d_*(P,F) &= -_* a +_* (c \cdot_* x)/_* a. && (3.36) \end{aligned}$$

Consider Eq. (3.30) and Eq. (3.35) together, we write as

$$(c +_* x)^{2*} + y^{2*} = (a +_* (c \cdot_* x)/_* a)^{2*}.$$

If both sides of this equation be done as in Eq. (2.1), we obtain

$$c^{2*} +_* e^2 \cdot_* c \cdot_* x +_* x^{2*} +_* y^{2*} = a^{2*} +_* e^2 \cdot_* a \cdot_* (c \cdot_* x)/_* a +_* (c^{2*} \cdot_* x^{2*})/_* a^{2*}$$

or

$$x^{2*} \cdot_* (c^{2*} -_* a^{2*}) = a^{2*} \cdot_* (c^{2*} -_* a^{2*}) +_* a^{2*} \cdot_* y^{2*}. \quad (3.37)$$

Also, in the multiplicative triangle PAO in Figure 12, from the multiplicative Pythagorean theorem

$$c^{2*} = a^{2*} +_* b^{2*}. \quad (3.38)$$

Let Eq. (3.38) be written in Eq. (3.37), so we get

$$x^{2*} \cdot_* b^{2*} -_* a^{2*} \cdot_* y^{2*} = a^{2*} \cdot_* b^{2*}. \quad (3.39)$$

If we arrange the equation, we can say

$$e^{(\log x)^2(\log b)^2/(\log a)^2(\log b)^2} -_* e^{(\log y)^2(\log a)^2/(\log a)^2(\log b)^2} = e^{(\log a)^2(\log b)^2/(\log a)^2(\log b)^2}$$

or

$$e^{\log e^{(\log x)^2/(\log a)^2} - \log e^{(\log y)^2/(\log b)^2}} = e. \quad (3.40)$$

Finally, it is obtained as follows

$$(x^{2*}/a^{2*}) -_* (y^{2*}/b^{2*}) = 1_*. \quad (3.41)$$

So, Eq. (3.41) is called the *general hyperbola equation.*

Conclusion 3.10 *Let the multiplicative hyperbola equation whose center is the multiplicative origin be given by Eq.* (3.41). *Then the parametric equation of the multiplicative hyperbola is*

$$x = a \cdot_* \sec_* \theta, \tag{3.42}$$
$$y = b \cdot_* \tan_* \theta. \tag{3.43}$$

Proof If we arrange the equation of the hyperbola whose center is the multiplicative origin given by Eq. (3.40), we get

$$e^{(\log x)^2/(\log a)^2} -_* e^{(\log y)^2/(\log b)^2}. \tag{3.44}$$

Let $x = a \cdot_* \sec_* \theta$ and $y = b \cdot_* \tan_* \theta$ be chosen in Eq. (3.44), we can se that

$$\begin{aligned} e^{(\log e^{\log a \log e^{\sec \log \theta} / \log a})^2 - (\log e^{\log b \log e^{\tan \log \theta} / \log b})^2} &= e^{(\sec \log \theta)^2 + (\tan \log \theta)^2} \\ &= e \\ &= 1_*. \end{aligned}$$

□

4. Conclusion

In order to study analytical geometry in multiplicative space, it is first necessary to create a coordinate system for this space. Because multiplicative space does not contain a line in the traditional sense. Therefore, the multiplicative coordinate system must consist of two multiplicative number lines that are multiplicative perpendicular to each other at their multiplicative starting points. The multiplicative coordinate system presented in Figure 3 is created taking these features into account. In the multiplicative coordinate system, distance is based on multiplication. For example, the multiplicative distance between e and e^2 is equal to the multiplicative distance between e^2 and e^3. Conical shapes are examined in this analytical plane. Although results parallel to the traditional results are obtained in terms of processing, completely different results are obtained from a geometrical perspective. Based on an approach different from the traditional concept of distance, different multiplicative conics are derived and basic definition theorems are presented.

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