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A sufficient condition for the wildness of an automorphism of a free Leibniz algebra

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Abstract: In this paper, we apply the criterion of Mikhalev and Umirbaev for the invertibility of an endomorphism of a finitely generated free Leibniz algebra via its Jacobian matrix to determine whether a given endomorphism is an automorphism. Moreover, it is shown that the invertibility of the determinant of the Jacobian matrix of an automorphism implies its wildness.

Key words: Leibniz algebra, Jacobian matrices, tame automorphism

1. Introduction

Leibniz algebras are algebraic structures that generalize the concept of Lie algebras by allowing nonantisymmetric products. The study of Leibniz algebras encompasses a wide range of topics, including their structure theory, automorphisms, representations, and subalgebras. In recent years, there has been significant progress in understanding the properties of Leibniz algebras and exploring various aspects of their theory. This progress has been driven by the efforts of many researchers who have made important contributions to the field.

Bloh [3] was the first to define the notions of a left and right Leibniz algebra (but called them left and right D-algebra, respectively). Much later, and independently, Loday reinvented the concept of a left and right Leibniz algebra. In particular, Loday and Pirashvili [6] made significant contributions to the theory of Leibniz algebras by studying the universal enveloping algebras of these structures and their (co)homology. Their work shed light on the algebraic properties and cohomological aspects of Leibniz algebras. In [1] Abdykhalikov et al. characterized tame automorphisms of free Leibniz algebras with two generators. This characterization offers an algorithm for identifying tame automorphisms. Building upon these findings, they have also devised a construction method for generating wild automorphisms. Drensky and Papistas established that the automorphism group of a free nilpotent Leibniz algebra with more than two generators can be generated by all tame automorphisms and one additional wild automorphism (see [5, Theorem 4.5]). In [7, Theorem2] Mikhalev and Umirbaev proved the analog of the Jacobian conjecture for Leibniz algebras, i.e. they proved that an endomorphism of a finitely generated free Leibniz algebra is an automorphism if, and only if, its Jacobian matrix is invertible.

Let $\mathfrak{F}$ be a finitely generated free Leibniz algebra over a field of characteristic zero. In this paper we start out by exploring some properties of partial derivatives of free Leibniz algebras. Then we apply the criterion of Mikhalev and Umirbaev for the invertibility of an endomorphism of a finitely generated free Leibniz algebra

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via its Jacobian matrix (see [7, Theorem 2]) to determine whether a given endomorphism is an automorphism. Moreover, we provide a sufficient condition for an automorphism of $\mathfrak{F}$ to be wild in terms of the determinant of its Jacobian matrix. In [1] Abdykhalikov et al. constructed a wild automorphism for free Leibniz algebras with two generators. Note that it is still unknown whether a free Leibniz algebra with more than two generators has wild automorphisms. We address this problem affirmatively by establishing the existence of a wild automorphism for any finitely generated free Leibniz algebra, and thus complementing Theorem 3 in [1]. Moreover, we construct another wild automorphism for finitely generated free Leibniz algebras. For three generators this automorphism is an analogue of the Anick automorphism. Umirbaev proved that the Anick automorphism for free associative algebras with three generators is wild (see [12]).

2. Preliminaries

A right Leibniz algebra $\mathfrak{L}$ over a field K is a nonassociative algebra with a K-bilinear product that satisfies the Leibniz identity

$$x(yz) = (xy)z - (xz)y$$

for all $x, y, z \in \mathfrak{L}$. In case $\mathfrak{L}$ is anticommutative, i.e. $x^2 = 0$ for all $x \in \mathfrak{L}$, the Leibniz identity is equivalent to the Jacobi identity. The Leibniz identity provides us with a powerful tool to simplify Leibniz products, as it enables us to represent any product as a linear combination of right-normed products. A right-normed product with 3 factors is written as $xyz = (xy)z$. To reference the ideal generated by elements $x^2, x \in \mathfrak{L}$ in $\mathfrak{L}$, we use the notation $\mathrm{Ann}(\mathfrak{L})$. It is well known by [6] that $r_a = 0$ if $a \in \mathrm{Ann}(\mathfrak{L})$. We denote the resulting Lie algebra as $\mathfrak{L}_{\mathrm{Lie}} = \mathfrak{L}/\mathrm{Ann}(\mathfrak{L})$ and the image of an element $x \in \mathfrak{L}$ under the canonical homomorphism $\pi : \mathfrak{L} \to \mathfrak{L}_{\mathrm{Lie}}$ as $\overline{x}$. Additionally, we refer to the automorphism group of $\mathfrak{L}$ as $\mathrm{Aut}\,\mathfrak{L}$.

In [6], the universal enveloping algebra of a Leibniz algebra was introduced. To elaborate, consider two distinct copies of the Leibniz algebra $\mathfrak{L}$, which we denote as $\mathfrak{L}^l$ and $\mathfrak{L}^r$. In these copies, we identify elements labeled as l_x and r_x, corresponding to the universal operators governing left and right multiplication by x. Let $I_{\mathfrak{L}}$ be the two-sided ideal of the associative tensor K-algebra $T(\mathfrak{L}^l \oplus \mathfrak{L}^r)$ over K with an identity element satisfying the following relations

$$r_{xy} = r_x r_y - r_y r_x \tag{2.1}$$

$$l_{xy} = l_x r_y - r_y l_x \tag{2.2}$$

$$(r_x + l_x) l_y = 0 \tag{2.3}$$

for any $x, y \in \mathfrak{L}$. Then the factor algebra $UL(\mathfrak{L}) = T(\mathfrak{L}^l \oplus \mathfrak{L}^r)/I_{\mathfrak{L}}$ is the universal enveloping algebra of the Leibniz algebra $\mathfrak{L}$. Using the homomorphism π, we obtain homomorphisms d_0 and d_1 such that

$$d_0, d_1 : UL(\mathfrak{L}) \to U(\mathfrak{L}_{\mathrm{Lie}}),$$

where $U(\mathfrak{L}_{\mathrm{Lie}})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{L}_{\mathrm{Lie}}$, and we have

$$d_1(r_x) = \overline{x},\ d_1(l_x) = -\overline{x}$$

$$d_0(r_x) = \overline{x},\ d_0(l_x) = 0.$$

The kernel $\mathrm{Ker}d_1$ (respectively $\mathrm{Ker}d_0$) is generated by the elements $r_x + l_x$ (respectively l_x) for $x \in UL(\mathfrak{L})$ and $\mathrm{Ker}d_1\mathrm{Ker}d_0 = 0$ (see [6, Proposition 2.5]).

Loday and Pirashvili [6] provided a comprehensive description of free Leibniz algebras. Consider a set X, and let $\mathfrak{L}(X)$ represent the free nonassociative algebra over K generated by X. Additionally, define I_X as the two-sided ideal in $\mathfrak{L}(X)$ generated by elements of the form:

$$a(bc) - (ab)c + (ac)b$$

for all $a, b, c \in \mathfrak{L}(X)$. Then the algebra $\mathfrak{F}(X) = \mathfrak{L}(X)/I_X$ is a free Leibniz algebra generated by X. For more details, we refer the reader to [6].

