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HEYBETKULU MUSTAFAYEV

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Individual stability of representations of abelian semigroups

Heybetkulu MUSTAFAYEV[1,2,*]
[1]General and Applied Mathematics Department, Azerbaijan State Oil and Industry University, Baku, Azerbaijan
[2]Western Caspian University, Baku, Azerbaijan

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Abstract: Let S be a suitable subsemigroup of a locally compact abelian group and let $\mathbf{T} = \{T(s)\}_{s\in S}$ be a bounded and strongly continuous representation of S on a Banach space X. In this note, we study the spectral conditions on $\mathbf{T}$ and the ergodic conditions on $x \in X$ which will imply that $T(s)x \to 0$ strongly through S.

Key words: Semigroup, representation semigroup, stability

1. Introduction and preliminaries

We will adopt the terminology of [3] and [4]. Let G be a locally compact abelian group equipped with the Haar measure dg and let S be a measurable subsemigroup of G such that $S - S = G$. We will assume that S contains the zero element of G and $\operatorname{int} S$ is dense in S. We will regard S as being ordered by " $\succcurlyeq$ ", where $s \succcurlyeq t$ if and only if $s - t \in S$. All limits over $s \in S$ will be with respect to this ordering. So, we will denote by $\lim_s$, the limit as $s \to \infty$ through S. The dual S^* of S, is the set of all nonzero, bounded and continuous homomorphisms of S into the multiplicative semigroup $\mathbb{C}$. By the *unitary characters* S_u^* of S we mean

$$S_u^* = \{\chi \in S^* : |\chi(s)| = 1, \ \forall s \in S\}.$$

Let $\widehat{G}$ be the dual group of G. Notice that each $\chi \in S_u^*$ can be extended uniquely to a character $\overline{\chi}$ of G in the following way: If $g \in G$, then as $g = s - t$ $(s, t \in S)$, we define

$$\overline{\chi}(g) := \chi(s)\overline{\chi(t)}.$$

So, we may identify S_u^* with the dual group of G. We will take S to be equipped with the restriction of the Haar measure. By $|E|$ we will denote the Haar measure of measurable subset E of S.

The space $L^1(S)$ will be identified with a subspace of $L^1(G)$. The space $L^1(S)$ is a commutative Banach algebra when convolution is taken as multiplication, where

$$(f_1 * f_2)(s) = \int\limits_{u+v=s} f_1(u) f_2(v)\, dv \quad \left(f_1, f_2 \in L^1(S)\right).$$

*Correspondence: hsmustafayev@yahoo.com
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The Gelfand space of $L^1(S)$ can be identified with S^*. The Gelfand transform of $f \in L^1(S)$ is just $\widehat{f}$, the Fourier transform of f;

$$\widehat{f}(\chi) = \int\limits_S \chi(s) f(s)\, ds \quad (\chi \in S^*).$$

The density of $\operatorname{int} S$ in S implies that the functions $\widehat{f}$ $\left(f \in L^1(S)\right)$ separate the points of S^* from each other and from zero.

Let us consider some examples.

a) Let $\mathbb{R}_+$ be the set of all nonnegative real numbers. If $G = \mathbb{R}^n$ and $S = \mathbb{R}^n_+$, then we can identify S^* with $\mathbb{C}^n_-$, where $\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$. The identification is given by $z \to \chi_z$, where

$$\chi_z(t) = \exp(-it \cdot z), \; t \cdot z = t_1 z_1 + ... + t_n z_n,$$

$t = (t_1, ..., t_n)$, and $z = (z_1, ..., z_n)$. For $f \in L^1\left(\mathbb{R}^n_+\right)$ and $z \in \mathbb{C}^n_-$, we have

$$\widehat{f}(z) = \int_{\mathbb{R}^n_+} f(t) \exp(-it \cdot z)\, dt$$

In this case, $S^*_u = \mathbb{R}^n$.

b) Let $\mathbb{Z}_+$ be the set of all nonnegative integers and $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. If $G = \mathbb{Z}^n$ and $S = \mathbb{Z}^n_+$, then we identify S^* with $\mathbb{D}^n$ by the relationship

$$\chi(k_1, ..., k_n) = z_1^{k_1} \cdot ... \cdot z_n^{k_n}.$$

For $f \in L^1\left(\mathbb{Z}^n_+\right)$, we have

$$\widehat{f}(z) = \sum_{k \in \mathbb{Z}^n_+} f(k)\, z_1^{k_1} \cdot ... \cdot z_n^{k_n}, \;\; k = (k_1, ..., k_n).$$

