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Transmission eigenvalues problem of a Schrödinger equation

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Abstract: In this paper, transmission eigenvalues of a Schrödinger equation have been studied by constructing a new inner product and using Weyl theory. Necessary conditions for these eigenvalues to be negative, real, and finite have been examined. This method has provided a new framework related to transmission eigenvalue problems and the investigation of their properties. The conclusions have been verified for the special case of the problem.

Key words: Spectral theory, transmission condition, eigenvalues, Schrödinger equation

1. Introduction

It is known that the differential equations with transmission conditions have an important place in the literature. They are used to quantitatively represent many physical systems, especially in quantum mechanics and atomic physics. For example, they are considered in studies of wave propagation in electrodynamics and, more generally, in some models of theoretical physics to obtain fully solvable models with a wide variety of applications. Therefore, there are many studies about this topic, recently [1–5, 9, 10, 15, 16, 19, 20, 26].

Especially, four-parameter point interactions are used to represent as self-adjoint extensions for nonrelativistic kinetic energy operator in one-dimensional quantum mechanics [10]. Depending on the choice of the parameter in this interaction, the physical, spectral, and scattering properties of this operator are affected. For example, this interaction may lead to P, T, or PT-symmetry. The effect of general point interaction on these symmetries has been investigated by Mostafazadeh [18].

If the determinant of the transfer matrix is not 1, the interaction is called anomalous point interaction [19]. There are three different types of symmetry breaking in physical systems with invariance, one of which is called an anomaly [6]. This concept, which emerges when the classical invariance principle in the system is broken during quantization, has been frequently used in such studies as it provides many contributions to the investigation of fundamental particles in the standart model and its derivatives. In addition to its phenomenological meaning, the investigation of anomalies has also been the subject of theoretical research [12, 21–24]. The problem in this study has been constructed under that assumption to give a perspective to new studies that is related to symmetries.

One of the most used theories to perform spectral analysis of a singular differential equation is the Weyl theory. This theory has been established in 1910 by Weyl [27]. He has obtained the singular second-order differential operators have two cases: limit-circle case and limit-point case. Then, he has shown that the

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solutions of the operators have different characteristics in these cases. This topic has been improved by many authors [7, 8, 13, 14, 23, 25]. Note that, in this paper, an important conclusion of Weyl theory has been used to examine characterizations of the transmission eigenvalues of the problem.

2. Main problem

Let us consider the one-dimensional Schrödinger equation

$$-y''(x) + \upsilon(x)y(x) = \rho^2 y(x), \quad x \in \mathbb{R}/\{0\} \tag{2.1}$$

with the transmission condition

$$\begin{aligned} y(0^+) &= \nu_1 y(0^-), \\ y'(0^+) &= \nu_2 y'(0^-), \end{aligned} \tag{2.2}$$

where $\nu_1\nu_2 \neq 1$, υ-is a real valued function and ρ is a spectral parameter. The equation (2.1) has bounded solutions $\xi_{\pm}(x,\rho)$ satisfying following limit conditions

$$\lim_{x\to\pm\infty} y(x)e^{\pm i\rho x} = 1, \quad \rho \in \overline{\mathbb{C}}_+ = \{\rho \colon \rho \in \mathbb{C}, \rho \geq 0\}. \tag{2.3}$$

These solutions $\xi_{\pm}(x,\rho)$ are called Jost solutions. Under the condition

$$\int_{-\infty}^{\infty} (1+|x|)|\upsilon(x)|dx < \infty, \tag{2.4}$$

the solutions $\xi_{\pm}(x,\rho)$ have the representations

$$\xi_-(x,\rho) = e^{-i\rho x} + \int_{-\infty}^{x} \kappa^-(x,t)e^{-i\rho t}dt, \tag{2.5}$$

and

$$\xi_+(x,\rho) = e^{i\rho x} + \int_{x}^{\infty} \kappa^+(x,t)e^{i\rho t}dt$$

for every $\rho \in \overline{\mathbb{C}}_+$. Here the kernel functions $\kappa^+(x,t)$ and $\kappa^-(x,t)$ satisfy