3. A sufficient condition for the wildness of an automorphism

Let $\mathfrak{F}$ be the free Leibniz algebra with the free generators $x_1, \ldots, x_n$ over a field K of characteristic 0. By definition, $UL(\mathfrak{F})$ is generated by the elements l_{x_i}, r_{x_i} $(i = 1, \ldots, n)$ that satisfy the relations (2.1), (2.2), (2.3). In addition, $\mathfrak{F}$ is a right $UL(\mathfrak{F})$-module with a right action given by

$$zl_x = xz,$$

$$zr_x = zx,$$

where $x, z \in \mathfrak{F}$. Define the *length* of a monomial $u = u_1 u_2 ... u_k \in UL(\mathfrak{F})$ as the number of elements $u_j \in \{r_{x_i}, l_{x_i} | i = 1, ..., n\}$ $(j = 1, ..., k$ for $k \geq 1)$ that u contains. Let I be the right ideal of the algebra $UL(\mathfrak{F})$ generated by the elements l_x, $x \in \mathfrak{F}$ which is the free $UL(\mathfrak{F})$-module with the basis $\{l_{x_1}, l_{x_2}, \ldots, l_{x_n}\}$. Partial derivatives in free Leibniz algebras are defined by Mikhalev and Umirbaev in [7] as follows. Consider mapping

$$d : \mathfrak{F} \to I$$

given by

$$d(x) = l_x, x \in \mathfrak{F}.$$

By the relations (2.2) and (2.3), it is obtained

$$d(xy) = l_{xy} = l_x r_y - r_y l_x = l_x r_y + l_y l_x = d(x) r_y + d(y) l_x$$

for all $x, y \in \mathfrak{F}$ (see [7, p. 437]). Therefore the mapping d is a universal derivation of the algebra $\mathfrak{F}$ with coefficients in the $UL(\mathfrak{F})$-module $UL(\mathfrak{F})$ (see [10, Definition 4]). The partial derivatives $\dfrac{\partial f}{\partial x_i}$ of an element f of the algebra $\mathfrak{F}$ can be expressed using the following formula;

$$d(f) = \sum_{i=1}^{n} d(x_i) \frac{\partial f}{\partial x_i}, \tag{3.1}$$

where $\dfrac{\partial f}{\partial x_i} \in UL(\mathfrak{F})$ (see [7], p. 440).

Lemma 3.1 *Let f_n be a monomial in $\mathfrak{F}$ of the form*

$$x_{i_1} x_{i_2} \ldots x_{i_n},$$

where $i_j \in \{1, \dots, n\}$ *and* $j = 1, \dots, n$. *Then*

$$d(f_n) = \sum_{j=1}^{n} d(x_{i_j}) l_{x_{i_1} x_{i_2} \dots x_{i_{j-1}}} r_{x_{i_{j+1}}} \dots r_{x_{i_n}}$$

and

$$\frac{\partial f_n}{\partial x_{i_j}} = l_{x_{i_1} x_{i_2} \dots x_{i_{j-1}}} r_{x_{i_{j+1}}} \dots r_{x_{i_n}}$$

Proof We use induction on n. Given a monomial $f_2 = x_{i_1} x_{i_2} \in \mathfrak{F}$, we obtain that

$$d(f_2) = d(x_{i_1} x_{i_2}) = l_{x_{i_1} x_{i_2}} = l_{x_{i_1}} r_{x_{i_2}} - r_{x_{i_2}} l_{x_{i_1}} = l_{x_{i_1}} r_{x_{i_2}} + l_{x_{i_2}} l_{x_{i_1}} = d(x_{i_1}) r_{x_{i_2}} + d(x_{i_2}) l_{x_{i_1}}$$

By the definition of partial derivatives, we get

$$\frac{\partial f_2}{\partial x_{i_1}} = r_{x_{i_2}}, \ \frac{\partial f}{\partial x_{i_2}} = l_{x_{i_1}}.$$

Now let us take $f_{n-1} = x_{i_1} x_{i_2} \dots x_{i_{n-1}} \in \mathfrak{F}$. By induction hypothesis, we write

$$d(f_{n-1}) = \sum_{j=1}^{n-1} d(x_{i_j}) l_{x_{i_1} x_{i_2} \dots x_{i_{j-1}}} r_{x_{i_{j+1}}} \dots r_{x_{i_{n-1}}}$$

and

$$\frac{\partial f_{n-1}}{\partial x_{i_j}} = l_{x_{i_1} x_{i_2} \dots x_{i_{j-1}}} r_{x_{i_{j+1}}} \dots r_{x_{i_{n-1}}}.$$

Then for the element $f_n = x_{i_1} x_{i_2} \dots x_{i_n} = f_{n-1} x_{i_n}$, we obtain that

$$\begin{aligned} d(f_n) &= d(f_{n-1}) r_{x_{i_n}} + d(x_{i_n}) l_{f_{n-1}} \\ &= \left(\sum_{j=1}^{n-1} d(x_{i_j}) l_{x_{i_1} x_{i_2} \dots x_{i_{j-1}}} r_{x_{i_{j+1}}} \dots r_{x_{i_{n-1}}}\right) r_{x_{i_n}} + d(x_{i_n}) l_{x_{i_1} x_{i_2} \dots x_{i_{n-1}}} \\ &= \sum_{j=1}^{n} d(x_{i_j}) l_{x_{i_1} x_{i_2} \dots x_{i_{j-1}}} r_{x_{i_{j+1}}} \dots r_{x_{i_n}}. \end{aligned}$$

Hence,

$$\frac{\partial f}{\partial x_{i_j}} = l_{x_{i_1} x_{i_2} \dots x_{i_{j-1}}} r_{x_{i_{j+1}}} \dots r_{x_{i_n}}.$$

□

The following lemma provides a chain rule for partial derivatives. Its proof is inspired by the proof of a similar chain rule for Lie algebras by Umirbaev (see [11, p. 1162]).

Lemma 3.2 *Given* $v = v(x_1, \dots, x_n) \in \mathfrak{F}$. *Then for* $y_1, \dots, y_n \in \mathfrak{F}$, $\frac{\partial v(y_1,\dots,y_n)}{\partial x_k} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial v(y_1,\dots,y_n)}{\partial y_i}$ *holds.*

Proof Applying the identity (3.1) to the element $v(x_1,...,x_n)$, we obtain

$$d(v(x_1,\ldots,x_n)) = \sum_{i=1}^{n} d(x_i)\frac{\partial v(x_1,\ldots,x_n)}{\partial x_i}. \tag{3.2}$$

If we substitute arbitrary y_i's for x_i's in (3.2), it follows

$$d(v(y_1,\ldots,y_n)) = \sum_{i=1}^{n} d(y_i)\frac{\partial v(y_1,\ldots,y_n)}{\partial y_i}.$$

Then it implies

$$d(v(y_1,...,y_n)) = \sum_{i=1}^{n}(\sum_{k=1}^{n} d(x_k)\frac{\partial y_i}{\partial x_k})\frac{\partial v(y_1,\ldots,y_n)}{\partial y_i}.$$

