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A short note on generalized Robertson Walker spacetimes

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Abstract: In this article, generalized Robertson Walker spacetimes are investigated in light of perfect fluid spacetimes. First, we establish that a perfect fluid spacetime with nonvanishing vorticity whose associated scalars are constant along the velocity vector field becomes a generalized Robertson Walker spacetime. Among others, it is also shown that a Ricci parallel perfect fluid spacetime is either a generalized Robertson Walker spacetime or a static spacetime. Finally, we acquire that in a conformally semisymmetric generalized Robertson Walker spacetime of dimension 4, the scalar curvature vanishes and the spacetime is locally isometric to the Minkowski spacetime, provided the electric part of the Weyl tensor vanishes. Moreover, it is established that the last result also holds in a conformally recurrent generalized Robertson Walker spacetime.

Key words: *GRW* spacetimes, conformally semisymmetry, perfect fluid spacetimes

1. Introduction

Lorentzian manifold is a particular instance of a semi (or pseudo)-Riemannian manifold with the metric tensor g whose signature is $(-,+,+,\ldots,+)$, that is, the index is one. Some of the most significant theories in contemporary physics, including string theory and general relativity, are based on the mathematical foundation of Lorentzian geometry. A Lorentzian manifold M, from a purely mathematical perspective, is a smooth manifold equipped with a nondegenerate bilinear symmetric metric g. Generally, there might not be a universally time-like vector on (M, g). If it accommodates a globally time-like vector, then it is referred to be a spacetime. Many researchers have investigated spacetimes in numerous ways, like [8, 13, 17, 20, 21, 30] and many others.

In a semi (or pseudo)-Riemannian manifold M^n $(n \geq 4)$, let g be a semi-Riemannian metric with signature (p, m), in which $p + m = n$. M^n equipped with g is referred to as a Lorentzian manifold [23] if g is a Lorentzian metric with signature $(n-1, 1)$ or $(1, n-1)$. If $M = -I \times_{\mathfrak{f}} \mathcal{M}$, then M is called a generalized Robertson-Walker (GRW) spacetime [2, 7], where $\mathcal{M}$ indicates an $(n-1)$-dimensional Riemannian manifold, $I \in \mathbb{R}$ (set of real numbers) is an open interval, and $\mathfrak{f} > 0$ stands for a smooth function. In particular, GRW spacetime becomes Robertson-Walker (RW) spacetime if we assume that $\mathcal{M}$ is a 3-dimensional Riemannian manifold of constant curvature. One is well aware that the GRW spacetimes include the Lorentzian Minkowski spacetimes, the Friedmann cosmological models, the Einstein-de Sitter spacetimes, the static Einstein spacetimes, the de Sitter spacetimes [24].

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The energy momentum tensor T_{kl} in a perfect fluid spacetime (PFS) is written by

$$T_{kl} = (\sigma + p)u_k u_l + p g_{kl}. \tag{1.1}$$

In here, p and σ denote the isotropic pressure and energy density [23]. In equation (1.1), the velocity vector is described as $g_{kl}u^k u^l = -1$ and $u_k = g_{kl}u^l$.

Without a cosmological constant, the Einstein's field equations (EFEs) are stated as

$$R_{kl} - \frac{R}{2} g_{kl} = \kappa T_{kl}, \tag{1.2}$$

where κ indicates gravitational constant, R^h_{kli} is the curvature tensor of type $(1,3)$, $R = g^{kl}R_{kl}$ and $R_{kl} = R^i_{kli}$ stand for the scalar curvature and the Ricci tensor, respectively.

M is called a PFS if it is a nonvanishing Ricci tensor R_{kl} obeys

$$R_{kl} = b u_k u_l + a g_{kl}, \tag{1.3}$$

where b and a indicate smooth functions. The equations (1.1) and (1.2) together provide equation (1.3) (see [21]).