$$\begin{aligned}\kappa^+(x,t) &= \frac{1}{2}\int\limits_{\frac{x+t}{2}}^{\infty} \upsilon(s)ds + \frac{1}{2}\int\limits_{x}^{\frac{x+t}{2}}\int\limits_{x+t-s}^{t-x+s} \upsilon(s)\kappa^+(s,r)dsdr \\ &+\frac{1}{2}\int\limits_{\frac{x+t}{2}}^{\infty}\int\limits_{s}^{t-x+s} \upsilon(s)\kappa^+(s,r)dsdr, \\ \kappa^-(x,t) &= \frac{1}{2}\int\limits_{-\infty}^{\frac{x+t}{2}} \upsilon(s)ds + \frac{1}{2}\int\limits_{x}^{\frac{x+t}{2}}\int\limits_{x+t-s}^{t-x+s} \upsilon(s)\kappa^-(s,r)dsdr \\ &+\frac{1}{2}\int\limits_{-\infty}^{\frac{x+t}{2}}\int\limits_{s}^{t-x+s} \upsilon(s)\kappa^-(s,r)dsdr.\end{aligned}$$

Furthermore, $\kappa^\pm(x,t)$ are continuously differentiable with respect to its arguments and the following inequalities

$$|\kappa^\pm(x,t)| \le c\sigma^\pm(\frac{x+t}{2})$$

$$|\kappa_x^\pm(x,t) \pm \frac{1}{4}|\upsilon(\frac{x+t}{2})|| \le c\sigma^\pm(\frac{x+t}{2})$$

$$|\kappa_t^\pm(x,t) \pm \frac{1}{4}|\upsilon(\frac{x+t}{2})|| \le c\sigma^\pm(\frac{x+t}{2})$$

hold, where

$$\sigma^+(t) = \int\limits_{x}^{\infty} |\upsilon(t)|dt, \quad \sigma^-(t) = \int\limits_{-\infty}^{x} |\upsilon(t)|dt$$

and $c > 0$ is a constant [17].

Lemma 2.1 *To investigate the structure of the transmission eigenvalues of the problem* (2.1)−(2.2)*, we present a new inner product in the Hilbert space*

$$H := \left\{ f : \mathbb{R} \to \mathbb{R},\ \langle f, f\rangle := \nu_1\nu_2 \int\limits_{-\infty}^{0} |f|^2dx + \int\limits_{0}^{\infty} |f|^2dx < \infty \right\} \tag{2.6}$$

which is defined by

$$\langle y, z\rangle_H = \nu_1\nu_2 \int\limits_{-\infty}^{0} y_1(x)\overline{z_1(x)}dx + \int\limits_{0}^{\infty} y_2(x)\overline{z_2(x)}dx, \tag{2.7}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in H \tag{2.8}$$

and we construct the operator $L : H \to H$ *whose domain is*

$$D(L) = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : \begin{array}{c} y_1' \in AC(-\infty,0), y_2' \in AC(0,\infty), \\ Ly \in L_2(-\infty,0) \oplus L_2(0,\infty), \\ y_2(0^+) = \nu_1 y_1(0^-), y_2'(0^+) = \nu_2 y_1'(0^-) \end{array} \right\}. \tag{2.9}$$

The problem (2.1) − (2.2) *can be considered as an eigenvalues problem in the operator form as*

$$Ly(x) = -y'(x) + \upsilon(x)y(x) = \rho^2 y(x) \tag{2.10}$$

where $y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D(L)$*. From* (2.6) *and* (2.7)*, it can be obtained that the eigenvalues of the operator* L *and the eigenvalues of transmission problem* (2.1) − (2.2) *are same.*

3. Transmission eigenvalues

Assume that $E(x,\rho)$ and $F(x,\rho)$ be two linearly independent solutions of the equation (2.1) satisfying

$$E(x,\rho) = \begin{cases} \xi_+(x,\rho), & x > 0 \\ \alpha(\rho)\xi_-(x,\rho) + \beta(\rho)\xi_-(x,-\rho), & x < 0 \end{cases},$$

$$F(x,\rho) = \begin{cases} \gamma(\rho)\xi_+(x,\rho) + \delta(\rho)\xi_+(x,-\rho), & x > 0 \\ \xi_-(x,\rho), & x < 0 \end{cases}. \tag{3.1}$$

for $\rho \in \mathbb{R}\backslash\{0\}$, where $\xi_+(x,\rho)$ and $\xi_-(x,\rho)$ are defined by (2.5). Note that the Wronskian of these solutions can be calculated by

$$W[E(x,\rho), F(x,\rho)] = \begin{cases} -2i\rho\nu_1\nu_2\beta(\rho), & x > 0 \\ -2i\rho\beta(\rho), & x < 0. \end{cases} \tag{3.2}$$