Therefore

$$\frac{\partial v(y_1,\ldots,y_n)}{\partial x_k} = \sum_{i=1}^{n}\frac{\partial y_i}{\partial x_k}\frac{\partial v(y_1,\ldots,y_n)}{\partial y_i}.$$

□

The following lemma characterizes the invertibility of certain elements in $UL(\mathfrak{F})$. In fact, the elements we consider are exactly the elements that do not belong to the augmentation ideal of $UL(\mathfrak{F})$. Note that $UL(\mathfrak{F})$ is an *augmented algebra*, i.e. there exists an algebra epimorphism $\varepsilon : UL(\mathfrak{F}) \to K$, which is defined by $\varepsilon(1) := 1$, $\varepsilon(l_x) := 0$, and $\varepsilon(r_x) := 0$ for any $x \in X$. The kernel $\mathrm{Ker}(\varepsilon)$ is called the *augmentation ideal* of $UL(\mathfrak{F})$, and one has $UL(\mathfrak{F}) = K\cdot 1 \oplus \mathrm{Ker}(\varepsilon)$ as vector spaces. The latter shows that the elements α and u in the following lemma are well-defined and independent of the set of generators.

Lemma 3.3 *Let $\alpha + u \in UL(\mathfrak{F})$ for some $\alpha \in K\backslash\{0\}$ and some $u \in \mathrm{Ker}(\varepsilon)$. Then $\alpha + u$ is invertible if, and only if, $u^2 = 0$. In this case $\frac{1}{\alpha} - \frac{1}{\alpha^2}u$ is the inverse of $\alpha + u$ in $UL(\mathfrak{F})$.*

Proof Firstly, suppose that $\alpha + u \in UL(\mathfrak{F})$ is invertible. Then

$$(\alpha + u)(\beta + v) = (\beta + v)(\alpha + u) = 1 \tag{3.3}$$

for some element $\beta + v \in UL(\mathfrak{F})$, where $\beta \in K$ and v belongs to the augmentation ideal of $UL(\mathfrak{F})$. We obtain that

$$\alpha\beta + \beta u + \alpha v + uv = \alpha\beta + \beta u + \alpha v + vu = 1.$$

Then it implies that $uv = vu$, and using the augmentation, we obtain $\alpha\beta = 1$ (which implies that $\beta \neq 0$) and

$$\beta u + \alpha v + uv = 0. \tag{3.4}$$

The image of the left-hand side of identity (3.4) is

$$d_t(\beta u) + d_t(\alpha v) + d_t(u)d_t(v) = 0. \tag{3.5}$$

for $t = 0, 1$. It is well known that $U(\mathfrak{F}_{\text{Lie}})$ is a graded algebra. If $d_t(u)$ and $d_t(v)$ are nonzero then the length of the element $d_t(u)d_t(v)$ is greater than the length of the both $d_t(u)$ and $d_t(v)$ for $t = 0, 1$. Therefore we obtain

$$d_t(\beta u) + d_t(\alpha v) = 0 \tag{3.6}$$

and

$$d_t(u)d_t(v) = 0 \tag{3.7}$$

by (3.5) for t=0.1. Since the algebra $U(\mathfrak{F}_{\text{Lie}})$ has no zero divisors, the identity (3.7) leads to a contradiction. At the same time we find that if $d_t(u) = 0$ then $d_t(v) = 0$ by (3.5). Similarly if $d_t(v) = 0$ then $d_t(u) = 0$. This implies that $d_t(u) = 0$ and $d_t(v) = 0$ for $t = 0.1$. Therefore u and v belong to the kernels $\text{Ker}d_0$ and $\text{Ker}d_1$. So u and v are elements of the intersection of these kernels, $\text{Ker}d_0 \cap \text{Ker}d_1$. It follows that u and v are elements of the ideals $\text{Ker}d_0\text{Ker}d_1$ or $\text{Ker}d_1\text{Ker}d_0$. If u and v are elements of $\text{Ker}d_1\text{Ker}d_0$ then $u = v = 0$ by the equation $\text{Ker}d_1\text{Ker}d_0 = 0$ (see [6, Proposition 2.5]). If u and v are elements of $\text{Ker}d_0\text{Ker}d_1$ then we obtain

$$u^2 = uu \in \text{Ker}d_0\text{Ker}d_1\text{Ker}d_0\text{Ker}d_1 = 0$$

which yields that $u^2 = 0$. Similarly we get $v^2 = 0$ and $uv = 0$. Subsequently, we obtain $v = -\dfrac{\beta}{\alpha}u$ by identity (3.4) which implies the inverse of $\alpha + u$ is determined as $\dfrac{1}{\alpha} - \dfrac{1}{\alpha^2}u$.

Conversely, let $\alpha + u \in UL(\mathfrak{F})$ for $\alpha \in K\backslash\{0\}$ and $u^2 = 0$. Then

$$(\alpha + u)(\frac{1}{\alpha} - \frac{1}{\alpha^2}u) = 1 - \frac{1}{\alpha^2}u^2 = 1.$$

□

As an example, take $1 + l_{x_2}(l_{x_1} + r_{x_1}) \in UL(\mathfrak{F})$. From the relation (2.3) in $UL(\mathfrak{F})$, we get

$$(1 + l_{x_2}(l_{x_1} + r_{x_1}))(1 - l_{x_2}(l_{x_1} + r_{x_1})) = 1 - l_{x_2}(l_{x_1} + r_{x_1})l_{x_2}(l_{x_1} + r_{x_1}) = 1,$$

which shows that $1 + l_{x_2}(l_{x_1} + r_{x_1})$ is an invertible element in $UL(\mathfrak{F})$.

Lemma 3.4 *If $y \in \text{Ann}(\mathfrak{F})$, then $\alpha + l_y$ $(\alpha \in K\backslash\{0\})$ is invertible in $UL(\mathfrak{F})$.*

Proof It follows from identity (2.1) that $r_{x^2} = r_x r_x - r_x r_x = 0$ for any $x \in \mathfrak{F}$ which immediately yields that $r_y = 0$ for any $y \in \text{Ann}(\mathfrak{F})$. The equality $0 = d(xy) = l_{xy} = l_x r_y + l_y l_x = l_y l_x$ is then deduced for all $x \in \mathfrak{F}$. Thus, we get $l_y^2 = 0$ and by Lemma 3.3, $\alpha + l_y$ is invertible. □

Let f be an element of $\mathfrak{F}$. By $\partial(f)$, we denote the column $(\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n})^T$, where T indicates transposition. Given an endomorphism φ of $\mathfrak{F}$ such that $\varphi(x_j) = f_j$ for $j = 1, \ldots, n$. The *Jacobian matrix* $J(\varphi)$ of the endomorphism φ is the matrix with rows $(\partial(f_1), \ldots, \partial(f_n))$ of the elements $f_1, f_2, \ldots, f_n$, i.e.