Jointly, the equations (1.1), (1.2) and (1.3) infer

$$b = \kappa(p+\sigma), \; a = \frac{\kappa(p-\sigma)}{2-n}. \tag{1.4}$$

It is known that every RW-spacetime is a PFS [23]. Also, a 4-dimensional GRW-spacetime is a PFS if and only if it is a RW-spacetime [12]. In [21], the authors prove that a PFS with $p + \sigma \neq 0$ satisfying $\nabla_h C^h_{ijk} = 0$, in which ∇ denotes the covariant differentiation is a GRW spacetime. Shepley and Taub [27] proved that a 4-dimensional PFS with $\nabla_h C^h_{ijk} = 0$ and obeying state equation $p = p(\sigma)$ is conformally flat, and the flow is irrotational, geodesic and shear-free, also the metric is RW. In [26], Sharma and Ghosh established that if T_{hk} is Killing in a PFS, then (i) σ and p are constant, and the PFS is (ii) expansion and shear-free, with a geodesic flow that may or may not be vorticity free. The characteristics of perfect fluid spacetimes have been seen in [3, 9, 10, 18, 19].

The Weyl tensor denoted by C is provided in local coordinates by

$$\begin{aligned} C_{ijkl} &= R_{ijkl} - \frac{1}{n-2}(g_{il}R_{jk} - g_{ik}R_{jl} + g_{jk}R_{il} - g_{jl}R_{ik}) \\ &\quad + \frac{R}{(n-1)(n-2)}\{g_{il}g_{jk} - g_{ik}g_{jl}\}, \end{aligned} \tag{1.5}$$

where R_{hijk} stands for the $(0,4)$ type curvature tensor.

Further, it is known that

$$\nabla_k C^k_{hji} = \frac{n-3}{n-2}[\{\nabla_i R_{hj} - \nabla_j R_{hi}\} - \frac{1}{2(n-1)}\{g_{hj}\nabla_i R - g_{hi}\nabla_j R\}]. \tag{1.6}$$

If $\nabla_k C^k_{lij} = 0$, then the Weyl tensor is named harmonic. The tensor's harmonicity can be seen in the conservation laws of physics.

Chen and Yano [6] obtain the subsequent curvature tensor's expression

$$R_{hijk} = \gamma(g_{hk}g_{ij} - g_{hj}g_{ik}) + \mu\{g_{hk}u_i u_j + g_{ij}u_h u_k - g_{hj}u_i u_k - g_{ik}u_h u_j\}, \tag{1.7}$$

where $R_{hijk} = g_{hl}R^l_{ijk}$, R^h_{ijk} denotes the $(1,3)$ type curvature tensor and u_i is a unit vector, named the generator and γ, μ are scalars. A space of quasi-constant curvature (briefly, $(QC)_n$) is a conformally flat space of dimension n that obeys the equation (1.7). Nevertheless, it is simple to confirm that the space is conformally flat if equation (1.7) holds. Thus, conformally flatness is not necessary according to the definition. A spacetime of quasi-constant curvature is a Lorentzian manifold whose curvature tensor satisfies equation (1.7) and u_i is a unit time-like vector.

In [19], Mantica et. al and in [7], Chen have shown the subsequent:

Theorem A. [19] A GRW spacetime obeys $\nabla_h C^h_{ijk} = 0$ if and only if the spacetime is a PFS.

Theorem B. [7] M is a GRW spacetime if and only if it admits a time-like concircular vector field.

Also, the characteristics of GRW spacetimes have been found in [2, 18, 21, 24] and references therein. The research mentioned above encourages us to investigate some features of GRW spacetimes and provide the followings:

Theorem 1.1 *A PFS with nonvanishing vorticity whose associated scalars are constant along the velocity vector field becomes a GRW spacetime.*

Theorem 1.2 *A $(QC)_n$ spacetime is a GRW spacetime.*

Let a Lorentzian manifold admit a Killing time-like vector ρ. Then, the spacetime is considered to be a stationary spacetime. If, furthermore, ρ is irrotational the spacetime is named as static ([25], [28, p. 283]). Let us define a metric

$$g[(t,y)] = g_S[y] - \beta(y)dt^2, \tag{1.8}$$

where g_S indicates a Riemannian metric on S. The product $\mathbb{R}\times S$, equipped with the previous metric is named a static spacetime.