Lemma 3.1 *The following equations hold.*

$$\begin{aligned} \alpha(\rho) &= \frac{\nu_2\xi_+(0,\rho)\xi_-'(0,-\rho) - \nu_1\xi_-(0,-\rho)\xi_+'(0,\rho)}{2i\rho\nu_1\nu_2} \\ \beta(\rho) &= \frac{\nu_1\xi_-(0,\rho)\xi_+'(0,\rho) - \nu_2\xi_+(0,\rho)\xi_-'(0,\rho)}{2i\rho\nu_1\nu_2} \\ \gamma(\rho) &= \frac{\nu_2\xi_+(0,-\rho)\xi_-'(0,\rho) - \nu_1\xi_-(0,\rho)\xi_+'(0,-\rho)}{2i\rho}, \end{aligned} \tag{3.3}$$

$$\delta(\rho)=\frac{\nu_1\xi_-(0,\rho)\xi_+'(0,\rho)-\nu_2\xi_+(0,\rho)\xi_-'(0,\rho)}{2i\rho},$$

$$\gamma(\rho)=\nu_1\nu_2\overline{\alpha(\rho)}=-\nu_1\nu_2\alpha(-\rho),\ \delta(\rho)=\nu_1\nu_2\beta(\rho).$$

Proof Since the Jost solutions of the equation (2.1) satisfy the transmission condition (2.2), the following can be written.

$$\begin{aligned}\xi_+(0,\rho) &= \nu_1\alpha(\rho)\xi_-(0,\rho)+\nu_1\beta(\rho)\xi_-(0,-\rho)\\ \xi_+'(0,\rho) &= \nu_2\alpha(\rho)\xi_-'(0,\rho)+\nu_2\beta(\rho)\xi_-'(0,-\rho)\\ \gamma(\rho)\xi_+(0,\rho)+\delta(\rho)\xi_+(0,-\rho) &= \nu_1\xi_-(0,\rho)\\ \gamma(\rho)\xi_+'(0,\rho)+\delta(\rho)\xi_+'(0,-\rho) &= \nu_2\xi_-'(0,\rho).\end{aligned}$$

Solving these equation systems by Cramer's rule, the expressions of $\alpha(\rho)$ $\beta(\rho)$, $\gamma(\rho)$, and $\delta(\rho)$ can be obtained. Moreover, $\overline{\xi_+(0,\rho)}=\xi_+(0,-\rho)$, $\overline{\xi_-(0,\rho)}=\xi_-(0,-\rho)$ are provided for $\rho\in\ \mathbb{R}$. Thus, the relations between these coefficients are written by

$$\gamma(\rho)=\nu_1\nu_2\overline{\alpha(\rho)}=-\nu_1\nu_2\alpha(-\rho), \tag{3.4}$$

and

$$\delta(\rho)=\nu_1\nu_2\overline{\beta(\rho)}=\nu_1\nu_2\beta(\rho). \tag{3.5}$$

□

The set of the transmission eigenvalues of the problem (2.1) − (2.2) can be given as

$$\sigma_d\ :\ =\{\mu:\ \mu=\rho^2,\rho\in C_+,\beta(\rho)=0\} \tag{3.6}$$

Then, it is clearly seen that the structure of the zeros $\beta(\rho)$ must be examined in order to obtain the properties of the eigenvalues of the problem (2.1)-(2.2) .

Theorem 3.2 *For all* $\rho\in\mathbb{R}$, $\beta(\rho)\neq 0$.

