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Spherical product hypersurfaces in Euclidean spaces

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Abstract: Spherical product surfaces are obtained with the help of a special product by considering two curves in $n-$dimensional space. One of their special cases is rotational surface. The reason why the present study is significant that the spherical product is used to construct hypersurfaces. $(n-1)-$curves are needed during this construction. Firstly, the spherical product hypersurfaces are defined in $\mathbb{E}^4$, Gaussian and mean curvature are yielded and then conditions being flat or minimal are examined. Moreover, superquadrics, which are associated with spherical product, are handled for the first time in hypersurface form and give some examples. Finally, spherical product hypersurfaces are generalized to $n-$dimensional Euclidean space and contribute to literature.

Key words: Hypersurface, spherical product, superquadrics

1. Introduction

In differential geometry, using the sum of the curves, many times using the product of the functions, some surfaces can be created. These are translation and factorable surfaces [3, 7]. In addition, the surfaces can also be created with the help of a product called spherical product. The concept of spherical product comes from the definition of rotational embedding (see, [12]). Using this product on the curves $\alpha(x) = (f_1(x), f_2(x))$ and $\beta(y) = (g_1(y), g_2(y))$, the parametrization

$$\alpha(x) \otimes \beta(y) = (f_1(x), f_2(x)g_1(y), f_2(x)g_2(y)) \tag{1.1}$$

is specified and this corresponds to a spherical product surface in $3-$dimensional Euclidean space [5]. Such a surface was also evaluated in Euclidean $4-$space $\mathbb{E}^4$ and remarkable results were obtained [6]. Among the special cases of this, the most familiar are the rotational surfaces and the superquadrics. These two concepts have a wide coverage in geometry with their visual examples [8, 11].

In (1.1), by taking $\beta(y) = (\cos y, \sin y)$, the surfaces of revolution are encountered. Some of them are ruled, developable, helicoidal, canal, tube surfaces and catenoid, also have many applications in different disciplines [1, 4, 13].

Especially, the other form, a superquadric is handled by spherical product of superellipses or superhyperbols whose simple forms we know from analytical geometry. Created surface is superellipsoid, superhyperboloid with one piece or hyperboloid with two pieces or toroid [11].

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Moreover, in geometry, surfaces can also be considered under the category of hypersurfaces. The concept of hypersurfaces are known as $(n-1)-$surface in $n-$dimensional Euclidean space and defined by the following set:

$$M = \{x \in U \subset \mathbb{E}^n : \ \ \varphi(x) = c, \ c \text{ is constant and } U \text{ is open set}\}. \tag{1.2}$$

Many related studies can be founded in [2, 9, 10].

In the present study, the spherical product is considered for the first time on hypersurfaces. Firstly, in section 2, basic concepts about hypersurfaces are given. In section 3, spherical product hypersurfaces are defined with the help of 3−curves in 4−dimensional Euclidean space and demonstrated to be regular. The mean and Gaussian curvatures are yielded. The necessary and sufficient condition for the hypersurface to be flat is that one of the curves forming the hypersurface is a straight line. The condition being minimal is analyzed. In section 4, superquadrics in hypersurface form are defined in $\mathbb{E}^4$, some examples are given and the projections to $\mathbb{E}^3$ are plotted. Finally, in section 5, the spherical product hypersurface are generalized and the related parametrization is presented in $n-$dimensional Euclidean space $\mathbb{E}^n$.

2. Preliminaries

In the present section, we mention some general expressions for hypersurfaces in $\mathbb{E}^4$.

Let $r = (r_1, r_2, r_3, r_4)$, $s = (s_1, s_2, s_3, s_4)$ and $t = (t_1, t_2, t_3, t_4)$ be the vector fields in Euclidean 4−space, the inner product $\langle r, s\rangle$ and the vector product $r \times s \times t$ is given by

$$\langle r, s\rangle = r_1 s_1 + r_2 s_2 + r_3 s_3 + r_4 s_4, \tag{2.1}$$

and

$$r \times s \times t = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ r_1 & r_2 & r_3 & r_4 \\ s_1 & s_2 & s_3 & s_4 \\ t_1 & t_2 & t_3 & t_4 \end{vmatrix}, \tag{2.2}$$

respectively.