Now, we consider Ricci parallel PFS and demonstrate the subsequent:

Theorem 1.3 *A Ricci parallel PFS is either a static spacetime or a GRW spacetime.*

In [14], Gray invented the concept of the Codazzi type of Ricci tensor. A spacetime fulfills Codazzi type of Ricci tensor if its nonzero Ricci tensor R_{ij} obeys

$$\nabla_k R_{ij} = \nabla_j R_{ik}.$$

Considering the Codazzi type of Ricci tensor, we reveal:

Theorem 1.4 *In a PFS if the Ricci tensor is of Codazzi type, then either the PFS represents a GRW spacetime or the flow is irrotational provided the gradient of the scalar a is parallel to the velocity vector.*

Cartan [4] introduced the notion of semisymmetric spaces. A Lorentzian manifold M is called semisymmetric [29] if it obeys the condition

$$\nabla_m \nabla_l R^h_{ijk} - \nabla_l \nabla_m R^h_{ijk} = 0, \tag{1.9}$$

where ∇ indicates the covariant differentiation. Semisymmetric spacetimes have been considered in [15].

M is named conformally semisymmetric [15] if C_{hijk} fulfills

$$\nabla_m \nabla_l C_{hijk} - \nabla_l \nabla_m C_{hijk} = 0, \tag{1.10}$$

and Ricci semisymmetric [22] if R_{ij} fulfills

$$\nabla_m \nabla_l R_{ij} - \nabla_l \nabla_m R_{ij} = 0. \tag{1.11}$$

Obviously, a semisymmetric manifold implies Ricci semisymmetry and conformally semisymmetry. However, in the literature examples of conformally semisymmetric manifolds in dimension 4 which are not semisymmetric are given. In [11], Erikson and Senovilla proved that in a 4-dimensional Lorentzian manifold conformally semisymmetric implies Ricci semisymmetric and conformally semisymmetric and semisymmetric manifolds are equivalent.

In dimension 4, choosing conformally semisymmetric GRW spacetime, we acquire:

Theorem 1.5 *In a conformally semisymmetric GRW spacetime of dimension* 4*, the scalar curvature vanishes, and the spacetime is locally isometric to the Minkowski spacetime, provided the electric part of the Weyl tensor vanishes.*

Conformally recurrent manifold is defined by $\nabla_l C^h_{ijk} = \lambda_l C^h_{ijk}$ [1], λ_l is a covariant vector. Finally, we consider a nonconformally flat conformally recurrent GRW spacetime and produce:

Theorem 1.6 *In a conformally recurrent GRW spacetime the scalar curvature vanishes and the spacetime is locally isometric to the Minkowski spacetime, provided the electric part of the Weyl tensor vanishes.*

2. Proof of the Theorems

Proof of the Theorem 1.1

At first, from equation (1.3), we can easily obtain

$$\nabla_l R_{hk} = (\nabla_l a) g_{hk} + (\nabla_l b) u_h u_k + b[u_k \nabla_l u_h + u_h \nabla_l u_k]. \tag{2.1}$$

Multiplying by g^{hl}, we infer

$$\frac{1}{2}\nabla_k R = (\nabla_k a) + b^h u_h u_k + b[\nabla_l u^l u_k + u^l \nabla_l u_k]. \tag{2.2}$$

Since $u_h u^h = -1$ implies $u^h \nabla_l u_h + u_h \nabla_l u^h = 0 \Rightarrow u_h \nabla_l u^h = 0$, we acquire from the above

$$\nabla_k R = 2(\nabla_k a) + 2b^h u_h u_k + 2b u_k \nabla_l u^l. \tag{2.3}$$

Again, from equation (1.3) we have

$$\nabla_k R = n(\nabla_k a) - (\nabla_k b). \tag{2.4}$$

Making use of the last two equations, one can easily get

$$(n-2)u^k(\nabla_k a) = -u^k(\nabla_k b) - 2b\nabla_l u^l. \tag{2.5}$$

Suppose the associated scalars a and b defined in equation (1.3) are constant along the velocity vector u_h, then equation (2.5) infers either (i) $b \neq 0$ or, (ii) $b = 0$.