Proof Let us assume that $\beta(\rho_0)=0$ for any $\rho_0\in\mathbb{R}$. In this case, $|\alpha(\rho_0)|^2(1+\nu_1\nu_2)=0$ can be written by using (3.1) and transmission condition (2.2).
Since $\nu_1\nu_2>0$, we can write $\alpha(\rho)=0$. From the last two equations of (3.3),

$$\begin{aligned}\gamma(\rho) &= \nu_1\nu_2\overline{\alpha(\rho)}=0,\\ \delta(\rho) &= \nu_1\nu_2\beta(\rho)=0.\end{aligned}$$

are obtained. Then the solution $F(x,\rho)$ is equal to zero identically, that is $F(x,\rho)$ is a trivial solution of (2.1) -(2.2) . It gives a contradiction, thus $\beta(\rho)\neq 0$ for each $\rho\in\mathbb{R}$. □

Theorem 3.3 *Under the following conditions*

$$\upsilon\in AC(-\infty,\infty),\quad \lim_{x\to\pm\infty}\upsilon(x)=0,\quad \int_{-\infty}^{\infty}x|\upsilon'(x)|dx<\infty, \tag{3.7}$$

$\beta(\rho)$ has the representation

$$\beta(\rho) = \frac{1}{2i\rho\nu_1\nu_2}(a\rho + b + \int_0^\infty p(t)e^{i\rho t}dt), \ \ \rho \in \mathbb{C}_+, \tag{3.8}$$

where

$$\begin{aligned} a &= i\nu_1 + i\nu_2 \\ b &= -\nu_1\kappa^+(0,0) - \nu_2\kappa^-(0,0) - \nu_2\kappa^+(0,0) - \nu_1\kappa^-(0,0) \\ p(t) &= \kappa_x^-(0,t) - \nu_1\kappa^+(0,0)\kappa^-(0,-t) + \nu_2\kappa^+(0,t) \ - \nu_2\kappa_x^-(0,-t) \\ & \quad -\nu_1\kappa_x^+(0,t) + \nu_2\kappa^-(0,0)\kappa^+(0,t) - \nu_2(\kappa^+(0,t) * \kappa_x^+(0,-t)) \\ & \quad -\nu_1(\kappa_x^+(0,t) * \kappa^+(0,-t)) \end{aligned}$$

$m \in \mathbb{C}$, $n \in \mathbb{R}$ and $p \in L_1(0,\infty)$.

Proof $\beta(\rho)$ satisfies the following integral equation by virtue of (2.5) and (3.1)

$$\begin{aligned} \beta(\rho) = {} & (i\nu_1 + i\nu_2)\rho - \nu_1\kappa^+(0,0) - \nu_2\kappa^-(0,0) - \nu_2\kappa^+(0,0) - \nu_1\kappa^-(0,0) \\ & + i\rho\nu_1 \int_{-\infty}^0 \kappa^-(0,t)e^{-i\rho t}dt - \nu_1\kappa^+(0,0) \int_{-\infty}^0 \kappa^-(0,t)e^{-i\rho t}dt \\ & - \nu_2 \int_{-\infty}^0 \kappa_x^-(0,t)e^{-i\rho t}dt + i\rho\nu_2 \int_0^\infty \kappa^+(0,t)e^{i\rho t}dt + \nu_1 \int_0^\infty \kappa_x^+(0,t)e^{i\rho t}dt \\ & + \nu_1 \int_0^\infty \kappa_x^+(0,t)e^{i\rho t}dt \int_{-\infty}^0 \kappa^-(0,-t)e^{-i\rho t}dt - \nu_2\kappa^-(0,0) \int_0^\infty \kappa^+(0,t)e^{i\rho t}dt \\ & + \nu_2 \int_0^\infty \kappa^+(0,t)e^{i\rho t}dt \int_{-\infty}^0 \kappa_x^-(0,-t)e^{-i\rho t}dt, \end{aligned} \tag{3.9}$$

and the followings are explicit from (3.7) as well

$$\kappa^+(0,t), \kappa_x^+(0,t), \kappa_t^+(0,t) \in L_1(0,\infty) \tag{3.10}$$

$$\kappa^-(0,t), \kappa_x^-(0,t), \kappa_t^-(0,t) \in L_1(-\infty,0).$$

Then we can write

$$p \in L_1(0,\infty). \tag{3.11}$$

□

Consequently, $\beta(\rho)$ has analytic continuation to $\mathbb{C}_+$ and continuous up to the real axis.

