Turkish Journal of Mathematics

Volume 48 | Number 5 Article 5

9-10-2024

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LIGHT, GRAYSON; SAYAR, CEM; and AKYOL, MEHMET AKİF (2024) "φ−pluriharmonicity and φ−invariance of pointwise bislant Riemanniansubmersions," *Turkish Journal of Mathematics*: Vol. 48: No. 5, Article 5.
https://doi.org/10.55730/1300-0098.3546
Available at: https://journals.tubitak.gov.tr/math/vol48/iss5/5

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Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2024) 48: 874 – 890
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doi:10.55730/1300-0098.3546

$\varphi-$pluriharmonicity and $\varphi-$invariance of pointwise bislant Riemannian submersions

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Received: 09.12.2023 • **Accepted/Published Online:** 02.07.2024 • **Final Version:** 10.09.2024

Abstract: In this research, we investigate the intriguing realm of pointwise bislant Riemannian submersions, a generalization of many previous submersions, such as antiinvariant, slant, semislant, pointwise slant, pointwise semislant, and bislant submersions, within the framework of almost product manifolds. After giving an original example, we delve into the submersion's integrability conditions and geodesics. We explore the concept of $\varphi-$pluriharmonicity and $\varphi-$invariance within this context. The study sheds light on the profound interplay between pointwise bislant submersions' fibers and their being either geodesic or mixed geodesic, offering valuable insights into these intriguing mappings' geometric properties.

Key words: Riemannian submersion, pluriharmonicity, harmonic, pointwise slant distribution, geodesic, integrable

1. Introduction

The theory of submanifolds has been shown to be quite useful in Differential Geometry. It generalizes the concept of curves and surfaces to higher dimensions, aids in representing configuration spaces of physical systems, allows for the representation of complex shapes and motion paths in an efficient, compact manner in robotics and computers, and so on. Overall, submanifolds provide a powerful and flexible framework for understanding complex geometries and their intrinsic properties. They offer a deeper insight into the structure of spaces, and crucially, they find applications across a wide range of disciplines, making them an essential concept in modern mathematics and its various applications.

The importance of submanifolds prompted the Geometers to define and study specific submanifolds. One of the ways to obtain a submanifold is by working with *submersions*. The most well-known and studied map of this kind is the *Riemannian Submersion*. The notion of Riemannian submersion was introduced first by O'Neill [9]. Riemannian submersions have significant implications in physics, particularly in the study of gauge theories and field theories. In the context of fiber bundles, Riemannian submersions often arise when dealing with the projection of a higher-dimensional physical space onto a lower-dimensional base manifold. This projection preserves certain geometric and metric properties, making it a valuable tool in modeling and understanding physical phenomena, such as gauge field theories and the geometry of spacetime in general relativity. Additionally, Riemannian submersions find applications in optimal control theory, providing insights into the dynamics and symmetries of physical systems.

*Correspondence: csayer@fiu.edu
2010 *AMS Mathematics Subject Classification:* 53C15; 53B20.

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Later, Watson considered Riemannian submersions between almost Hermitian manifolds and called them *almost Hermitian submersions* [18] , where the submersion is a complex mapping. Consequently, the vertical and horizontal distributions are invariant with respect to the almost complex structure of the total manifold of the submersion. Another submersion, called an *antiinvariant Riemannian submersion*, was defined also in a complex context by Şahin [15]: in this case, the fibers are horizontal under the action of the almost complex structure, i.e. they are antiinvariant submanifolds of the total space. Outside of these specific cases, the notion of a Riemannian submersion has been considered in many other contexts, such as contact [2], complex [6, 17], almost product [13], and more. In all of these studies, submersions were defined based on the action of the structure of the manifold on the fibers.