In this case, $S^*_u = \mathbb{T}^n$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. A family of operators $\mathbf{T} := \{T(s)\}_{s \in S}$ in $B(X)$ is said to be a *representation* of S on X if the following conditions are satisfied:

(*i*) $T(0) = I$, the identity operator on X;

(*ii*) $T(s+t) = T(s) T(t)$ for all $s, t \in S$;

(*iii*) $s \to T(s) x$ is a continuous function on S for every $x \in X$.

A representation $\mathbf{T} = \{T(s)\}_{s \in S}$ is said to be *bounded* if $\sup_{s \in S} \|T(s)\| < \infty$. By changing to an equivalent norm given by

$$\|x\|_1 = \sup_{s \in S} \|T(s) x\| \quad (x \in X),$$

a bounded representation $\mathbf{T}$ can be made contractive, i.e. $\|T(s)\| \leq 1$ for all $s \in S$. If $\mathbf{T} = \{T(s)\}_{s \in S}$ is a contractive representation of S on X, then for every $x \in X$, the limit $\lim_s \|T(s) x\|$ exists and is equal to $\inf_{s \in S} \|T(s) x\|$.

Note that the adjoint semigroup $\mathbf{T}^* := \left\{T(s)^*\right\}_{s\in S}$ may not be a representation of S, as strong continuity may fail. But $\mathbf{T}^*$ is a $\sigma(X^*, X)$-continuous representation of S on X^*. A character $\chi \in S^*$ is said to be an eigenvalue of $\mathbf{T}^*$ if there exists a nonzero functional $\varphi \in X^*$ such that $T^*(s)\varphi = \chi(s)\varphi$ for all $s \in S$.

Let $\mathbf{T} = \{T(s)\}_{s\in S}$ be a bounded representation of S on a Banach space X. Then, for an arbitrary $f \in L^1(S)$, we can define $\widehat{f}(\mathbf{T}) \in B(X)$ by

$$\widehat{f}(\mathbf{T})x = \int\limits_S f(s)T(s)x ds \quad (x \in X).$$

The map $f \mapsto \widehat{f}(\mathbf{T})$ is a continuous homomorphism from $L^1(S)$ into $B(X)$. The *spectrum* $\mathrm{sp}(\mathbf{T})$ of $\mathbf{T}$ is defined by

$$\mathrm{sp}(\mathbf{T}) = \left\{\chi \in S^* : \left|\widehat{f}(\chi)\right| \le \left\|\widehat{f}(\mathbf{T})\right\|, \ \forall f \in L^1(S)\right\}.$$

By $\mathrm{sp}_u(\mathbf{T})$ we will denote the *unitary spectrum* of $\mathbf{T}$;

$$\mathrm{sp}_u(\mathbf{T}) := \mathrm{sp}(\mathbf{T}) \cap S_u^*.$$

Recall that a Hausdorff topological space is said to be *scattered* if it does not contain a nonempty perfect subset. If S is second countable, then G and $\widehat{G}$ are also second countable. In this case, scattered subsets of $\widehat{G}$ are countable sets only.

Assume that unitary spectrum of the bounded representation $\mathbf{T} = \{T(s)\}_{s\in S}$ is a scattered set. The celebrated Arendt-Batty-Lyubich-Phóng (ABLP) Theorem [1, 4, 8, 11] asserts that if the adjoint semigroup $\mathbf{T}^*$ has no unitary eigenvalues, then $\mathbf{T}$ is *stable*, that is,

$$\lim_s \|T(s)x\| = 0 \ \text{ for all } x \in X.$$

For related results see also, [9, 10].

In this note, for the individual stability of $\mathbf{T}$ at $x \in X$, some suitable ergodic and spectral conditions are found on $\mathbf{T}$ and on x.