Theorem 3.4 *The set of eigenvalues of the (2.1) − (2.2) problem is bounded and has at most a countable number of elements under condition (2.5). In addition, the boundary points of this cluster can only be located within a bounded subinterval on the real axis.*

Proof By using (3.11), we get that

$$\int_0^\infty p(t)e^{i\rho t}dt = o(1),\ \rho \in \mathbb{C}_+.$$

Hence the asymptotic behaviour of $\beta(\rho)$ as $|\rho| \to \infty$ is

$$\beta(\rho) = \frac{1}{2i\rho\nu_1\nu_2}(a\rho + b + o(1)),\ \rho \in \mathbb{C}_+. \tag{3.12}$$

This asymptotic equation indicates the boundedness of the set of eigenvalues, and its limit points can lie only in a bounded subinterval of the real axis. □

Lemma 3.5 *The condition (2.4) implies that the function $\upsilon(x)$ in equation (2.1) has the representation*

$$\upsilon(x) \sim \left(\frac{1}{|x|^{2+\epsilon}}\right),\ x \to \pm\infty \tag{3.13}$$

for $\epsilon > 0$. Thus, $\upsilon(x) \in L_2(-\infty,\infty)$ is satisfied and so the equation (2.1) is in the limit point case at $\pm\infty$ [8].

Theorem 3.6 *All eigenvalues of the problem (2.1) – (2.2) are real.*

Proof Let L be defined by

$$L(y) = -y''(x) + \upsilon(x)y(x) \tag{3.14}$$

in the Hilbert space $L_2(-\infty,\infty) := L_2(-\infty,0)\oplus L_2(0,\infty)$, where $\upsilon(x)$ is a real valued function. The eigenvalues problem of (2.1) – (2.2) which is defined on the Hilbert space $L_2(-\infty,\infty)$ can be considered as the eigenvalue problem of L. Now, we define the special inner product in the $L_2(-\infty,\infty)$ as follow

$$\langle y,z\rangle_{L_2(-\infty,\infty)} = \nu_1\nu_2\langle y_1,z_1\rangle_{L_2(-\infty,0)} + \langle y_2,z_2\rangle_{L_2(0,\infty)}, \tag{3.15}$$

where $y = (y_1,y_2)$, $z = (z_1,z_2) \in L_2(-\infty,\infty)$, and $\nu_1\nu_2 > 0$. Our first aim is to show that the operator L is symmetric. From (3.15), we get

$$\langle Ly,z\rangle_{L_2(-\infty,\infty)} = \nu_1\nu_2\langle Ly_1,z_1\rangle_{L_2(-\infty,0)} + \langle Ly_2,z_2\rangle_{L_2(0,\infty)}$$

Furthermore,

$$\langle Ly_1,z_1\rangle_{L_2(-\infty,0)} = W[y_1,\overline{z_1};0^-] - W[y_1,\overline{z_1};-\infty] + \langle y_1,Lz_1\rangle_{L_2(-\infty,0)} \tag{3.16}$$

and

$$\langle Ly_2,z_2\rangle_{L_2(0,\infty)} = W[y_2,\overline{z_2};\infty] - W[y_2,\overline{z_2};0^+] + \langle y_2,Lz_2\rangle_{L_2(0,\infty)}, \tag{3.17}$$

can be written where $W[y,z;x]$ denote the Wronskian of the functions y and z:

$$W[y,z;x] \quad = \quad y(x)z^{'}(x) - y^{'}(x)z(x).$$

Using (3.15), (3.16), and (3.17), we obtain

$$\langle Ly, z\rangle_{L_2(-\infty,\infty)} = \langle y_2, Lz_2\rangle_{L_2(0,\infty)} + \nu_1\nu_2\langle y_1, Lz_1\rangle_{L_2(-\infty,0)} \tag{3.18}$$

$$+\nu_1\nu_2 W[y_1, \overline{z_1}; 0] - W[y_2, \overline{z_2}; 0] W[y_2, \overline{z_2}; \infty] - \nu_1\nu_2 W[y_1, \overline{z_1}; -\infty]$$