$$J(\varphi) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

An endomorphism ψ of $\mathfrak{F}$ induces an endomorphism $\overline{\psi}$ of $UL(\mathfrak{F})$ and the endomorphism $\overline{\psi}$ induces an endomorphism on the matrix algebra with components from $UL(\mathfrak{F})$ that this endomorphism is denoted by the same symbol $\overline{\psi}$. Given endomorphisms ψ and φ of $\mathfrak{F}$ such that $\varphi(x_j) = v_j = v_j(x_1, ..., x_n)$ and $\psi(x_j) = y_j = y_j(x_1, ..., x_n)$ for $j = 1, \dots, n$. The image $\psi(v_j(x_1, ..., x_n))$ is obtained substituting arbitrary y_j's for x_j's in $v_j(x_1, ..., x_n)$ that is $v_j(y_1, ..., y_n)$. The Jacobian matrix of the composition $\psi \circ \varphi$ of ψ and φ is

$$J(\psi \circ \varphi) = \begin{pmatrix} \frac{\partial \psi\circ\varphi(x_1)}{\partial x_1} & \frac{\partial \psi\circ\varphi(x_2)}{\partial x_1} & \cdots & \frac{\partial \psi\circ\varphi(x_n)}{\partial x_1} \\ \frac{\partial \psi\circ\varphi(x_1)}{\partial x_2} & \frac{\partial \psi\circ\varphi(x_2)}{\partial x_2} & \cdots & \frac{\partial \psi\circ\varphi(x_n)}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \psi\circ\varphi(x_1)}{\partial x_n} & \frac{\partial \psi\circ\varphi(x_2)}{\partial x_n} & \cdots & \frac{\partial \psi\circ\varphi(x_n)}{\partial x_n} \end{pmatrix}$$

Lemma 3.2 implies that

$$\frac{\partial \psi \circ \varphi(x_j)}{\partial x_k} = \frac{\partial \psi(v_j(x_1, ..., x_n))}{\partial x_k} = \frac{\partial v_j(y_1, ..., y_n)}{\partial x_k} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_k} \frac{\partial v_j(y_1, \dots, y_n)}{\partial y_i} \tag{3.8}$$

for $j = 1, \dots, n$. Rewriting the components of $J(\psi \circ \varphi)$ using (3.8) yields

$$J(\psi \circ \varphi) = \begin{pmatrix} \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_1} \frac{\partial v_1(y_1,...,y_n)}{\partial y_i} & \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_1} \frac{\partial v_2(y_1,...,y_n)}{\partial y_i} & \cdots & \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_1} \frac{\partial v_n(y_1,...,y_n)}{\partial y_i} \\ \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_2} \frac{\partial v_1(y_1,...,y_n)}{\partial y_i} & \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_2} \frac{\partial v_2(y_1,...,y_n)}{\partial y_i} & \cdots & \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_2} \frac{\partial v_n(y_1,...,y_n)}{\partial y_i} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_n} \frac{\partial v_1(y_1,...,y_n)}{\partial y_i} & \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_n} \frac{\partial v_2(y_1,...,y_n)}{\partial y_i} & \cdots & \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_n} \frac{\partial v_n(y_1,...,y_n)}{\partial y_i} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial v_1(y_1,...,y_n)}{\partial y_1} & \frac{\partial v_2(y_1,...,y_n)}{\partial y_1} & \cdots & \frac{\partial v_n(y_1,...,y_n)}{\partial y_1} \\ \frac{\partial v_1(y_1,...,y_n)}{\partial y_2} & \frac{\partial v_2(y_1,...,y_n)}{\partial y_2} & \cdots & \frac{\partial v_n(y_1,...,y_n)}{\partial y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial v_1(y_1,...,y_n)}{\partial y_n} & \frac{\partial v_2(y_1,...,y_n)}{\partial y_n} & \cdots & \frac{\partial v_n(y_1,...,y_n)}{\partial y_n} \end{pmatrix}.$$

It implies the following composition rule for the Jacobian matrices

$$J(\psi \circ \varphi) = J(\psi)\overline{\psi}(J(\varphi)). \tag{3.9}$$

We denote by (u_{ij}) a $n \times n$ matrix with components u_{ij} for $i, j = 1, \dots, n$. An elementary matrix $E_{kl}(w)$ is defined as a matrix that differs from the identity matrix only by having an element w in the k-th row and l-th column. The inverse of $E_{kl}(w)$ is $E_{kl}(-w)$.

Lemma 3.5 *Let $E_{kl}(w)$ be an elementary matrix and D be a diagonal matrix with invertible elements over $UL(\mathfrak{F})$. Then there exists an elementary matrix E' such that $DE_{kl}(w) = E'D$.*

Proof Let D be a diagonal matrix, where the diagonal elements $d_i = \alpha_{ii} + u_{ii}$, $\alpha_{ii} \in K\backslash\{0\}$, $u_{ii} \in Ker(\varepsilon)$ for $i = 1, \dots, n$ are invertible over $UL(\mathfrak{F})$. Without lost of generality we consider elementary matrix $E_{1n}(w)$

over $UL(\mathfrak{F})$. Then we get

$$DE_{1n}(w) = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & w \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} d_1 & 0 & \cdots & d_1 w \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & d_1 w(\frac{1}{\alpha_{nn}} - \frac{1}{\alpha_{nn}^2} u_{nn}) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} = E_{1n}(d_1 w(\frac{1}{\alpha_{nn}} - \frac{1}{\alpha_{nn}^2} u_{nn}))D,$$

where $\frac{1}{\alpha_{nn}} - \frac{1}{\alpha_{nn}^2} u_{nn}$ is the inverse of d_n by Lemma 3.3. □

The homomorphısm d_t (t=0,1) induces a homomorphism of the $n \times n$ matrix algebras

$$M_{n\times n}(UL(\mathfrak{L})) \to M_{n\times n}(U(\mathfrak{L}_{\text{Lie}})),$$

denote by the same symbol d_t $(t = 0, 1)$, such that $d_t((w_{ij})) = (d_t(w_{ij}))$ for every $(w_{ij}) \in M_{n\times n}(UL(\mathfrak{L}))$

Lemma 3.6 *Given an elementary matrix $E_{kl}(u)$ over $U(\mathfrak{F}_{Lie})$. Then*

$$E_{kl}(u) = d_t\Big(E_{kl}(w_{kl}^t) + (v_{ij}^t)\Big)$$

holds, where $v_{ij}^t \in \text{Ker}d_t$, $w_{kl}^t \in UL(\mathfrak{F})$ does not involve elements of Kerd_t *for $t = 0, 1$ and $i, j = 1, \ldots, n$.*