(i) For $b \neq 0$, we get divergence of the velocity vector vanishes which implies the vorticity of the fluid vanishes.

(ii) If $b = 0$, then equation (1.3) implies the spacetime becomes Einstein. Hence, from equation (1.6) it follows that $\nabla_h C^h_{ijk} = 0$. Thus, by Theorem A, we get the desired result. □

Proof of the Theorem 1.2

The $(QC)_n$ spacetime is defined as

$$R_{hijk} = p_1(g_{hk}g_{ij} - g_{hj}g_{ik}) + q_1(g_{hk}u_iu_j + g_{ij}u_hu_k - g_{hj}u_iu_k - g_{ik}u_hu_j), \tag{2.6}$$

in which p_1, q_1 are scalars and vector u_i is time-like and unit, that is, $u^iu_i = -1$.

Therefore, the foregoing equation reveals

$$R_{ij} = \{p_1(n-1) - q_1\}g_{ij} + (n-2)q_1u_iu_j, \tag{2.7}$$

which provides that the spacetime represents PFS.

It is easy to see that for a $(QC)_n$, the Weyl tensor C^h_{ijk} vanishes. Hence, $\nabla_h C^h_{ijk} = 0$. Therefore, using Theorem A we acquire the required result. □

Proof of the Theorem 1.3

Since the PFS is Ricci parallel, we have $\nabla_l R_{hk} = 0$ and hence

$$(\nabla_l a)g_{hk} + (\nabla_l b)u_hu_k + b[u_k\nabla_l u_h + u_h\nabla_l u_k] = 0, \tag{2.8}$$

which implies

$$(\nabla_l a)u_k - (\nabla_l b)u_k - b\nabla_l u_k = 0. \tag{2.9}$$

Since $\nabla_l R_{hk} = 0$, we get $R =$ constant and hence $na - b = 0$.

From the last equation, it follows

$$((\nabla_l a) - n(\nabla_l a))u_k - b\nabla_l u_k = 0. \tag{2.10}$$

Transvecting (2.10) with u^k, we find $(\nabla_l a) = 0$ which implies $a =$ constant and hence $b =$ constant.

Therefore, equation (2.9) infers either (i) $b \neq 0$ or (ii) $b = 0$.

(i) For $b \neq 0$, we acquire $\nabla_l u_k = 0$. Thus, for a smooth vector field v we infer

$$\pounds_v g_{ij} = \nabla_i v_j + \nabla_j v_i,$$

where $\pounds$ indicates Lie derivative operator. Now, $\nabla_l u_k = 0$ implies $\pounds_u g_{ij} = 0$ which implies that u is Killing. Further, $\nabla_l u_k = 0$ entails u_k is irrotational. Therefore, the PFS becomes static [28].

(ii) If $b = 0$, from equation (1.3) the spacetime becomes Einstein and hence, from equation (1.6) we acquire $\nabla_h C^h_{ijk} = 0$. Thus, by Theorem A, we get the result. □

Proof of the Theorem 1.4

Differentiation (1.3) covariantly yields

$$\nabla_l R_{ij} = (\nabla_l a) g_{ij} + (\nabla_l b) u_i u_j + b(u_j \nabla_l u_i + u_i \nabla_l u_j). \tag{2.11}$$