Let $M : F(x, y, z)$ be a hypersurface in $\mathbb{E}^4$, then M is expressed as

$$\begin{aligned} F &: \ \mathbb{E}^3 \to \mathbb{E}^4 \\ (x, y, z) &\to F(x, y, z) \\ F(x, y, z) &= (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z), F_4(x, y, z)) \end{aligned} \tag{2.3}$$

and the unit normal vector field of M is calculated by

$$\eta = \frac{F_x \times F_y \times F_z}{\|F_x \times F_y \times F_z\|}. \tag{2.4}$$

The first fundamental form of M is given with the help of the coefficients

$$\begin{aligned} e &= \langle F_x, F_x\rangle, \quad f = \langle F_x, F_y\rangle, \quad a = \langle F_x, F_z\rangle, \\ g &= \langle F_y, F_y\rangle, \quad b = \langle F_y, F_z\rangle, \quad c = \langle F_z, F_z\rangle, \end{aligned} \tag{2.5}$$

and the second fundamental form of M is written by the coefficients

$$\begin{aligned} l &= \langle F_{xx}, \eta \rangle, \quad m = \langle F_{xy}, \eta \rangle, \quad p = \langle F_{xz}, \eta \rangle, \\ n &= \langle F_{yy}, \eta \rangle, \quad t = \langle F_{yz}, \eta \rangle, \quad v = \langle F_{zz}, \eta \rangle, \end{aligned} \tag{2.6}$$

(see, [9]).

Suppose I and II are the matrices corresponding to the 1st and 2nd fundamental form. Then, the shape operator matrix can be obtained by

$$S = (I)^{-1} II. \tag{2.7}$$

Definition 2.1 *Let M be a hypersurface given by (2.3) in $\mathbb{E}^4$. Then, the mean curvature and the Gaussian curvature of M is defined by*

$$H = \frac{tr(S)}{3} \tag{2.8}$$

and

$$K = \frac{\det(II)}{\det(I)} = \det(S) \tag{2.9}$$

respectively [2, 9].

3. Spherical product hypersurfaces in $4-$dimensional Euclidean space

Definition 3.1 *Let $f_i,\ g_i,\ h_i\ (i = 1, 2)$ be smooth functions and $\alpha,\ \beta,\ \gamma : I \subset \mathbb{R} \to \mathbb{E}^2$ be regular curves in $\mathbb{E}^2$ given by $\alpha(x) = (f_1(x), f_2(x))$, $\beta(y) = (g_1(y), g_2(y))$, $\gamma(z) = (h_1(z), h_2(z))$. Spherical product of these curves ($\alpha(x) \otimes \beta(y) \otimes \gamma(z)$) defines a $3-$surface and is called spherical product hypersurface in $\mathbb{E}^4$.*

Hence, for the parametrization of this, we write

$$\begin{aligned} F(x,y,z) &= \alpha(x) \otimes \beta(y) \otimes \gamma(z) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \otimes \begin{bmatrix} g_1(y) \\ g_2(y) \end{bmatrix} \otimes \begin{bmatrix} h_1(z) \\ h_2(z) \end{bmatrix} \\ &= \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \otimes \begin{bmatrix} g_1(y) \\ g_2(y)h_1(z) \\ g_2(y)h_2(z) \end{bmatrix}. \end{aligned}$$

It follows that

$$F(x,y,z) = (f_1(x), f_2(x)g_1(y), f_2(x)g_2(y)h_1(z), f_2(x)g_2(y)h_2(z)). \tag{3.1}$$

It is clear that the spherical product $\beta(y) \otimes \gamma(z)$ is congruent to spherical product surface in $\mathbb{E}^3$ as

$$G(y,z) = (g_1(y), g_2(y)h_1(z), g_2(y)h_2(z)). \tag{3.2}$$

Example 3.2 *Choosing the curves $\alpha(x) = (f_1(x), x)$, $\beta(y) = (\cos y, \sin y)$ and $\gamma(z) = (\cos z, \sin z)$, the spherical product hypersurface*

$$M : F(x,y,z) = (f_1(x), x\cos y, x\sin y\cos z, x\sin y\sin z)$$

corresponds to a rotational hypersurface in Euclidean $4-$space $\mathbb{E}^4$.