Proof Without lost of generality we consider elementary matrix $E_{1n}(u)$ over $U(\mathfrak{F}_{Lie})$. Let $U = (u_{ij})$ be a matrix over $U(\mathfrak{F}_{Lie})$, where $u_{ij} = \alpha_{ij} + v_{ij}$ $(\alpha_{ij} \in K)$ for $i, j = 1, \ldots, n$. Then by the surjectivity of the homomorphisms d_t $(t = 0, 1)$ for every element $u_{ij} \in U(\mathfrak{F}_{Lie})$ there exist elements $\beta_{ij} + v_{ij}^t \in UL(\mathfrak{F})$ $(\beta_{ij} \in K)$ such that

$$u_{ij} = \alpha_{ij} + v_{ij} = d_t(\beta_{ij} + v_{ij}^t) = \beta_{ij} + d_t(v_{ij}^t)$$

for $i, j = 1, \ldots, n$. Then $\alpha_{ij} = \beta_{ij}$ and $v_{ij} = d_t(v_{ij}^t)$ are obtained for $i, j = 1, \ldots, n$. Thus we obtain

$$E_{1n}(u) = \begin{pmatrix} 1 & 0 & \cdots & u \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 + d_t(v_{11}^t) & d_t(v_{12}^t) & \cdots & \beta_{1n} + d_t(v_{1n}^t) \\ d_t(v_{21}^t) & 1 + d_t(v_{22}^t) & \cdots & d_t(v_{2n}^t) \\ \vdots & \vdots & \ddots & \vdots \\ d_t(v_{n1}^t) & d_t(v_{n2}^t) & \cdots & 1 + d_t(v_{nn}^t) \end{pmatrix},$$

where $\beta_{1n} + d_t(v_{1n}^t) = u$, $d_t(v_{ij}^t) = 0$ for $v_{ij}^t \neq v_{1n}^t$ for $i, j = 1, \ldots, n$, $t = 0, 1$. It follows $v_{ij}^t \in \text{Ker}d_t$ for $v_{ij}^t \neq v_{1n}^t$. We can write $\beta_{1n} + v_{1n}^t = w_1^t + w_2^t$, where $w_2^t \in \text{Ker}d_t$ and $w_1^t \in UL(\mathfrak{F})$ does not involve elements of Kerd_t. Therefore we obtain

$$E_{1n}(u) = d_t\Big(\begin{pmatrix} 1 & 0 & \cdots & w_1^t \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} v_{11}^t & v_{12}^t & \cdots & w_2^t \\ v_{21}^t & v_{22}^t & \cdots & v_{2n}^t \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1}^t & v_{n2}^t & \cdots & v_{nn}^t \end{pmatrix}\Big)$$

where $v_{ij}^t \in \mathrm{Ker}d_t$ for $v_{ij}^t \neq v_{1n}^t$, $i,j=1,\dots,n$, $d_t(w_1^t+w_2^t)=u$, $w_1^t+w_2^t \in UL(\mathfrak{F})$ such that w_2^t is belongs to $\mathrm{Ker}d_t$ and w_1^t does not involve elements of $\mathrm{Ker}d_t$. □

Let A be a matrix over $U(\mathfrak{F}_{Lie})$. The preimage of the matrix A under the homomorphism d_t, denoted by $d_t^{-1}(A)$, is the set of all matrices M over $UL(\mathfrak{F})$ such that the images are the matrix A. There exists an augmentation map $\bar{\varepsilon}: U(\mathfrak{F}_{\mathfrak{Lie}}) \to K$ which is defined by $\bar{\varepsilon}(1):=1$, $\bar{\varepsilon}(\bar{x}):=0$ for any $x \in X$. We denote $\bar{\varepsilon}(A)$ and $\varepsilon(M)$ by the matrices of the augmentations of the components of A and M, respectively.

Proposition 3.7 *Every invertible Jacobian matrix on $UL(\mathfrak{F})$ can be written of the form $D.E$ where E is a product of elementary matrices and D is a diagonal matrix.*

Proof Given an invertible Jacobian matrix M on $UL(\mathfrak{F})$. The image $\varepsilon(M)$ is invertible on the ground field K that it is of the form $D.E$, where $D=\mathrm{diag}(d_1,\dots,d_n)$ is the diagonal matrix with elements on the diagonal $d_i \in K\backslash\{0\}$ for $i=1,\dots,n$ and E is a product of elementary matrices over the field K. The images $d_0(M)$ and $d_1(M)$ are invertible matrices on $U(\mathfrak{F}_{Lie})$ and

$$\varepsilon(M)=\bar{\varepsilon}(d_0(M))=\bar{\varepsilon}(d_1(M)))=D.E.$$

Then $d_t(M)$ is of the form $D.E_t$, where $D=\mathrm{diag}(d_1,\dots,d_n)$ for $d_i \in K\backslash\{0\}$, $i=1,\dots,n$ and E_t is a product of elementary matrices over $U(\mathfrak{F}_{Lie})$ for $t=0,1$ (see the proof of Corollary 3.4, [8]). Then we can write

$$d_t(M)=D.\Pi E_{kl}(d_t(v_{kl})), \tag{3.10}$$

where $D=\mathrm{diag}(d_1,\dots,d_n)$ for $d_i \in K\backslash\{0\}$, $i=1,\dots,n$ and $\Pi E_{kl}(d_t(v_{kl}))$ is products of elementary matrices for $d_t(v_{kl}) \in U(\mathfrak{F}_{Lie})$, $t=0,1$. Every element of $UL(\mathfrak{F})$ is in the form $\alpha+u$ for $\alpha \in K$, $u \in \mathrm{Ker}\varepsilon$. Then we get

$$M=\varepsilon(M)+(u_{ij})=D.E+(u_{ij}), \tag{3.11}$$

where $u_{ij} \in \mathrm{Ker}\varepsilon$. It follows that

$$d_t(M)=DE+d_t((u_{ij})) \tag{3.12}$$

for $t=0,1$. By (3.10) and (3.12) we have

$$d_t(M)=DE+d_t((u_{ij}))=D.\Pi E_{kl}(d_t(v_{kl})), \tag{3.13}$$

Multiplying the both sides of the equations (3.13) and (3.11) by the inverse matrix D^{-1} of D on the left-hand side yields

$$E+D^{-1}d_t((u_{ij}))=\Pi E_{kl}(d_t(v_{kl})). \tag{3.14}$$

and

$$D^{-1}M=E+D^{-1}(u_{ij}). \tag{3.15}$$

Applying d_t to the both sides of (3.15) and by (3.14) we get

$$d_t(D^{-1}M)=E+D^{-1}d_t((u_{ij}))=\Pi E_{kl}(d_t(v_{kl}))$$

for $t = 0,1$. Then $D^{-1}M$ is an element of the intersection of the preimages $d_0^{-1}\Big(\Pi E_{kl}(d_0(v_{kl}))\Big)$ and $d_1^{-1}\Big(\Pi E_{kl}(d_1(v_{kl}))\Big)$. Using this fact let us determine $D^{-1}M$. By Lemma 3.6 we have

$$E_{kl}(d_t(v_{kl})) = d_t\Big(E_{kl}(w^t_{kl}) + (u^t_{ij})\Big)$$

for $w^t_{kl} \in UL(\mathfrak{F})$ does not involve elements of $\mathrm{Ker}d_t$, $u^t_{ij} \in \mathrm{Ker}d_t$ for $i,j,k,l = 1,\dots,n$, $t = 0,1$. Then we obtain

$$\Pi E_{kl}(d_t(v_{kl})) = \Pi d_t\Big(E_{kl}(w^t_{kl}) + (u^t_{ij})\Big) = d_t\Big(\Pi\Big(E_{kl}(w^t_{kl}) + (u^t_{ij})\Big)\Big) = d_t\Big(\Pi E_{kl}(w^t_{kl}) + (q^t_{ij})\Big) \tag{3.16}$$