Let us consider that $y = (y_1, y_2)$, $z = (z_1, z_2)$ satisfy the transmission condition (2.2), so

$$W[y_2, \overline{z_2}; 0] = y_2(0)\overline{z_2}'(0) - y_2'(0)\overline{z_2}(0) = \nu_1\nu_2 W[y_1, \overline{z_1}; 0] \tag{3.19}$$

is provided. According to Lemma 3.5, equation (2.1) is in the limit point case and so, the Wronskian at $\pm\infty$ is equal to zero by Hartman Theorem [11], i.e.

$$W[y_2, \overline{z_2}; \infty] = 0, W[y_1, \overline{z_1}; -\infty] = 0. \tag{3.20}$$

From (3.15), (3.16) and (3.17), it can easily be seen that the operator L is symmetric. Consequently, L is self-adjoint operator and all transmission eigenvalues of problem (2.1) – (2.2) are real. □

The results of Theorem 3.2 and Theorem 3.6 give the following theorem.

Theorem 3.7 *All eigenvalues of problem* (2.1) – (2.2) *are negative.*

Theorem 3.8 *The set of transmission eigenvalues of problem* (2.1) – (2.2) *is finite.*

Proof Let us consider that μ_n and $\widetilde{\mu_n}$ are eigenvalues of problem (2.1) -(2.2) and $\delta = \inf_{n\in\mathbb{N}} |\rho_n - \widetilde{\rho_n}|$, where $\widetilde{\rho_n} \geq \rho_n > 0$ and $\max_n \rho_n \leq$M. If δ is positive, the proof has been completed. Assume that $\delta = 0$. Since $\mu_n = \rho_n^2$ and $\widetilde{\mu_n} = \widetilde{\rho_n}^2$ are negative and real, the following equalities hold.

$$\rho_n = i\gamma_n \text{ and } \widetilde{\rho_n} = i\widetilde{\gamma_n}$$

where γ_n, $\widetilde{\gamma_n} > 0$. From (2.5) and (2.6), there exists for $A > 0$ such that $|\xi_+(x, i\gamma_n)| > \frac{1}{2}e^{-\gamma_n x}$ for $x \in (A, \infty)$ and $|\xi_-(x, i\gamma_n)| > \frac{1}{2}e^{\gamma_n x}$ for $x \in (-\infty, -A)$ are satisfied. By using these inequalities, the following inequalities also can be obtained

$$\int_A^\infty \overline{\xi_+(x, \rho_n)}\xi_+(x, \widetilde{\rho}_n)dx \frac{e^{-A(\gamma_n + \widetilde{\gamma_n})}}{4(\gamma_n + \widetilde{\gamma_n})} > \frac{e^{-2AM}}{8M} > 0 \tag{3.21}$$

$$\int_{-\infty}^{-A} \overline{\xi_-(x, \rho_n)}\xi_-(x, \widetilde{\rho}_n)dx \frac{e^{-A(\gamma_n + \widetilde{\gamma_n})}}{4(\gamma_n + \widetilde{\gamma_n})} > \frac{e^{-2AM}}{8M} > 0. \tag{3.22}$$

As μ_n is an eigenvalues, $\beta(\rho_n) = W[E(\rho_n, x), F(\rho_n, x)] = 0$. So $E(x, \rho_n)$ and $F(x, \rho_n)$ should be dependent. Hence,

$$E(x, \rho_n) = c_n F(x, \rho_n), c_n \neq 0 \tag{3.23}$$

is written for each ρ_n are real. Since L is a self-adjoint operator, we find

$$\int_{-\infty}^{\infty} L[E(x,\widetilde{\rho_n})]\overline{E(x,\rho_n)}dx \quad = \quad \int_{-\infty}^{\infty} E(x,\widetilde{\rho_n})\overline{L[E(x,\rho_n)]}dx$$

$$(\widetilde{\rho_n}^{\,2} - \overline{\rho}_n^2)\int_{-\infty}^{\infty} E(x,\widetilde{\rho_n})\overline{E(x,\rho_n)}dx \quad = 0.$$