Let M be a spherical product hypersurface given by (3.1) in $\mathbb{E}^4$. Then, the vectors

$$
\begin{aligned}
F_x &= \frac{\partial F(x,y,z)}{\partial x} = \left(f_1', f_2'g_1, f_2'g_2h_1, f_2'g_2h_2\right), \\
F_y &= \frac{\partial F(x,y,z)}{\partial y} = \left(0, f_2g_1', f_2g_2'h_1, f_2g_2'h_2\right), \\
F_z &= \frac{\partial F(x,y,z)}{\partial z} = \left(0, 0, f_2g_2h_1', f_2g_2h_2'\right),
\end{aligned} \tag{3.3}
$$

span the tangent space $T(M)$.

The normal vector field is obtained by the vector product of F_x, F_y, and F_z as

$$
\eta = \frac{f_2^2 g_2}{W}\left(f_2'\left(g_1g_2' - g_1'g_2\right)\left(h_1h_2' - h_1'h_2\right), -f_1'g_2'\left(h_1h_2' - h_1'h_2\right), f_1'g_1'h_2', -f_1'g_1'h_1'\right). \tag{3.4}
$$

Here, $W = \|F_x \times F_y \times F_z\|$ and obtained as

$$
W^2 = f_2^4 g_2^2\left[\left(h_1h_2' - h_1'h_2\right)^2\left(f_2'^2\left(g_1g_2' - g_1'g_2\right)^2 + f_1'^2 g_2'^2\right) + f_1'^2 g_1'^2\left(h_1'^2 + h_2'^2\right)\right].
$$

The matrix I corresponding the 1st fundamental form is

$$
I = \begin{bmatrix} e & f & a \\ f & g & b \\ a & b & c \end{bmatrix}, \tag{3.5}
$$

where the coefficients are calculated as

$$
\begin{aligned}
e &= f_1'^2 + f_2'^2\left[g_1^2 + g_2^2\left(h_1^2 + h_2^2\right)\right] = f_1'^2 + f_2'^2\left\|G(y,z)\right\|^2, \\
f &= f_2'f_2\left[g_1'g_1 + g_2'g_2\left(h_1^2 + h_2^2\right)\right] = f_2'f_2\left\langle G(y,z), G_y(y,z)\right\rangle, \\
a &= f_2'f_2g_2^2\left(h_1'h_1 + h_2'h_2\right) = f_2'f_2\left\langle G(y,z), G_z(y,z)\right\rangle, \\
g &= f_2^2\left[g_1'^2 + g_2'^2\left(h_1^2 + h_2^2\right)\right] = f_2^2\left\|G_y(y,z)\right\|^2, \\
b &= f_2^2g_2'g_2\left(h_1'h_1 + h_2'h_2\right) = f_2^2\left\langle G_y(y,z), G_z(y,z)\right\rangle, \\
c &= f_2^2g_2^2\left(h_1'^2 + h_2'^2\right) = f_2^2\left\|G_z(y,z)\right\|^2.
\end{aligned} \tag{3.6}
$$

It can be seen from the equations (3.5) and (3.6) that $\det I = W^2$. Since this expression is positive definite, M is regular.

The second partial derivatives are

$$
\begin{aligned}
F_{xx} &= \left(f_1'', f_2''g_1, f_2''g_2h_1, f_2''g_2h_2\right), \\
F_{xy} &= \left(0, f_2'g_1', f_2'g_2'h_1, f_2'g_2'h_2\right), \\
F_{xz} &= \left(0, 0, f_2'g_2h_1', f_2'g_2h_2'\right), \\
F_{yy} &= \left(0, f_2g_1'', f_2g_2''h_1, f_2g_2''h_2\right), \\
F_{yz} &= \left(0, 0, f_2g_2'h_1', f_2g_2'h_2'\right), \\
F_{zz} &= \left(0, 0, f_2g_2h_1'', f_2g_2h_2''\right).
\end{aligned} \tag{3.7}
$$