Recently, in the complex context, Sepet defined and studied pointwise bislant Riemannian submersion [14], while two of the current paper's authors defined and studied bislant Riemannian submersion [11]. The current paper attempts to fill a gap in the literature by studying a corresponding notion of pointwise bislant Riemannian submersion in the almost product context, which is a generalization of many submersions defined before such as slant [16], pointwise slant [4], semislant [7], pointwise semislant [10], and conformal quasi bislant [8]. It is structured as follows. In Section 2, we establish the groundwork for acquiring a thorough comprehension of Riemannian submersion and almost product Riemannian manifolds within the field of Differential Geometry. Section 3 starts with the definition and an original example of a pointwise bislant Riemannian submersion in the almost product context. Following the customary focus on prior research, our inquiry examines the integrability of the fibers. We delve into the exploration of totally geodesic fibers within the context of a pointwise bislant Riemannian submersion. By investigating the properties of totally geodesic fibers, we aim to gain deeper insights into the geometric structures of the underlying submersion. The last part of our paper is devoted to the notion of $\varphi-$pluriharmonicity and $\varphi-$invariance, which are new approaches to investigate the mixed geodesics of the fibers and generalize the notion of harmonicity.

2. Preliminaries

In this section, we lay the foundation for a comprehensive understanding of Riemannian submersion and almost product Riemannian manifolds in differential geometry. The preliminary concepts presented here serve as essential building blocks to grasp the more advanced aspects of these topics. We will introduce key ideas related to differential geometry, Riemannian metrics, submersions, and some special types of Riemannian manifolds such as almost product Riemannian manifold and locally product Riemannian manifold. These fundamental concepts are crucial for comprehending the geometry and properties of Riemannian submersions and will pave the way for exploring their applications and implications.

2.1. Riemannian submersions

This section is devoted to the basics of Riemannian submersions.

Let (M,g) and $(N,\bar{g})$ be Riemannian manifolds. A surjective mapping $\pi:(M,g)\to(N,\bar{g})$ is called a *Riemannian submersion* [9] if

i) π has maximal rank;

ii) the restriction of the differential map π_* on $(\ker\pi_*)^{\perp}$ is a linear isometry.

In this case, we recall the following observations and concepts;

- For each $q \in N$, $\pi^{-1}(q)$ is a k-dimensional submanifold of M and called a *fiber*, where $k = \dim(M) - \dim(N)$.
- A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers.
- We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $\ker \pi_*$ and the horizontal distribution $\ker \pi_*^{\perp}$, respectively.
- The manifold (M, g) is called *total manifold* and the manifold $(N, \bar{g})$ is called *base manifold* of the submersion $\pi : (M, g) \to (N, \bar{g})$.
- A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e.

$$\pi_* X_p = X_{*\pi(p)}, \, \forall p \in M.$$

The last fact given above yields the following Lemma [9], which explains the preservation of brackets, inner products, and covariant derivatives;

Lemma 2.1 *Let $\pi : (M, g) \to (N, \bar{g})$ be a Riemannian submersion between Riemannian manifolds. If X and Y are basic vector fields, then*

- $g(X, Y) = \bar{g}(X_*, Y_*) \circ \pi$,
- *the horizontal part $\mathcal{H}[X, Y]$ of $[X, Y]$ is a basic vector field corresponding to $[X_*, Y_*]$,*
- *the horizontal part $\mathcal{H}(\nabla^M_X Y)$ of $\nabla^M_X Y$ is the basic vector field corresponding to $\nabla^N_{X_*} Y_*$,*
- *$[U, X]$ is vertical for any vector field U of $\ker \pi_*$.*

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_E G = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}G + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}G, \tag{2.1}$$

$$\mathcal{A}_E G = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}G + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}G \tag{2.2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection of g. One can see that a Riemannian submersion π has totally geodesic fibers if and only if $\mathcal{T}$ vanishes. On the other side, $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution. Moreover, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. Now we give the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$.

Let V, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{2.3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{2.4}$$

On the other hand, from (2.1) and (2.2), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.5}$$

$$\nabla_V X = \mathrm{T}_V X + \mathcal{H}\nabla_V X, \tag{2.6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.8}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic

$$\mathcal{H}\nabla_V X = \mathcal{A}_X V.$$

Remark 2.2 *In this paper, we will assume all horizontal vector fields as basic vector fields.*

For more details, we refer to O'Neill's paper [9] and the book [6].

Let π be a C^∞-map from a Riemannian manifold (M, g) to a Riemannian manifold $(N, \overline{g})$. The second fundamental form of π is given by

$$(\nabla\pi_*)(X, Y) = \nabla^\pi_X \pi_* Y - \pi_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM), \tag{2.9}$$

where ∇^π is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g and $\overline{g}$, [5].