Similarly, we obtain

$$\nabla_j R_{il} = (\nabla_j a) g_{il} + (\nabla_j b) u_i u_l + b(u_l \nabla_j u_i + u_i \nabla_j u_l). \tag{2.12}$$

Suppose the Ricci tensor is of Codazzi type. Then, $\nabla_l R_{ij} = \nabla_j R_{il}$. Hence, using the last two equations, we acquire

$$\begin{aligned}&(\nabla_l a) g_{ij} - (\nabla_j a) g_{il} + (\nabla_l b) u_i u_j - (\nabla_j b) u_i u_l \\ &+ b[u_j \nabla_l u_i - u_l \nabla_j u_i + u_i\{\nabla_l u_j - \nabla_j u_l\}] = 0.\end{aligned} \tag{2.13}$$

The result of multiplying with u^i is

$$(\nabla_l a) u_j - (\nabla_j a) u_l - (\nabla_l b) u_j + (\nabla_j b) u_l - b\{\nabla_l u_j - \nabla_j u_l\} = 0. \tag{2.14}$$

Since $\nabla_l R_{ij} = \nabla_j R_{il}$ provides $R =$ constant, from the equation (2.4) it is easily follows that $n(\nabla_k a) = (\nabla_k b)$.

Hence, equation (2.14) yields

$$(1-n)\{(\nabla_l a) u_j - (\nabla_j a) u_l\} - b\{\nabla_l u_j - \nabla_j u_l\} = 0. \tag{2.15}$$

Suppose the gradient of the scalar a is parallel to the velocity vector. Then the foregoing equation entails either (i) $b \neq 0$ or (ii) $b = 0$.

(i) For $b \neq 0$, we say that the flow is irrotational.

(ii) If $b = 0$, then from equation (1.3) it follows that the spacetime is Einstein. Hence, from equation (1.6) we obtain $\nabla_k C^k_{lij} = 0$. Thus, by Theorem A, we have a PFS, which is a GRW spacetime. □

Proof of the Theorem 1.5

In a GRW spacetime [18], we have the subsequent shape of the Ricci tensor R_{ij}:

$$R_{ij} = \frac{nf - R}{(n-1)} \mu_i \mu_j - \frac{f - R}{(n-1)} g_{ij} + (n-2) C_{hijk} \mu^h \mu^k, \tag{2.16}$$

in which $\mu_i = \frac{X_i}{\sqrt{X_p X^p}}$ and $R_{ij}\mu^j = f\mu_i$, f is a scalar. Equation (2.16) can be written as

$$R_{ij} = \alpha \mu_i \mu_j - \beta g_{ij} + (n-2) C_{hijk} \mu^h \mu^k, \tag{2.17}$$

where $\alpha = \frac{nf-R}{(n-1)}$ and $\beta = \frac{f-R}{(n-1)}$.

From the last equation, one can easily acquire

$$\begin{aligned}\nabla_l\nabla_m R_{ij}-\nabla_m\nabla_l R_{ij} &= \alpha[\nabla_l\nabla_m\mu_i\mu_j+\mu_i\nabla_l\nabla_m\mu_j-\mu_j\nabla_m\nabla_l\mu_i-\mu_i\nabla_m\nabla_l\mu_j]\\ &\quad +(n-2)\{\nabla_l\nabla_m C_{hijk}-\nabla_m\nabla_l C_{hijk}\}\mu^h\mu^k\\ &\quad +(n-2)C_{hijk}\{(\nabla_l\nabla_m\mu^h-\nabla_m\nabla_l\mu^h)\mu^k+\mu^h(\nabla_l\nabla_m\mu^k\\ &\quad -\nabla_m\nabla_l\mu^k)\}. \end{aligned} \tag{2.18}$$

In dimension 4, let us suppose that GRW spacetime is conformally semisymmetric.