From now on, we will use the following abbreviations

$$\begin{aligned} A(x) &= f_1''f_2' - f_2''f_1', \\ B(y) &= g_1g_2' - g_1'g_2, \\ C(y) &= g_1'g_2'' - g_1''g_2', \\ D(z) &= h_1h_2' - h_1'h_2, \\ E(z) &= h_1''h_2' - h_1'h_2''. \end{aligned} \tag{3.8}$$

By the use of (3.7), (3.4), and (2.6), we can write the matrix II corresponding the 2nd fundamental form as

$$II = \begin{bmatrix} l & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & v \end{bmatrix}, \tag{3.9}$$

where l, n, and v are given by

$$\begin{aligned} l &= \frac{f_2^2 g_2}{W} A(x)B(y)D(z), \\ n &= \frac{f_1' f_2^3 g_2}{W} C(y)D(z), \\ v &= \frac{f_1' g_1' f_2^3 g_2^2}{W} E(z). \end{aligned} \tag{3.10}$$

Theorem 3.3 *Let M be a spherical product hypersurface in Euclidean $4-$space $\mathbb{E}^4$. Then, the Gaussian curvature of M is presented by*

$$K = \frac{f_1'^2 f_2^8 g_2^4 g_1' A(x)B(y)C(y)D^2(z)E(z)}{W^3}, \tag{3.11}$$

where the functions $A(x)$, $B(y)$, $C(y)$, $D(z)$, and $E(z)$ are specified by (3.8).

Proof With the help of the equalities (3.5), (3.6), (3.9), (3.10) with (2.9), we get the desired result. □

Theorem 3.4 *Let M be a spherical product hypersurface in Euclidean $4-$space $\mathbb{E}^4$. Then, M has zero Gaussian curvature (flat) if and only if one of the curves forming the hypersurface is a straight line.*

Proof Let M be a spherical product hypersurface given by (3.1). If M is flat ($K = 0$), then by using the equation (3.11), we obtain that at least one of the following equalities is satisfied:

$$\begin{aligned} A(x) &= 0, \\ B(y) &= 0, \\ C(y) &= 0, \\ D(z) &= 0, \\ E(z) &= 0. \end{aligned}$$

□

This means that the curve $\alpha(x)$ or $\beta(y)$ or $\gamma(z)$ is congruent to a straight line. In addition, the converse statement is trivial.

Theorem 3.5 *Let M be a spherical product hypersurface in Euclidean 4− space $\mathbb{E}^4$. Then, the mean curvature of M is presented by*

$$H = \frac{f_2^5 g_2}{3W^3} \left\{ \begin{array}{c} D(z) \left[\begin{array}{c} f_2 A(x) B(y) \left(\|G_y\|^2 \|G_z\|^2 - \langle G_y, G_z \rangle^2 \right) \\ + f_1' C(y) \left(f_1'^2 \|G_z\|^2 + f_2'^2 \left(\|G\|^2 \|G_z\|^2 - \langle G, G_z \rangle^2 \right) \right) \end{array} \right] \\ + g_2 f_1' g_1' E(z) \left(f_1'^2 \|G_y\|^2 + f_2'^2 \left(\|G\|^2 \|G_y\|^2 - \langle G, G_y \rangle^2 \right) \right) \end{array} \right\}, \tag{3.12}$$

where $A(x),\ B(y),\ C(y),\ D(z),\ E(z)$ are indicated in (3.8) and $G = G(y, z)$ is a $3D-$spherical product surface parametrization specified as (3.2).

Proof Let M be a spherical product hypersurface given by (3.1) in $\mathbb{E}^4$. By the use of (2.7), (3.5), and (3.9), we get

$$tr\,(S) = l \left(\frac{gc - b^2}{W^2} \right) + n \left(\frac{ec - a^2}{W^2} \right) + v \left(\frac{eg - f^2}{W^2} \right). \tag{3.13}$$

Also, substituting the 1st and 2nd fundamental form coefficients (3.6), (3.10) into (3.13) and using (2.8), we yield the mean curvature of M as (3.12) and complete the proof. □

Corollary 3.6 *Let M be a spherical product hypersurface given by (3.1). With the help of (3.12), the following cases occur:*

(a) If the curve $\gamma(z)$ is a straight line passing through the origin, then M has zero mean curvature (minimal).