If $(\nabla\pi_*)(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$, π is called a *totally geodesic map*. In particular, if $(\nabla\pi_*)(X, Y) = 0,\ X, Y \in \Gamma(D)$ for any subset D of TM, π is called a $D-$ *totally geodesic map*, [5].

2.2. Almost product Riemannian and locally product Riemannian manifolds

An m-dimensional manifold M is called *almost product manifold* if it is equipped with an *almost product structure* φ, which is a tensor field of type (1,1) satisfying

$$\varphi^2 = id, (\varphi \neq \pm id)\ , \tag{2.10}$$

denoted by (M, φ). Also for $E, G \in \Gamma(TM)$, if (M, φ) admits a Riemannian metric g satisfying

$$g(\varphi E, \varphi G) = g(E, G), \tag{2.11}$$

then M is said to be an *almost product Riemannian manifold*.
Let ∇ be the Riemannian connection with respect to the metric g on M. Then M is called a *locally product Riemannian manifold* (briefly, *l.p.R.*) if φ is parallel with respect to the connection, i.e. [19]

$$\nabla\varphi = 0. \tag{2.12}$$

3. Pointwise bislant submersions

In this section, we define and study pointwise bislant Riemannian submersion in almost product context.

Definition 3.1 *Let (M, g, φ) be an almost product Riemannian manifold and $(N, \bar{g})$ be a Riemannian manifold. A Riemannian submersion $\pi : (M, g, \varphi) \to (N, \bar{g})$ is called a pointwise bislant Riemannian submersion if the vertical distribution $ker\pi_*$ of φ decomposes into two orthogonal complementary (pointwise slant) distributions*

$\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$.
In this case, we have the decomposition

$$ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}, \tag{3.1}$$

where $\mathcal{D}^{\theta_i}$ is a pointwise slant distribution and the angle θ_i between φU and the space $(D^{\theta_i})_q$, $(\forall q \in M)$, which is independent of the choice of nonzero vector $U \in \Gamma(\mathcal{D}^{\theta_i})_q$, is called slant function of the pointwise bislant Riemannian submersion, for $i = 1, 2$.

Now, we give an example to prove the existence of the pointwise bislant Riemannian submersion concept.

Example 3.2 *Consider the standard Euclidean space $\mathbb{R}^8$ with the standard metric g. One can see that*

$$\varphi_1(x_1, x_2, ..., x_8) = (-x_3, x_4, -x_1, x_2, -x_7, x_8, -x_5, x_6)$$

and

$$\varphi_2(x_1, x_2, ..., x_8) = (x_2, x_1, x_4, x_3, x_6, x_5, x_8, x_7)$$

are almost product Riemannian structures on $\mathbb{R}^8$, where $\varphi_1\varphi_2 = -\varphi_2\varphi_1$. We can define a new almost product Riemannian structure such that

$$\varphi_{1,2} = f\varphi_1 + g\varphi_2,$$

where f and g defined by

$$\begin{aligned} f : \mathbb{R}^8 - \{-1\} &\rightarrow \mathbb{R} \\ f(x_1, x_2, ..., x_8) &= -\frac{x_1}{\sqrt{(x_1)^2 + 1}} \end{aligned}$$

$$\begin{aligned} g : \mathbb{R}^8 &\rightarrow \mathbb{R} \\ g(x_1, x_2, ..., x_8) &= \frac{1}{\sqrt{(x_1)^2 + 1}}. \end{aligned}$$

Therefore, $(\mathbb{R}^8, \varphi_{1,2}, g)$ is an almost product Riemannian manifold.