where $w^t_{kl} \in UL(\mathfrak{F})$ does not involve elements of $\mathrm{Ker}d_t$, q^t_{ij}, $u^t_{ij} \in \mathrm{Ker}d_t$ for $i,j = 1,\dots,n$, $t = 0,1$. Thus we obtain the preimages

$$d_t^{-1}\Big(\Pi E_{kl}(d_t(v_{kl}))\Big) = \Big\{\Pi E_{kl}(w^t_{kl}) + (q^t_{ij}) : q^t_{ij} \in \mathrm{Ker}d_t, w^t_{kl} \in UL(\mathfrak{F}) \text{ does not involve elements of } \mathrm{Ker}d_t\Big\}$$

for $t = 0,1$. The matrices in the intersection $d_0^{-1}\Big(\Pi E_{kl}(d_0(v_{kl}))\Big) \cap d_1^{-1}\Big(\Pi E_{kl}(d_1(v_{kl}))\Big)$ holds the following equality

$$\Pi E_{kl}(w^0_{kl}) + (q^0_{ij}) = \Pi E_{kl}(w^1_{kl}) + (q^1_{ij}) \tag{3.17}$$

for $q^t_{ij} \in \mathrm{Ker}d_t$, $w^t_{kl} \in UL(\mathfrak{F})$ does not involve elements of $\mathrm{Ker}d_t$ for $t = 0,1$. Multiplying the both side of (3.17) by the inverse matrix $\Pi E_{kl}(-w^0_{kl})$ of $\Pi E_{kl}(w^0_{kl})$ on the left-hand side gives

$$I + (p^0_{ij}) = \Pi E_{kl}(-w^0_{kl})\Pi E_{kl}(w^1_{kl}) + (p^1_{ij}) \tag{3.18}$$

where I is the identity matrix, $p^0_{ij} \in \mathrm{Ker}d_0$, $p^1_{ij} \in \mathrm{Ker}d_1$. Since $\varepsilon(p^0_{ij}) = \varepsilon(p^1_{ij}) = 0$, we have

$$I = \Pi E_{kl}(-w^0_{kl})\Pi E_{kl}(w^1_{kl}) \tag{3.19}$$

that follows

$$\Pi E_{kl}(w^0_{kl}) = \Pi E_{kl}(w^1_{kl}). \tag{3.20}$$

This holds $q^0_{ij} = q^1_{ij}$ by (3.17) which implies q^0_{ij}, $q^1_{ij} \in \mathrm{Ker}d_0\mathrm{Ker}d_1$. Subsequently u^t_{ij} are elements of $\mathrm{Ker}d_0\mathrm{Ker}d_1$ by (3.16) for $i,j = 1,\dots,n$, $t = 0,1$. Hence $D^{-1}M$ is in the following form

$$\Pi\Big(E_{kl}(w^t_{kl}) + (u^t_{ij})\Big)$$

for $u^t_{ij} \in \mathrm{Ker}d_0\mathrm{Ker}d_1, w^t_{kl} \in UL(\mathfrak{F})$. Since $(u^t_{ij})^2 = 0$, every matrix $E_{kl}(w^t_{kl}) + (u^t_{ij})$ in this product has the diagonal elements $1 + u^t_{ii}$ that are invertible by Lemma 3.3 for $i = 1,\dots,n$. This allows us to reduce these matrices to a diagonal form by applying elementary transformations to its rows. Subsequently we can write $E_{kl}(w^t_{kl}) + (u^t_{ij})$ of the form $D_{kl}.E_{kl}$ where D_{kl} is a diagonal matrix with invertible diagonal elements and E_{kl} is a product of elementary matrices over $UL(\mathfrak{F})$ by Lemma 3.5. Consequently, we have

$$M = D\Pi\Big(E_{kl}(w^t_{kl}) + (u^t_{ij})\Big) = D\Pi D_{kl}.E_{kl}$$

where $D = \mathrm{diag}(d_1, \dots, d_n)$ is the diagonal matrix with elements on the diagonal $d_i \in K\backslash\{0\}$, D_{kl} is a diagonal matrix with invertible diagonal elements and E_{kl} is a product of elementary matrices over $UL(\mathfrak{F})$. The proof is completed by Lemma 3.5. □

Let φ be an automorphism of $\mathfrak{F}$. According to the result by Mikhalev and Umirbaev (see [7, Theorem 2]) the Jacobian matrix $J(\varphi)$ is an invertible matrix over $UL(\mathfrak{F})$. If $J(\varphi)$ has at least one invertible element in every row and column, this allows us to reduce the matrix $J(\varphi)$ to a diagonal form by applying elementary transformations to its rows. Although we cannot say that this property holds for every invertible Jacobian matrix over $UL(\mathfrak{F})$, we can write each invertible Jacobian matrix as the product of a diagonal matrix and a product of elementary matrices by Proposition 3.7. Let $J(\varphi)$ be expressed in the form $J(\varphi) = D(\mu).E$, where E is a product of elementary matrices and $D(\mu)$ is the diagonal matrix that μ is the product of the elements on the diagonal. Take the group $UL(\mathfrak{F})^*$ of all invertible elements of $UL(\mathfrak{F})$, and the quotient group $\overline{UL(\mathfrak{F})}$ of $UL(\mathfrak{F})^*$ by its commutator subgroup that the commutator subgroup is generated by the elements of the form

$$(1-v)(1-w)(1+v)(1+w)$$

for $v, w \in \mathrm{Ker}(\varepsilon)$ and $v^2 = w^2 = 0$. Then the image $\overline{\mu}$ of μ in $\overline{UL(\mathfrak{F})}$ is defined uniquely and is called the *determinant* of the matrix $J(\varphi)$ in the sense of Dieudonn´e (see [2], [9]), denoted as $\det(J(\varphi))$. It has some usual properties of the determinant; in particular, the determinant of the product of two invertible matrices is equal to the product of their determinants. By the uniqueness and multiplicativity of the determinant for invertible matrices, the invertibility of the determinant of $J(\varphi)$ implies the invertibility of the matrix $J(\varphi)$.

The automorphisms φ_i $(i = 1, \dots, n)$ defined by

$$\varphi_i(x_j) = \begin{cases} \alpha x_i + g & if \quad j = i \\ x_j & if \quad j \neq i, \end{cases}$$

where $\alpha \in K\backslash\{0\}$ and g belongs to the subalgebra of $\mathfrak{F}$ generated by $\{x_j | j \neq i\}$, are called *elementary automorphism* of $\mathfrak{F}$. Moreover, the subgroup of the automorphism group $\mathrm{Aut}\,\mathfrak{F}$ generated by all the elementary automorphisms is said to be the *tame subgroup* of $\mathrm{Aut}\,\mathfrak{F}$, and its elements are called *tame automorphisms*. Finally, an automorphism of $\mathfrak{F}$ that is not tame is said to be *wild*.