(b) If $\alpha(x)$, $\beta(y)$, and $\gamma(z)$ are straight lines, then M has zero mean curvature (minimal).

4. Superquadrics in hypersurface form

The concept of superellipse is associated with the definition of Lame curve that is represented by

$$\left(\frac{x_1}{a_1} \right)^m + \left(\frac{x_2}{a_2} \right)^m = 1. \tag{4.1}$$

Lame curves which are studied by Loria, 1910 and named by Gabriel Lame have nine types. While the number m increases, the curve gets closer to rectangularity. Superellipses are the special case of these curves and given by

$$\left(\frac{x_1}{a_1} \right)^{\frac{2}{\epsilon}} + \left(\frac{x_2}{a_2} \right)^{\frac{2}{\epsilon}} = 1. \tag{4.2}$$

Also, the parametric form is represented by

$$\alpha(x) = \left(a_1 \cos^{\varepsilon} x, a_2 \sin^{\epsilon} x \right). \tag{4.3}$$

It is clear that the case $\epsilon = 1$ is the simplest form known from analytical geometry. In addition to superellipses, superhiperbola has a similar definition.

Using the spherical product of these types of curves, we encounter with superellipsoid, superhyperboloid, and supertoroid which are generally called superquadrics. Now, we define superquadrics with the form of hypersurfaces in $4-$dimensional Euclidean space.

Definition 4.1 *Let $\alpha,\ \beta,\ \gamma$ be the superellipses or the superhyperbolas in $\mathbb{E}^2$. The spherical product of these curves ($\alpha \otimes \beta \otimes \gamma$) defines a $3-$surface called superquadrics in hypersurface form in $\mathbb{E}^4$.*

Example 4.2 *Suppose the related curves (superellipses) are chosen as*

$$\begin{aligned} \alpha(x) &= \left(a_1 \cos^{\varepsilon_1} x, \sin^{\varepsilon_1} x\right), \\ \beta(y) &= \left(a_2 \cos^{\varepsilon_2} y, \sin^{\epsilon_2} y\right), \\ \gamma(z) &= \left(a_3 \cos^{\varepsilon_3} z, a_4 \sin^{\varepsilon_3} z\right). \end{aligned} \tag{4.4}$$

Then, the spherical product of these curves is presented as

$$F(x,y,z) = \left(a_1 \cos^{\varepsilon_1} x, a_2 \sin^{\varepsilon_1} x \cos^{\varepsilon_2} y, a_3 \sin^{\varepsilon_1} x \sin^{\epsilon_2} y \cos^{\varepsilon_3} z, a_4 \sin^{\varepsilon_1} x \sin^{\epsilon_2} y \sin^{\varepsilon_3} z\right).$$

This parametrization is congruent to superellipsoid in hypersurface form. Actually, it satisfies the equation

$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 + \left(\frac{x_4}{a_4}\right)^2 = 1. \tag{4.5}$$

We can plot the projection of a superellipsoid in $\mathbb{E}^3$ (as shown in Figure 1) by taking $a_1 = 1,\ a_2 = 2,\ a_3 = a_4 = 3,\ \epsilon_1 = 3,\ \epsilon_2 = 2,\ \epsilon_3 = 1,$ and $z = \pi,$ with Maple command

$$plot3d\,(x_1(u,v), x_2(u,v), x_3(u,v) + x_4(u+v)],\ u = -2 * Pi...2 * Pi,\ v : -2 * Pi..2 * Pi)$$

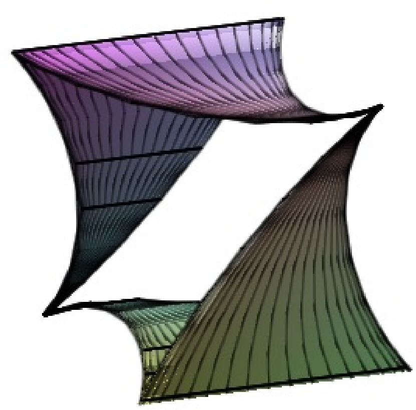

Figure 1. Projection of a superellipsoid in hypersurface form

Example 4.3 *Suppose the curves (superhyperbola and superellipses) are chosen as*

$$\begin{aligned} \alpha(x) &= (a_1 \tan^{\epsilon_1} x, \sec^{\epsilon_1} x), \\ \beta(y) &= (a_2 \cos^{\epsilon_2} y, \sin^{\epsilon_2} y), \\ \gamma(z) &= (a_3 \cos^{\epsilon_3} z, a_4 \sin^{\epsilon_3} z). \end{aligned} \tag{4.6}$$