2. The main result

Recall that a net $\{K_i\}_{i\in I}$ of compact subsets of the abelian semigroup S is called a *Følner net* for S if $|K_i| > 0$ $(\forall i \in I)$ and

$$\lim_i \frac{|(s + K_i)\Delta K_i|}{|K_i|} = 0 \ \text{ uniformly for } s \text{ in a compact subsets of } S.$$

As is known [12, pp.131,145], there exists a Følner net for S. Moreover if S is σ-compact, then the net may be chosen to be a sequence. Notice that if $\{K_i\}_{i\in I}$ is a Følner net for S and $\chi \in S_u^*$, then

$$\lim_i \frac{1}{|K_i|} \int\limits_{K_i} \chi(s) ds = \begin{cases} 1, & \chi = 1; \\ 0, & \chi \ne 1. \end{cases}$$

The following result is an individual version of the ABLP Theorem.

Theorem 2.1 *Let* $\mathbf{T} = \{T(s)\}_{s\in S}$ *be a bounded representation of the abelian semigroup* S *on a Banach space* X *and let* $x \in X$. *Assume that:*

(*i*) *The unitary spectrum of* $\mathbf{T}$ *is a scattered set.*

(*ii*) *For a Følner net* $\{K_i\}_{i\in I}$ *for* S,

$$\lim_i \frac{1}{|K_i|}\int\limits_{K_i} |\langle \varphi, T(s)x\rangle|^{\alpha}\, ds = 0 \quad \text{for some } \alpha > 0 \text{ and for each } \varphi \in X^*.$$

Then, $\lim_s \|T(s)x\| = 0$.

It follows from the condition (ii) of Theorem 2.1 that the adjoint semigroup $\mathbf{T}^*$ has no unitary eigenvalues. For the proof of Theorem 2.1, we need some preliminary results.

Lemma 2.2 *Let* $\mathbf{T} = \{T(s)\}_{s\in S}$ *be a bounded representation of the abelian semigroup* S *on a Banach space* X *and let* $x \in X$. *Assume that for a Følner net* $\{K_i\}_{i\in I}$ *for* S,

$$\lim_i \frac{1}{|K_i|}\int\limits_{K_i} |\langle \varphi, T(s)x\rangle|^{\alpha}\, ds = 0 \quad \text{for some } \alpha > 0 \text{ and for each } \varphi \in X^*.$$

Then,

$$\lim_i \frac{1}{|K_i|}\int\limits_{K_i} |\langle \varphi, T(s)x\rangle|^{\beta}\, ds = 0 \quad \text{for every } \beta > 0 \text{ and for each } \varphi \in X^*.$$

Proof Let K be a compact subset of S and let f be a continuous function on K. If $0 < \beta \le \alpha$, then from the Hölder inequality

$$\int_K |f(s)|^{\beta}\, ds \le \left(\int_K |f(s)|^{\alpha}\, ds\right)^{\frac{\beta}{\alpha}} |K|^{1-\frac{\beta}{\alpha}},$$

we can write

$$\frac{1}{|K_i|}\int\limits_{K_i} |\langle \varphi, T(s)x\rangle|^{\beta}\, ds \le \left(\frac{1}{|K_i|}\int\limits_{K_i} |\langle \varphi, T(s)x\rangle|^{\alpha}\, ds\right)^{\frac{\beta}{\alpha}}.$$

If $\beta > \alpha$, then as $\beta = \alpha + \gamma$ $(\gamma > 0)$, we get

$$\frac{1}{|K_i|}\int\limits_{K_i} |\langle \varphi, T(s)x\rangle|^{\beta}\, ds \le (C\|x\|\|\varphi\|)^{\gamma} \frac{1}{|K_i|}\int\limits_{K_i} |\langle \varphi, T(s)x\rangle|^{\alpha}\, ds,$$

where $C = \sup_{s\in S}\|T(s)\|$. □

Let G be a locally compact abelian group and let $M(G)$ be the convolution measure algebra of G. Recall that the convolution product $\mu * \nu$ of two measures μ, $\nu \in M(G)$ is defined by

$$(\mu * \nu)(B) = \int\limits_G \mu(B - g)\, d\nu(g) \quad \text{for every Borel subset } B \text{ of } G.$$

The Fourier-Stieltjes transform of $\mu \in M(G)$ is defined by

$$\widehat{\mu}(\chi) = \int_G \overline{\chi(g)} d\mu(g) \quad \left(\chi \in \widehat{G}\right).$$