It is known that all eigenvalues of L are real and μ_n, $\widetilde{\mu_n}$ are different eigenvalues, so $\rho_n^2 \neq \widetilde{\rho_n}^{\,2}$ is satisfied. Hence, $E(x,\widetilde{\rho_n})$ and $\overline{E(x,\rho_n)}$ are orthogonal in $L_2(-\infty,\infty)$. By using (2.1) and (3.23), the following integral equation is obtained

$$\begin{aligned}
0 = & \int_0^{\infty} E(x,\widetilde{\rho_n})\overline{E(x,\rho_n)}dx + c_n\widetilde{c_n}\int_{-\infty}^{0} F(x,\widetilde{\rho_n})\overline{F(x,\rho_n)}dx \\
= & \int_A^{\infty} \xi_+(x,\widetilde{\rho_n})\overline{\xi_+(x,\rho_n)}dx + c_n\widetilde{c_n}\int_{-\infty}^{-A} \xi_-(x,\widetilde{\rho_n})\overline{\xi_-(x,\rho_n)}dx \\
& + \int_0^{A} |\xi_+(x,\widetilde{\rho_n})|^2 dx + \int_0^{A} \xi_+(x,\widetilde{\rho_n})[\overline{\xi_+(x,\rho_n)} - \overline{\xi_+(x,\widetilde{\rho_n})}]dx \\
& + c_n\widetilde{c_n}\int_{-A}^{0} |\xi_-(x,\widetilde{\rho_n})|^2 dx + c_n\widetilde{c_n}\int_{-A}^{0} \xi_-(x,\widetilde{\rho_n})[\overline{\xi_-(x,\rho_n)} - \overline{\xi_-(x,\widetilde{\rho_n})}]dx.
\end{aligned}$$

Since δ is equal to zero, i.e. $\lim_{n\to\infty} \rho_n - \widetilde{\rho_n} = 0$, we can write

$$\lim_{n\to\infty}\int_0^{A} \xi_+(x,\widetilde{\rho_n})[\overline{\xi_+(x,\rho_n)} - \overline{\xi_+(x,\widetilde{\rho_n})}]dx = 0, \tag{3.24}$$

and similarly,

$$\lim_{n\to\infty}\int_{-A}^{0} \xi_-(x,\widetilde{\rho_n})[\overline{\xi_-(x,\rho_n)} - \overline{\xi_-(x,\widetilde{\rho_n})}]dx = 0. \tag{3.25}$$

As $E(x,\rho_n)$ and $F(x,\rho_n)$ are continuous for $\rho_n \in \mathbb{R}$,

$$\lim_{n\to\infty}(c_n\widetilde{c_n}) = [\frac{E(x,\rho_0)}{F(x,\rho_0)}]^2 > 0 \tag{3.26}$$

can be obtained. Using (3.21), (3.22), and (3.23), the following inequalities must be satisfied

$$\int_A^{\infty} \xi_+(x,\widetilde{\rho_n})\overline{\xi_+(x,\rho_n)}dx < 0, \tag{3.27}$$

$$\int_{-\infty}^{-A} \xi_-(x,\widetilde{\rho_n})\overline{\xi_-(x,\rho_n)}dx < 0. \tag{3.28}$$

But the relations between (3.21), (3.22) and (3.27), (3.28) give us a contradiction. Then, it can be seen that $\delta > 0$. As $\delta > 0$ and σ_d are bounded, the set of transmission eigenvalues is finite. □

3.1. Special case

Example 3.9 *Let us consider the transmission boundary value problem*

$$-y''(x) = \rho^2 y(x), \quad x \in \mathbb{R}/\{0\} \tag{3.29}$$

$$\begin{aligned} y(0^+) &= y(0^-), \\ y'(0^+) &= 2y'(0^-). \end{aligned} \tag{3.30}$$

In this case, it can be seen that the Jost solutions of the problem (3.29) − (3.30) *are*

$$\xi_-(x,\rho) = e^{-i\rho x} \quad \textit{and} \quad \xi_+(x,\rho) = e^{i\rho x}. \tag{3.31}$$

So, using the expression of $\beta(\rho)$ *in the equation* (3.3)*, we get* $\beta(\rho) = \frac{3}{4}$*. Consequently, the set of the eigenvalues of this problem is*

$$\sigma_d = \{\mu : \ \mu = \rho^2, \rho \in C_+, \beta(\rho) = 0\} = \emptyset. \tag{3.32}$$

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