Then, the spherical product of these curves is presented as

$$F(x,y,z) = (a_1 \tan^{\epsilon_1} x, a_2 \sec^{\epsilon_1} x \cos^{\epsilon_2} y, a_3 \sec^{\epsilon_1} x \sin^{\epsilon_2} y \cos^{\epsilon_3} z, a_4 \sec^{\epsilon_1} x \sin^{\epsilon_2} y \sin^{\epsilon_3} z).$$

The parametrization above is congruent to superhyperboloid with one piece in $\mathbb{E}^4$ and following equation is hold:

$$-\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 + \left(\frac{x_4}{a_4}\right)^2 = 1. \tag{4.7}$$

The projection of a hyperboloid one piece can be yielded in $\mathbb{E}^3$ by taking $a_1 = 2,\ a_2 = 3,\ a_3 = a_4 = 1,\ \epsilon_1 = 2,$ $\epsilon_2 = \epsilon_3 = 1,$ and $z = \pi,$ with Maple. It is observed as in Figure 2.

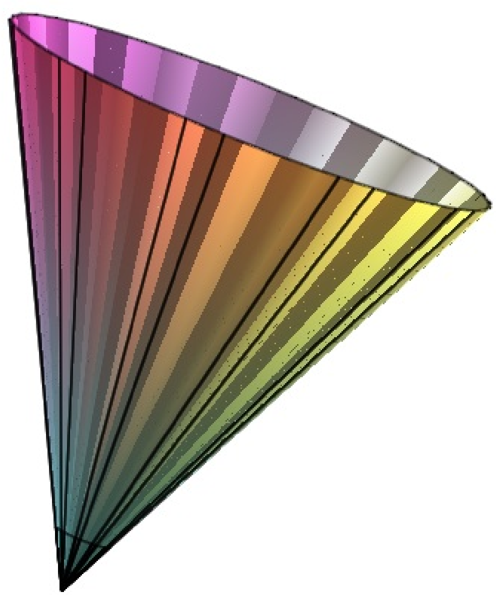

Figure 2. Projection of a superhyperboloid (one piece)

Example 4.4 *Suppose the curves (superhyperbolas) are chosen as*

$$\begin{aligned} \alpha(x) &= (a_1 \tan^{\epsilon_1} x, \sec^{\epsilon_1} x), \\ \beta(y) &= (a_2 \tan^{\epsilon_2} y, \sec^{\epsilon_2} y), \\ \gamma(z) &= (a_3 \tan^{\epsilon_3} z, a_4 \sec^{\epsilon_3} z). \end{aligned} \tag{4.8}$$

Then, the spherical product of these curves is presented as

$$F(x,y,z) = (a_1 \tan^{\epsilon_1} x, a_2 \sec^{\epsilon_1} x \tan^{\epsilon_2} y, a_3 \sec^{\epsilon_1} x \sec^{\epsilon_2} y \tan^{\epsilon_3} z, a_4 \sec^{\epsilon_1} x \sec^{\epsilon_2} y \sec^{\epsilon_3} z).$$

This representation is congruent to superhyperboloid with two piece in $\mathbb{E}^4$ due to satisfying the equation

$$-\left(\frac{x_1}{a_1}\right)^2 - \left(\frac{x_2}{a_2}\right)^2 - \left(\frac{x_3}{a_3}\right)^2 + \left(\frac{x_4}{a_4}\right)^2 = 1. \tag{4.9}$$

In addition, by taking $a_1 = a_2 = a_3 = a_4 = 1,\ \epsilon_1 = \epsilon_3 = 1,\ \epsilon_2 = 2,$ and $z = \pi,$ the visiluation of the projection of this type of hypersurface is encountered as Figure 3.