It is well known that if $\widehat{\mu}(\chi) = 0$ for all $\chi \in \widehat{G}$, then $\mu = 0$. Also, note that if $\{K_i\}_{i \in I}$ is a Følner net for S, then for an arbitrary $\mu \in M\left(\widehat{G}\right)$,

$$\lim_i \frac{1}{|K_i|} \int_{K_i} \widehat{\mu}(s)\, ds = \mu\{0\}. \tag{2.1}$$

For a closed subset K of $\widehat{G}$,

$$I_K := \left\{ f \in L^1(G) : \widehat{f}(\chi) = 0,\ \forall \chi \in K \right\}$$

is the largest closed ideal of $L^1(G)$ whose hull is K and $J_K := \overline{J_K^o}$ is the smallest closed ideal of $L^1(G)$ whose hull is K [6, Section 8.3], where

$$J_K^o = \left\{ f \in L^1(G) : supp \widehat{f} \cap K = \emptyset \right\}.$$

Let $\mathbf{T} = \{T(g)\}_{g \in G}$ be a representation of G by isometries on a Banach space X. The *Arveson spectrum* $\mathrm{sp}(\mathbf{T})$ of $\mathbf{T}$ [2] is defined as the hull of the closed ideal

$$I_{\mathbf{T}} := \left\{ f \in L^1(G) : \widehat{f}(\mathbf{T}) = 0 \right\},$$

where

$$\widehat{f}(\mathbf{T})x = \int_G f(g) T(g) x dg \quad (x \in X).$$

In other words,

$$\mathrm{sp}(\mathbf{T}) = \left\{ \chi \in \widehat{G} : \widehat{f}(\chi) = 0,\ \forall f \in I_{\mathbf{T}} \right\}.$$

Clearly,

$$J_{\mathrm{sp}(\mathbf{T})} \subseteq I_T \subseteq I_{\mathrm{sp}(\mathbf{T})}. \tag{2.2}$$

Note that the definition of $\mathrm{sp}(\mathbf{T})$ in the preceeding paragraph coincides with the definition of the Arveson spectrum [4].

Lemma 2.3 *Let G be a locally compact abelian group and let $\mathbf{U} = \{U(g)\}_{g \in G}$ be a unitary representation of G on a Hilbert space H. Assume that the Arveson spectrum of $\mathbf{U}$ is a scattered set. If $x \in H$ satisfies the condition*

$$\lim_i \frac{1}{|K_i|} \int_{K_i} |\langle U(s) x, x \rangle|^2\, ds = 0,$$

for a Følner net $\{K_i\}_{i \in I}$ for S, then $x = 0$.

Proof By Stone's theorem, there exists a spectral measure E on $\widehat{G}$ such that

$$U(g) = \int_{\widehat{G}} \overline{\chi(g)} dE(\chi) \quad (\forall g \in G).$$

Let $y \in H$ and let μ_y be the scalar measure defined on the Borel subsets of $\widehat{G}$, by

$$\mu_y(B) = \langle E(B)y, y\rangle = \|E(B)y\|^2.$$

For an arbitrary $f \in L^1(G)$, from the identity

$$\widehat{f}(\mathbf{U})y = \int_{\widehat{G}} \widehat{f}(\chi) d\mu_y(\chi),$$

we can write

$$\left\|\widehat{f}(\mathbf{U})y\right\|^2 = \int_{\widehat{G}} \left|\widehat{f}(\chi)\right|^2 d\mu_y(\chi).$$

Since

$$suppE = \bigcup_{y\in H} supp\mu_y,$$

it follows from the previous identity that $\widehat{f}(\mathbf{U}) = 0$ if and only if $\widehat{f}$ vanishes on $suppE$. Hence we have

$$I_{\mathbf{U}} = I_{suppE}$$

and so,

$$\mathrm{sp}(\mathbf{U}) = \mathrm{hull}(I_{suppE}) = suppE.$$

Further, since

$$supp\mu_x \subseteq suppE = \mathrm{sp}(\mathbf{U}),$$

$supp\mu_x$ is a scattered set. On the other hand,

$$\begin{aligned}\widehat{\mu_x}(g) &= \int_{\widehat{G}} \overline{\chi(g)} d\mu_x(\chi) \\ &= \int_{\widehat{G}} \overline{\chi(g)} d\langle E(\chi)x, x\rangle = \langle U(g)x, x\rangle.\end{aligned}$$