With the help of the above result equation (2.18), we have

$$\begin{aligned}&\alpha[(\nabla_l\nabla_m\mu_i-\nabla_m\nabla_l\mu_i)\mu_j+\mu_i(\nabla_l\nabla_m\mu_j-\nabla_m\nabla_l\mu_j)]\\ &+2C_{hijk}\{(\nabla_l\nabla_m\mu^h-\nabla_m\nabla_l\mu^h)\mu^k\\ &+\mu^h(\nabla_l\nabla_m\mu^k-\nabla_m\nabla_l\mu^k)\}=0.\end{aligned} \tag{2.19}$$

Applying Ricci identity, we get

$$\begin{aligned}&\alpha[\mu_r R^r_{ilm}\mu_j+\mu_r R^r_{jlm}\mu_i]\\ &+2C_{hijk}\{\mu^r R^h_{rlm}\mu^k+\mu^r R^k_{rlm}\mu^h\}=0.\end{aligned} \tag{2.20}$$

Multiplying equation (2.20) by g^{il} we get

$$\begin{aligned}&\alpha[\mu_r R^r_m\mu_j+\mu_r R^r_{jlm}\mu^l]\\ &+2C_{hijk}g^{il}\{\mu^r R^h_{rlm}\mu^k+\mu^r R^k_{rlm}\mu^h\}=0.\end{aligned} \tag{2.21}$$

Multiplication with μ^j and using $\mu^j\mu_j=-1$ we obtain

$$\begin{aligned}&-\alpha\mu_r R^r_m+\alpha\mu_r\mu^l\mu^j R^r_{jlm}\\ &=2C_{hijk}g^{il}\mu^r R^h_{rlm}\mu^k\mu^j+2C_{hijk}g^{il}\mu^r R^k_{rlm}\mu^h\mu^j=0.\end{aligned} \tag{2.22}$$

Now, using $\mu_r R^r_i=f\mu_i$ in the foregoing equation, we have

$$-\alpha f\mu_m-2C_{hijk}\mu^h\mu^j R^k_{rlm}\mu^r g^{il}=0. \tag{2.23}$$

Suppose the electric part of the Weyl tensor vanishes [16], then $C_{hijk}\mu^h\mu^j=0$.

Thus, under the above assumption, we infer $R_{ij}=0$, that is, the spacetime is a vacuum, which implies that the GRW spacetime is locally isometric to the Minkowski spacetime. □

Remark 2.1 *Since semisymmetry and conformally semisimmetry are equivalent in a spacetime, the above result holds for a semisymmetric GRW spacetime.*

Remark 2.2 *A spacetime is said to be conformally symmetric [5] if* $\nabla_l C^h_{ijk}=0$*. Conformally symmetric spacetime implies conformally semisymmetric spacetime; therefore, the above theorem also holds for conformally symmetric GRW spacetime of dimension* 4*.*

Proof of the Theorem 1.6

Let us consider a nonconformally flat conformally recurrent GRW spacetime and $f^2 = C_{hijk}C^{hijk}$, from which we get

$$2ff_l = \nabla_l C_{hijk}C^{hijk} + C_{hijk}C_l^{hijk} = 2f^2\lambda_l, \tag{2.24}$$

where $f_l = \nabla_l f$.

From equation (2.24), we infer $f_l = f\lambda_l$. From the above we can acquire $\nabla_l f = f\lambda_l$ which implies

$$\nabla_m\nabla_l f = (\nabla_m f)\lambda_l + f\nabla_m\lambda_l = f\lambda_m\lambda_l + f\nabla_m\lambda_l. \tag{2.25}$$

Using the last equation, we obtain

$$\nabla_l\nabla_m f - \nabla_m\nabla_l f = f\lambda_l\lambda_m + f\nabla_l\lambda_m - f\lambda_m\lambda_l - f\nabla_m\lambda_l, \tag{2.26}$$

which provides

$$0 = f(\nabla_l\lambda_m - \nabla_m\lambda_l). \tag{2.27}$$

Hence, we have $\nabla_l\lambda_m = \nabla_m\lambda_l$, since $f \neq 0$.