Proposition 3.8 *If φ is an elementary automorphism of $\mathfrak{F}$, then $\det J(\varphi)$ is a nonzero element in the ground field K.*

Proof Without loss of generality, we consider the elementary automorphism φ_1. Since the Jacobian matrix of φ_1 is

$$J(\varphi_1) = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ \frac{\partial g}{\partial x_2} & 1 & 0 & \cdots & 0 \\ \frac{\partial g}{\partial x_3} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial x_n} & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} E,$$

where E is a product of elementary automorphisms such that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{\partial g}{\partial x_2} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \frac{\partial g}{\partial x_3} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial x_n} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

we obtain that $\det J(\varphi_1) = \alpha,\ \alpha \in K\backslash\{0\}$. □

Proposition 3.9 *If φ is an elementary automorphism and ψ is an arbitrary automorphism of $\mathfrak{F}$, then $\det \overline{\psi}(J(\varphi))$ is a nonzero element in the ground field K.*

Proof Without loss of generality, we consider the elementary automorphism φ_1. Since the image of Jacobian matrix of φ_1 under $\overline{\psi}$ is

$$\overline{\psi}(J(\varphi_1)) = \begin{pmatrix} \overline{\psi}(\alpha) & 0 & 0 & \cdots & 0 \\ \overline{\psi}(\frac{\partial g}{\partial x_2}) & 1 & 0 & \cdots & 0 \\ \overline{\psi}(\frac{\partial g}{\partial x_3}) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\psi}(\frac{\partial g}{\partial x_n}) & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \overline{\psi}(\alpha) & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} E,$$

where E is a product of elementary automorphisms such that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \overline{\psi}(\frac{\partial g}{\partial x_2}) & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \overline{\psi}(\frac{\partial g}{\partial x_3}) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\psi}(\frac{\partial g}{\partial x_n}) & 0 & 0 & \cdots & 1 \end{pmatrix},$$

we obtain that $\det \overline{\psi}(J(\varphi_1)) = \overline{\psi}(\alpha),\ \alpha \in K\backslash\{0\}$. □

Corollary 3.10 *If φ is a tame automorphism, then* $\det J(\varphi)$ *is a nonzero element in the ground field K.*

Proof Let φ be a tame automorphism. By definition, φ is a composition of elementary automorphisms. Then the proof can be completed by applying identity (3.9) and Proposition 3.8. □

By combining Theorem 2 in [7] and Corollary 3.10, we obtain the following corollary.

Corollary 3.11 *Let φ be an endomorphism of $\mathfrak{F}$. If $\det J(\varphi) = \alpha + u$ is invertible for some $\alpha \in K\backslash\{0\}$ and some $u \neq 0$, then φ is a wild automorphism of $\mathfrak{F}$.*

For two generators the existence of wild automorphisms was obtained by Abdykhalikov et al. (see [1, Theorem 3]). The following theorem establishes the existence of a wild automorphism of a free Leibniz algebra with any finite number of generators. Our construction generalizes the wild automorphism a free Leibniz algebra with two generators found by Abdykhalikov et al.

Theorem 3.12 *Let φ be the endomorphism of $\mathfrak{F}$ with generators $x_1, \ldots, x_n$ defined by*

$$\varphi(x_j) = \begin{cases} x_1 + vx_1 & if \quad j = 1, \\ x_j & if \quad j \neq 1, \end{cases}$$

where $v \in \mathrm{Ann}(\mathfrak{F})$ depends only on $x_2, \ldots, x_n$. Then φ is a wild automorphism of $\mathfrak{F}$.

Proof According to Theorem 2 in [7], it is enough to verify that the Jacobian matrix of φ is invertible. We calculate $d(\varphi(x_i))$ for $i = 1, ..., n$: $d(\varphi(x_1)) = l_{x_1 + vx_1} = l_{x_1}(1 + l_v) + l_v r_{x_1}$ and $d(\varphi(x_j)) = l_{x_j}$ for $j \neq 1$. Consequently, the Jacobian matrix of φ is

$$\begin{bmatrix} 1+l_v & 0 & 0 & 0 & ... & 0 \\ \frac{\partial\varphi(x_1)}{\partial x_2} & 1 & 0 & 0 & ... & 0 \\ \frac{\partial\varphi(x_1)}{\partial x_3} & 0 & 1 & 0 & ... & 0 \\ \frac{\partial\varphi(x_1)}{\partial x_4} & 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varphi(x_1)}{\partial x_n} & 0 & 0 & 0 & ... & 1 \end{bmatrix} = \begin{bmatrix} 1+l_v & 0 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & ... & 1 \end{bmatrix} E,$$

where E is the product of elementary matrices in the following form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{\partial\varphi(x_1)}{\partial x_2} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \frac{\partial\varphi(x_1)}{\partial x_3} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varphi(x_1)}{\partial x_n} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

thus its determinant is $1 + l_v$, where $v \in \mathrm{Ann}(\mathfrak{F})$ depends only on the elements x_j for $j \neq 1$. By Lemma 3.4, $1 + l_v$ is invertible. Then it follows from Corollary 3.11 that φ is a wild automorphism of $\mathfrak{F}$. □

The following example establishes the existence of a wild automorphism of a free Leibniz algebra with any finite number of generators. For three generators the automorphism in Example 3.13 is an analog of the Anick automorphism (see [4, p. 343]). Umirbaev proved that the Anick automorphism is wild (see [12]).

Example 3.13 *The endomorphism γ of the free Leibniz algebra with generators $x_1, \ldots, x_n$ defined by*

$$\gamma : \begin{cases} x_1 \to x_1 + x_3(x_1x_3 - x_3x_2) \\ x_2 \to x_2 + (x_1x_3 - x_3x_2)x_3 \\ x_i \to x_i, i \neq 1, 2 \end{cases}$$

is wild. The endomorphism γ is the composition of the automorphisms $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9$ such *that* $\gamma = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8\alpha_9$, *where the automorphisms are defined by*

$$\begin{aligned}
\alpha_1 &: x_2 \to x_2 + x_1, \\
&\quad x_i \to x_i, i \neq 2, \\
\alpha_2 &: x_1 \to x_1 + x_3(x_2x_3), \\
&\quad x_i \to x_i, i \neq 1, \\
\alpha_3 &: x_2 \to x_2 - x_1, \\
&\quad x_i \to x_i, i \neq 2, \\
\alpha_4 &: x_2 \to x_2 + x_3(x_1x_3) + x_1x_3x_3, \\
&\quad x_i \to x_i, i \neq 2, \\
\alpha_5 &: x_2 \to x_2 - x_3x_3x_2, \\
&\quad x_i \to x_i, i \neq 2, \\
\alpha_7 &: x_1 \to x_1 + (x_3x_3)(x_2x_3)x_3, \\
&\quad x_i \to x_i, i \neq 1, \\
\alpha_9 &: x_1 \to x_1 - (x_3x_3)((x_2 - x_1)x_3)x_3, \\
&\quad x_i \to x_i, i \neq 1,
\end{aligned}$$