Now, let ν_x be the measure defined on the Borel subsets of $\widehat{G}$, by $\nu_x(B) = \mu_x(-B)$. Since

$$\widehat{\nu_x}(g) = \overline{\langle U(g)x, x\rangle},$$

we have

$$\widehat{\nu_x * \mu_x}(g) = |\langle U(g)x, x\rangle|^2 \quad (\forall g \in G).$$

Taking into account the identity (2.1), we can write

$$\begin{aligned} 0 &= \lim_i \frac{1}{|K_i|} \int_{K_i} |\langle U(s)x, x\rangle|^2 \, ds \\ &= \lim_i \frac{1}{|K_i|} \int_{K_i} \widehat{\nu_x * \mu_x}(s)\, ds \\ &= (\nu_x * \mu_x)\{0\} = \int_{\widehat{G}} \nu_x\{-\chi\}\, d\mu_x(\chi) \\ &= \sum_{\chi \in \widehat{G}} (\mu_x\{\chi\})^2 . \end{aligned}$$

It follows that μ_x is a continuous measure. But, in view of [5, p.52, Theorem 10], there is no nonzero continuous measure supported by scattered set. Thus we have $\mu_x = 0$. This clearly implies that $x = 0$. □

In the following result we use the method of [4, 8, 11] to construct an isometric representation $\mathbf{V} = \{V(s)\}_{s\in S}$ of the abelian semigroup S on a different Banach space.

Lemma 2.4 *Let $\mathbf{T} = \{T(s)\}_{s\in S}$ be a representation of the abelian semigroup S by contractions on a Banach space X. Then there exists a Banach space Y, a bounded linear map $J : X \to Y$ with dense range, and a representation $\mathbf{V} = \{V(s)\}_{s\in S}$ of S by isometries on Y with the following properties:*

(*a*) $\|Jx\| = \lim_s \|T(s)x\|$ *for every* $x \in X$.

(*b*) $V(s)J = JT(s)$ *for every* $s \in S$.

(*c*) $sp(\mathbf{V}) \subseteq sp(\mathbf{T})$.

The triple $(Y, J, \mathbf{V})$ will be called *isometric representation associated with* $\mathbf{T}$.

Let φ be a bounded and uniformly continuous function on a locally compact abelian group G. The w*-*spectrum* $\sigma_*(\varphi)$ of φ is defined as the hull of the closed ideal

$$I_\varphi := \left\{ f \in L^1(G) : \varphi * f = 0 \right\},$$

that is,

$$\sigma_*(\varphi) = \left\{ \chi \in \widehat{G} : \widehat{f}(\chi) = 0, \ \forall f \in I_\varphi \right\}.$$

The well-known theorem of Loomis [7] states that if the w*-spectrum of φ is a scattered set, then φ is an almost periodic function.

Next, we have the following.

Lemma 2.5 *Let G be a locally compact abelian group and let $\mathbf{V} = \{V(g)\}_{g\in G}$ be a representation of G by isometries on a Banach space X. Assume that the Arveson spectrum of $\mathbf{V}$ is a scattered set. Then, for an arbitrary $\varphi \in X^*$, there exists a Hilbert space H_φ, a bounded linear operator $J_\varphi : X \to H_\varphi$ with dense range, and a unitary representation $\mathbf{U}_\varphi = \{U_\varphi(g)\}_{g\in G}$ of G on H_φ, with the following properties:*

(*a*) $U_\varphi(g)J_\varphi = J_\varphi V(g)$, $\forall g \in G$.

(b) $sp(\mathbf{U}_\varphi) \subseteq sp(\mathbf{V})$.

(c) $\bigcap_{\varphi \in X^*} \ker J_\varphi = \{0\}$.