Now,

$$\begin{aligned}\nabla_l\nabla_m C^h_{ijk} &= \lambda_l\nabla_l C^h_{ijk} + \lambda_l\nabla_m C^h_{ijk} \\ &= \nabla_m\lambda_l C^h_{ijk} + \lambda_l\lambda_m C^h_{ijk}.\end{aligned} \tag{2.28}$$

Therefore, we get

$$\nabla_l\nabla_m C^h_{ijk} - \nabla_m\nabla_l C^h_{ijk} = (\nabla_m\lambda_l - \nabla_l\lambda_m)C^h_{ijk}. \tag{2.29}$$

Hence, $\nabla_l C^h_{ijk} = \lambda_l C^h_{ijk}$ implies $\nabla_l\nabla_m C^h_{ijk} - \nabla_m\nabla_l C^h_{ijk} = 0$.

Hence, using the last theorem we acquire the desired result. □

3. Discussion

Spacetime, a time-oriented, torsion-less Lorentzian manifold, is a platform of the physical world that is being modelled at the moment. General relativity theory, originally laid out by Albert Einstein in 1915, tells that the matter content of the cosmos can be expressed by selecting a suitable energy momentum tensor and in cosmological models, this matter behaves like a PFS. Perfect fluid meaning it cannot transport heat. Since a perfect fluid has no viscosity, it is unable to endure a tangential force even when it is flowing. Perfect fluids are employed in general relativity to model idealized matter distributions, as those seen inside stars or in an isotropic universe. In the second case, the state equation of the perfect fluid can be utilised to describe the evolution of the cosmos using the Friedmann-Lemaître-Robertson-Walker equations. Large scale cosmology is performed in GRW spacetimes, which are a logical and expansive extension of RW spacetimes.

In this article, we determine the conditions under which a PFS turns into a GRW spacetime. In the near future, we or perhaps other researchers will look at the various characteristics of GRW spacetimes.

References

[1] Adati T, Miyazawa T. On a Riemannian space with recurrent conformal curvature, Tensor (N. S.) 1967; 18: 348-354.

[2] Alias L, Romero A, Sanchez M. Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes. General Relativity and Gravitation 1995; 27: 71–84. https://doi.org/10.1007/BF02105675

[3] Blaga AM. Solitons and geometrical structures in a perfect fluid spacetime. The Rocky Mountain Journal of Mathematics 2020; 50: 41-53. https://doi.org/10.1216/rmj.2020.50.41

[4] Cartan E. Leçons sur la Geometrie des Espaces de Riemannian. 2nd Edition, Gauthier-Villars, Paris, 1946.

[5] Chaki MC, Gupta B. On conformally symmetric spaces. Indian Journal of Mathematics 1963; 5: 113-122.

[6] Chen BY, Yano K. Hypersurfaces of a conformally flat space. Tensor (N.S.) 1972; 26: 318-322.

[7] Chen BY. Pseudo-Riemannian Geometry, δ-invariants and Applications. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. https://doi.org/10.1142/8003

[8] De UC, Chaubey SK, Shenawy S. Perfect fluid spacetimes and Yamabe solitons. Journal of Mathematical Physics 2021; 62 (3): 032501. https://doi.org/10.1063/5.0033967

[9] De K, De UC, Gezer A. Perfect fluid spacetimes and k-almost Yamabe solitons. Turkish Journal of Mathematics 2023; 47 (4): 1236-1246. https://doi.org/10.55730/1300-0098.3423

[10] De K, De UC, Velimirovic JS. Some curvature properties of perfect fluid spacetimes. Quaestiones Mathematicae 2024; 47 (4): 751-764. https://doi.org/10.2989/16073606.2023.2255933

[11] Eriksson I, Senovilla JMM. Note on (conformally) semi-symmetric spacetimes. Classical and Quantum Gravity 2010; 27 (2): 027001, 5. https://doi.org/10.1088/0264-9381/27/2/027001