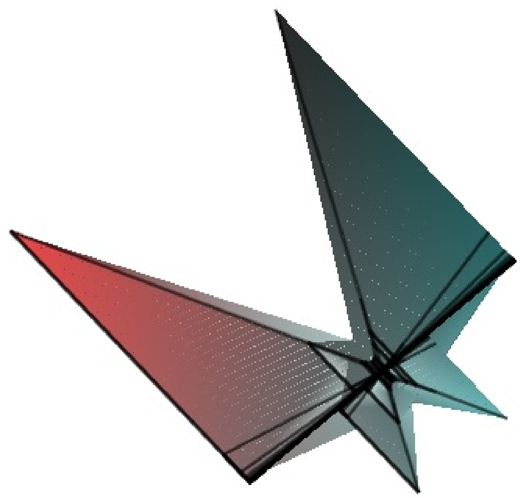

Figure 3. Projection of a superhyperboloid (two-piece)

5. The Generalization of spherical product hypersurfaces

In the present section, we purpose to obtain the parametrization of spherical product hypersurfaces in $n-$dimensional Euclidean space $\mathbb{E}^n$.

In $3-$dimension using two curves:

$$\alpha_1 \otimes \alpha_2 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \otimes \begin{bmatrix} f_3 \\ f_4 \end{bmatrix},$$

$$\alpha_1 \otimes \alpha_2 = (f_1, f_2 f_3, f_2 f_4).$$

In $4-$dimension using three curves:

$$\alpha_1 \otimes \alpha_2 \otimes \alpha_3 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \otimes \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \otimes \begin{bmatrix} f_5 \\ f_6 \end{bmatrix},$$

$$\alpha_1 \otimes \alpha_2 \otimes \alpha_3 = (f_1, f_2 f_3, f_2 f_4 f_5, f_2 f_4 f_6).$$

In $5-$ **dimension using four curves:**

$$\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_4 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \otimes \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \otimes \begin{bmatrix} f_5 \\ f_6 \end{bmatrix} \otimes \begin{bmatrix} f_7 \\ f_8 \end{bmatrix},$$

$$\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_4 = (f_1, f_2 f_3, f_2 f_4 f_5, f_2 f_4 f_6 f_7, f_2 f_4 f_6 f_8).$$

In $6-$ **dimension using five curves:**

$$\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_4 \otimes \alpha_5 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \otimes \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \otimes \begin{bmatrix} f_5 \\ f_6 \end{bmatrix} \otimes \begin{bmatrix} f_7 \\ f_8 \end{bmatrix} \otimes \begin{bmatrix} f_9 \\ f_{10} \end{bmatrix},$$

$$\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_4 \otimes \alpha_5 = (f_1, f_2 f_3, f_2 f_4 f_5, f_2 f_4 f_6 f_7, f_2 f_4 f_6 f_8 f_9, f_2 f_4 f_6 f_8 f_{10}).$$

Corollary 5.1 *Let* $\alpha_1 = (f_1, f_2), \alpha_2 = (f_3, f_4), \ \alpha_3 = (f_5, f_6), ..., \alpha_{n-1} = (f_{2n-3}, f_{2n-2})$ *be regular curves in* $\mathbb{E}^2$*. The spherical product of these curves defines a hypersurface in* $\mathbb{E}^n$ *called spherical product hypersurface. The parametrization of this hypersurface is given by*

$$\begin{aligned} \alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes ... \otimes \alpha_{n-1} &= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \otimes \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \otimes \begin{bmatrix} f_5 \\ f_6 \end{bmatrix} \otimes ... \otimes \begin{bmatrix} f_{2n-3} \\ f_{2n-2} \end{bmatrix} \\ &= f_1 Y_1 + \sum_{i=2}^{n-1} \left(\prod_{j=1}^{i-1} f_{2j} \right) f_{2i-1} Y_i + \left(\prod_{j=1}^{n-2} f_{2j} \right) f_{2n-2} Y_n, \end{aligned}$$

where $Y_1, Y_2, ..., Y_n$ *are coordinate functions in* $\mathbb{E}^n$*.*

6. Conclusion

In this study, we achieve the general parametrization of spherical product hypersurfaces and give significant results in four-dimensional space and especially for superquadrics. We hope this work will be the base for further studies.

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