Proof Let $\varphi \in X^*$ be given. For $x \in X$, define the function x_φ on G, by

$$x_\varphi(g) := \langle \varphi, V(-g)\, x \rangle.$$

Then, x_φ is a bounded and uniformly continuous function on G. We claim that

$$\sigma_*(x_\varphi) \subseteq \mathrm{sp}(\mathbf{V}).$$

Suppose that there exists $\xi_0 \in \sigma_*(x_\varphi)$, but $\xi_0 \notin \mathrm{sp}(\mathbf{V})$. Then there exists $f \in L^1(G)$ such that $\widehat{f}(\xi_0) \neq 0$ and $\widehat{f}$ vanishes on a neighborhood of $\mathrm{sp}(\mathbf{V})$. In other words, f belongs to the smallest closed ideal of $L^1(G)$ whose hull is $\mathrm{sp}(\mathbf{V})$. It follows from (2.2) that $\widehat{f}(\mathbf{V}) = 0$. Consequently, we can write

$$\begin{aligned}(x_\varphi * f)(g) &= \int_G f(s)\, \langle \varphi, V(s-g)\, x \rangle ds \\ &= \langle \varphi, V(-g) \int_G f(s)\, \langle V(s)\, x \rangle ds \\ &= \langle \varphi, V(-g)\, \widehat{f}(\mathbf{V})\, x \rangle = 0 \quad (\forall g \in G).\end{aligned}$$

Since $\xi_0 \in \sigma_*(x_\varphi)$, we have $\widehat{f}(\xi_0) = 0$. This contradiction proves the claim. Hence, $\sigma_*(x_\varphi)$ is a scattered set. By the Loomis Theorem, x_φ is an almost periodic function.

Let H_φ^0 denote the linear set $\{x_\varphi : x \in X\}$ with the inner product defined by

$$\langle x_\varphi, y_\varphi \rangle = \Phi_g \left[x_\varphi(g)\, \overline{y_\varphi(g)} \right] \quad (y \in X),$$

where Φ is the invariant mean on the space of almost periodic functions on G. Let H_φ be the completion of H_φ^0 with respect to the induced norm. Then, H_φ is a Hilbert space. Notice also that

$$\|x_\varphi\| \leq \|x_\varphi\|_\infty \leq \|\varphi\| \, \|x\|.$$

It follows that the map $J_\varphi : X \to H_\varphi$, defined by $J_\varphi x = x_\varphi$, is a bounded linear operator with dense range.

For an arbitrary $g \in G$, define the map $U_\varphi(g) : H_\varphi \to H_\varphi$, by

$$U_\varphi(g)\, x_\varphi = (V(g)\, x)_\varphi.$$

Then, $\mathbf{U}_\varphi := \{U_\varphi(g)\}_{g \in G}$ is a unitary representation of G on H_φ and

$$U_\varphi(g)\, J_\varphi = J_\varphi V(g) \quad (\forall g \in G). \tag{2.3}$$

Moreover, we have

$$\bigcap_{\varphi \in X^*} \ker J_\varphi = \{0\}.$$

Now, let us show that $\mathrm{sp}(\mathbf{U}_\varphi) \subseteq \mathrm{sp}(\mathbf{V})$. To see this, let $\chi \in \mathrm{sp}(\mathbf{U}_\varphi)$ and let $f \in L^1(G)$ be such that $\widehat{f}(\mathbf{V}) = 0$. We must show that $\widehat{f}(\chi) = 0$. Indeed, from the identity (2.3), we have

$$\widehat{f}(\mathbf{U}_\varphi) J_\varphi = J_\varphi \widehat{f}(\mathbf{V}),$$

which implies $\widehat{f}(\mathbf{U}_\varphi) J_\varphi = 0$. Since J_φ has dense range, $\widehat{f}(\mathbf{U}_\varphi) = 0$. Also, since $\chi \in \mathrm{sp}(\mathbf{U}_\varphi)$, we have $\widehat{f}(\chi) = 0$. □

The triple $(H_\varphi, J_\varphi, \mathbf{U}_\varphi)$ will be called *unitary representation associated with the pair* $(\mathbf{V}, \varphi)$.

Lemma 2.6 *Let G be a locally compact abelian group and let $\mathbf{V} = \{V(g)\}_{g \in G}$ be a representation of G by isometries on a Banach space X. Assume that the Arveson spectrum of $\mathbf{V}$ is a scattered set. If $x \in X$ satisfies the condition*

$$\lim_i \frac{1}{|K_i|} \int_{K_i} |\langle \varphi, V(s)x \rangle|^2 \, ds = 0,$$

for a Følner net $\{K_i\}_{i \in I}$ for S and for every $\varphi \in X^*$, then $x = 0$.