[12] Gutiérrez M, Olea B. Global decomposition of a Lorentzian manifold as a generalized Robertson-Walker space. Differential Geometry and its Applications 2009; 27 (1): 146-156. https://doi.org/10.1016/j.difgeo.2008.06.015

[13] Güler S, Demirbağ SA. A study of generalized quasi-Einstein spacetimes with applications in general relativity. n International Journal of Theoretical Physics 2016; 55: 548-562. https://doi.org/10.1007/s10773-015-2692-1

[14] Gray A. Einstein-like manifolds which are not Einstein. Geometriae Dedicata 1978; 7 (3): 259-280. https://doi.org/10.1007/BF00151525

[15] Haesen S, Verstraelen L. Classification of the pseudosymmetric space–times. Journal of Mathematical Physics 2004; 45 (6): 2343-2346. https://doi.org/10.1063/1.1745129

[16] Hervik S, Ortaggio M, Wylleman L. Minimal tensors and purely electric or magnetic spacetimes of arbitrary dimension. Classical and Quantum Gravity 2013; 30 (16): 165014, 50.

[17] Mallick S, Suh YJ, De UC. A spacetime with pseudo-projective curvature tensor. Journal of Mathematical Physics 2016; 57 (6): 062501, 10. https://doi.org/10.1063/1.4952699

[18] Mantica CA, Molinari LG. Generalized Robertson Walker spacetimes-A survey. International Journal of Geometric Methods in Modern Physics 2017; 14 (3): 1730001, 27. https://doi.org/10.1142/S021988781730001X

[19] Mantica CA, De UC, Suh YJ, Molinari LG. Perfect fluid spacetimes with harmonic generalized curvature tensor. Osaka Journal of Mathematics 2019; 56 (1): 173-182.

[20] Mantica CA, Molinari LG, Suh YJ, Shenawy S. Perfect fluid, generalized Robertson Walker spacetimes and Gray's decomposition. Journal of Mathematical Physics 2019; 60: 052506, 9. https://doi.org/10.1063/1.5089040

[21] Mantica CA, Molinari LG, De UC. A condition for a perfect fluid spacetime to be a generalized Robertson-Walker spacetime. Journal of Mathematical Physics 2016; 57 (2): 022508. https://doi.org/10.1063/1.4941942

[22] Mirzoyan VA. Structure theorems for Riemannian Ric-semisymmetric spaces. Izvestiya Vysshikh Uchebnykh Zavedeniĭ Matematika 1992; 36: 80–89.

[23] O'Neil B. Semi-Riemannian Geometry with Applications to the Relativity. Academic Press, New York-London, 1983.

[24] Sánchez M. On the geometry of generalized Robertson Walker spacetimes: geodesics. General Relativity and Gravitation 1998; 30: 915-932. https://doi.org/10.1023/A:1026664209847

[25] Sánchez M. On the geometry of static spacetimes, Nonlinear Analysis: Theory, Methods and Applications 2005; 63 (5–7): 455-463. https://doi.org/10.1016/j.na.2004.09.009

[26] Sharma R, Ghosh A. Perfect fluid space-times whose energy-momentum tensor is conformal Killing. Journal of Mathematical Physics 2010; 51 (2): 022504, 5. https://doi.org/10.1063/1.3319562

[27] Shepley LC, Taub AH. Spacetimes containing perfect fluids and having a vanishing conformal divergence. Communications in Mathematical Physics 1967; 5: 237-256. https://doi.org/10.1007/BF01646477

[28] Stephani H, Kramer D, Mac-Callum M, Hoenselaers C, Hertl E. Exact solutions of Einstein's field equations. 2nd Edn. Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2009.

[29] Szabo ZI. Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$. Journal of Differential Geometry 1982; 17: 531-582. https://doi.org/10.4310/jdg/1214437486

[30] Zengin FO. M-projectively flat spacetimes. Mathematical Reports 2012; 14 (64): 363-370.