$\alpha_6 = \alpha_3$ *and* $\alpha_8 = \alpha_1$. *Let us take* $\psi_i = \alpha_1\alpha_2\cdots\alpha_i$ *for* $i = 1, \ldots, 9$. *It is obtained that* $\psi_i(x_3) = x_3$. *Then identity (3.9) implies that*

$$J_\gamma = J_{\alpha_1}\overline{\psi}_1(J_{\alpha_2})\overline{\psi}_2(J_{\alpha_3})\overline{\psi}_3(J_{\alpha_4})\overline{\psi}_4(J_{\alpha_5})\overline{\psi}_5(J_{\alpha_6})\overline{\psi}_6(J_{\alpha_7})\overline{\psi}_7(J_{\alpha_8})\overline{\psi}_8(J_{\alpha_9}).$$

Since the determinant is multiplicative for invertible matrices, we obtain the determinant of the Jacobian matrix J_γ *as*

$$\det J_\gamma = \det J_{\alpha_1} \det\overline{\psi}_1(J_{\alpha_2}) \det\overline{\psi}_2(J_{\alpha_3}) \det\overline{\psi}_3(J_{\alpha_4}) \det\overline{\psi}_4(J_{\alpha_5}) \det\overline{\psi}_5(J_{\alpha_6}) \det\overline{\psi}_6(J_{\alpha_7}) \det\overline{\psi}_7(J_{\alpha_8}) \det\overline{\psi}_8(J_{\alpha_9}) \tag{3.21}$$

The automorphisms $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7$ *and* α_8 *are elementary. Therefore we have* $\det J_{\alpha_1}$, $\det\overline{\psi}_1(J_{\alpha_2})$, $\det\overline{\psi}_2(J_{\alpha_3})$, $\det\overline{\psi}_3(J_{\alpha_4})$, $\det\overline{\psi}_5(J_{\alpha_6})$, $\det\overline{\psi}_6(J_{\alpha_7})$ *and* $\det\overline{\psi}_7(J_{\alpha_8})$ *are nonzero elements of the ground field* K *by Proposition 3.8 and 3.9. Furthermore* $\det J_{\alpha_5} = 1 - l_{x_3x_3}$ *and* α_5 *is wild by Theorem 3.12. The image* $\psi_4(x_3) = x_3$ *holds* $\det\overline{\psi}_4(J_{\alpha_5}) = 1 - l_{x_3x_3}$. *Now let us find the* $\det\overline{\psi}_8(J_{\alpha_9})$. *The Jacobian matrix of* α_9 *is*

$$J(\alpha_9) = \begin{bmatrix} 1 - r_{x_3}l_{x_3x_3}r_{x_3} & 0 & 0 & 0 & \ldots & 0 \\ r_{x_3}l_{x_3x_3}r_{x_3} & 1 & 0 & 0 & \ldots & 0 \\ (r_{x_3} + l_{x_3})r_{(x_2-x_1)x_3}r_{x_3} + l_{(x_2-x_1)}l_{x_3x_3}r_{x_3} + l_{(x_3x_3)((x_2-x_1)x_3)} & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Then

$$\overline{\psi}_8(J_{\alpha_9}) = \begin{bmatrix} 1 - r_{x_3} l_{x_3x_3} r_{x_3} & 1 & 0 & 0 & \dots & 0 \\ r_{x_3} l_{x_3x_3} r_{x_3} & 1 & 0 & 0 & \dots & 0 \\ (r_{x_3} + l_{x_3}) r_{\psi_8(x_2-x_1)x_3} r_{x_3} + l_{\psi_8(x_2-x_1)} l_{x_3x_3} r_{x_3} + l_{(x_3x_3)(\psi_8(x_2-x_1)x_3)} & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

This matrix is equal to

$$\begin{bmatrix} 1 - r_{x_3} l_{x_3x_3} r_{x_3} & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} E,$$

where E *is the product of elementary matrices such that*

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ r_{x_3} l_{x_3x_3} r_{x_3} & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ (r_{x_3} + l_{x_3}) r_{\psi_8(x_2-x_1)x_3} r_{x_3} + l_{\psi_8(x_2-x_1)} l_{x_3x_3} r_{x_3} + l_{(x_3x_3)(\psi_8(x_2-x_1)x_3)} & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Hence, we get $\det(\overline{\psi}_8(J(\alpha_9))) = 1 - r_{x_3} l_{x_3x_3} r_{x_3}$. *It follows from*

$$r_{x_3} l_{x_3x_3} r_{x_3} r_{x_3} l_{x_3x_3} r_{x_3} = r_{x_3} l_{x_3x_3} l_{x_3} l_{x_3} l_{x_3x_3} r_{x_3} = 0,$$

and Lemma 3.3 that $1 - r_{x_3} l_{x_3x_3} r_{x_3}$ *is invertible. By the identity (3.21) the determinant of* J_γ *is*

$$\begin{aligned} \det(J(\gamma)) &= k(1 + l_{x_3x_3})(1 - r_{x_3} l_{x_3x_3} r_{x_3}) \\ &= k(1 + l_{x_3x_3} - r_{x_3} l_{x_3x_3} r_{x_3} - l_{x_3x_3} r_{x_3} l_{x_3x_3} r_{x_3}) \\ &= k(1 + l_{x_3x_3} - r_{x_3} l_{x_3x_3} r_{x_3}) \end{aligned}$$

where $k \in K \backslash \{0\}$. *It follows from*

$$(l_{x_3x_3} - r_{x_3} l_{x_3x_3} r_{x_3})(l_{x_3x_3} - r_{x_3} l_{x_3x_3} r_{x_3}) = 0,$$

and Lemma 3.3 that $k(1 + l_{x_3x_3} - r_{x_3} l_{x_3x_3} r_{x_3})$ *is invertible, and thus Theorem 2 in [7] shows that* γ *is an automorphism. Finally, we conclude from Corollary 3.11 that* γ *is wild.*

4. Conclusion

This paper starts with investigating various properties of partial derivatives of a finitely generated free Leibniz algebra over a field of characteristic zero. Then we characterize the invertibility of certain elements of the universal enveloping algebra of the free Leibniz algebra. Next, we prove that every invertible Jacobian matrix over the universal enveloping algebra of a finitely generated free Leibniz algebra can be written as the product of a diagonal matrix and a product of elementary matrices. Thus, using this result, we give an analog of the Dieudonné determinant of the invertible Jacobian matrices. After that, we apply the criterion of Mikhalev and Umirbaev (see [7, Theorem 2]) for the invertibility of an endomorphism of a finitely generated free Leibniz algebra via its Jacobian matrix to determine whether a given endomorphism is an automorphism. Then we establish a sufficient condition for an automorphism to be considered wild based on the determinant of its Jacobian matrix. Building on the work of Abdykhalikov et al. [1] who constructed a wild automorphism for free Leibniz algebras with two generators, we address the open question regarding the existence of wild automorphisms for free Leibniz algebras with more than two generators. At the end of our paper, we establish the existence of wild automorphisms for any finitely generated free Leibniz algebra, thereby complementing Theorem 3 in [1]. Finally, we construct another wild automorphism for a free Leibniz algebra with n generators analogous to the Anick automorphism when $n = 3$. Umirbaev proved that the Anick automorphism of a free associative algebra with three generators is wild (see [12]).

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Conflicts of interest

The author declares no conflict of interest.

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