Proof Let $\varphi \in X^*$ and let $(H_\varphi, J_\varphi, \mathbf{U}_\varphi)$ be the unitary representation associated with the pair $(\mathbf{V}, \varphi)$. In view of Lemma 2.5,

$$\begin{aligned} \langle U_\varphi(g) J_\varphi x, J_\varphi x \rangle &= \overline{\langle J_\varphi x, U_\varphi(g) J_\varphi x \rangle} \\ &= \overline{\langle J_\varphi x, J_\varphi V(g) x \rangle} = \overline{\langle J_\varphi^* J_\varphi x, V(g) x \rangle} \quad (\forall g \in G). \end{aligned}$$

Consequently, we have

$$\lim_i \frac{1}{|K_i|} \int_{K_i} |\langle U_\varphi(s) J_\varphi x, J_\varphi x \rangle|^2 \, ds = \lim_i \frac{1}{|K_i|} \int_{K_i} \left| \langle J_\varphi^* J_\varphi x, V(s)x \rangle \rangle \right|^2 ds = 0.$$

By Lemma 2.3, $J_\varphi x = 0$ for all $\varphi \in X^*$. Taking into account Lemma 2.5, we get that $x = 0$. □

Let $\mathbf{V} = \{V(s)\}_{s \in S}$ be a representation of the abelian semigroup S by isometries on a Banach space X. In [3, Theorem 5.1], it was proved that if $\mathrm{sp}_u(\mathbf{V})$ is a scattered set, then each $V(s)$ $(s \in S)$ is an invertible isometry. Consequently, $\mathbf{V}$ extends to an isometric representation $\mathbf{W} = \{W(g)\}_{g \in G}$ of G on X by defining

$$W(g) = V(s) V(t)^{-1}, \text{ where } g = s - t.$$

In this case, $\mathrm{sp}_u(\mathbf{V})$ coincides with the Arveson spectrum of $\mathbf{W}$ [3].

Now, we are in a position to prove Theorem 2.1.

Proof [Proof of Theorem 2.1.] By changing to an equivalent norm, the representation $\mathbf{T}$ can be made contractive (renorming does not change the spectral assumptions). By Lemma 2.2, we may assume that

$$\lim_i \frac{1}{|K_i|} \int_{K_i} |\langle \varphi, T(s)x \rangle|^2 \, ds = 0 \text{ for all } \varphi \in X^*.$$

Let $(Y, J, \mathbf{V})$ be the isometric representation associated with $\mathbf{T}$. By Lemma 2.4,

$$\mathrm{sp}_u(\mathbf{V}) \subseteq \mathrm{sp}_u(\mathbf{T}),$$

which implies that the unitary spectrum of $\mathbf{V}$ is also a scattered set. As we have noted above, $\mathbf{V}$ extends to an isometric representation $\mathbf{W} = \{W(g)\}_{g\in G}$ of G on Y and $\mathrm{sp}_u(\mathbf{V}) = \mathrm{sp}(\mathbf{W})$. Consequently, we have $\mathrm{sp}(\mathbf{W}) \subseteq \mathrm{sp}_u(\mathbf{T})$ and therefore, $\mathrm{sp}(\mathbf{W})$ is a scattered set. If $y^* \in Y^*$, then by Lemma 2.4,

$$\langle y^*, V(s)Jx\rangle = \langle y^*, JT(s)x\rangle = \langle J^*y^*, T(s)x\rangle \quad (\forall s \in S).$$

Since $W(s) = V(s)$ $(\forall s \in S)$, we can write

$$\begin{aligned} \lim_i \frac{1}{|K_i|} \int\limits_{K_i} |\langle y^*, W(s)Jx\rangle|^2 \, dg &= \lim_i \frac{1}{|K_i|} \int\limits_{K_i} |\langle y^*, V(s)Jx\rangle|^2 \, ds \\ &= \lim_i \frac{1}{|K_i|} \int\limits_{K_i} |\langle J^*y^*, T(s)x\rangle|^2 \, ds = 0. \end{aligned}$$

By Lemma 2.6, $Jx = 0$. By Lemma 2.4, this means that $\lim_s \|T(s)x\| = 0$. □

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.7 *Let* $\mathbf{T} = \{T(s)\}_{s\in S}$ *be a bounded representation of the abelian semigroup* S *on a Banach space* X *with scattered unitary spectrum. If* $x \in X$ *and* $\lim_s T(s)x = 0$ *weakly, then* $\lim_s \|T(s)x\| = 0$.

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