Now, let π be a map between $\mathbb{R}^8$ and $\mathbb{R}^4$ defined by

$$\pi(x_1, x_2, ..., x_8) = \left(\frac{x_1 - x_3}{\sqrt{2}}, \frac{x_2 - x_4}{\sqrt{2}}, \frac{x_5 + x_8}{\sqrt{2}}, \frac{-x_6 + x_7}{\sqrt{2}}\right).$$

The following decomposition of $\ker\pi_*$

$$\ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2},$$

where

$$\begin{aligned} \mathcal{D}^{\theta_1} &= span\left\{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\right\}, \\ \mathcal{D}^{\theta_2} &= span\left\{\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7}\right\} \end{aligned}$$

shows that π is a pointwise bislant submersion with the slant functions

$$\theta_1 = \cos^{-1}(g), \quad and \quad \theta_2 = \cos^{-1}(-f).$$

Remark 3.3 *The following table shows the importance of our work. It gives that pointwise bislant submersion is a generalization of some other submersions defined and studied before in the literature.*

Table 1. Subclasses of a pointwise bislant Riemannian submersion.

$\dim D^{\theta_1}$	$\dim D^{\theta_2}$	Submersion	Reference
$\neq 0\,(\theta_1\, constant)$	0	slant	[16]
$\neq 0\,(\theta_1\, function)$	0	pointwise slant	[4]
$\neq 0\,(\theta_1 = \frac{\pi}{2})$	0	anti invariant	[15]
$\neq 0\,(\theta_1 = 0)$	$\neq 0\,(\theta_2\, constant)$	semislant	[7]
$\neq 0\,(\theta_1 = 0)$	$\neq 0\,(\theta_2\, function)$	pointwise semislant	[10]
$\neq 0\,(\theta_1\, constant)$	$\neq 0\,(\theta_2\, constant)$	bislant	[11]

Let $\pi : (M, g, \varphi) \to (N, \bar{g})$ be a pointwise bislant submersion from an almost product Riemannian manifold M onto a Riemannian manifold N. Then, for any $V \in \Gamma(\ker \pi_*)$, we may decompose φV into vertical and horizontal parts:

$$\varphi V = tV + nV, \tag{3.2}$$

where $tV \in \Gamma(\ker \pi_*)$ and $nV \in \Gamma(\ker {\pi_*}^{\perp})$. Similarly, for any $\xi \in \Gamma(\ker {\pi_*}^{\perp})$,

$$\varphi\xi = \mathfrak{T}\xi + N\xi, \tag{3.3}$$

where $\mathfrak{T}\xi \in \Gamma(\ker \pi_*)$ and $N\xi \in \Gamma(\ker {\pi_*}^{\perp})$.

Remark 3.4 *The concept of pointwise bislant Riemannian submersion in the complex context was given first by Sepet, [14], which is a special case of the work of Sayar et al., [12]. In this case, while the canonical structure t is not symmetric, t^2 shows up symmetric, which yields us to define such submersions. In our current work, t becomes symmetric and all the results appear different than the previous works done before.*
Moreover, the last two sections include the original results, which have not been given in the literature.

Under these circumstances, we have the following identities for the canonical structures given above.

Lemma 3.5 *Let π be a pointwise bislant submersion from an almost product Riemannian manifold (M, g, φ) onto a Riemannian manifold $(N, \bar{g})$. Then, we have*

$$X = t^2X + \mathfrak{T}nX, \tag{3.4}$$
$$0 = NnX + ntX, \tag{3.5}$$
$$Y = N^2Y + n\mathfrak{T}Y, \tag{3.6}$$
$$0 = t\mathfrak{T}Y + \mathfrak{T}NY, \tag{3.7}$$

where $X \in \Gamma(\ker \pi_*)$, $Y \in \Gamma(\ker {\pi_*}^{\perp})$.

Proof The proof follows from (2.10). □

The following lemma gives similar results to the previous one in the case of almost product structure φ is parallel.

Lemma 3.6 *Let* π *be a pointwise bislant submersion from a locally product Riemannian (l.p.R.) manifold* (M,g,φ) *onto a Riemannian manifold* $(N,\bar{g})$*. Then, we have*

$$
\begin{aligned}
\hat{\nabla}_X tY + \mathcal{T}_X nY &= \mathfrak{T}\mathcal{T}_X Y + t\hat{\nabla}_X Y,\\
\mathcal{T}_X tY + \mathcal{A}_{nY} X &= N\mathcal{T}_X Y + n\hat{\nabla}_X Y,\\
\hat{\nabla}_\beta \mathfrak{T}\xi + \mathcal{A}_\beta N\xi &= \mathfrak{T}\mathcal{H}\nabla_\beta \xi + t\mathcal{A}_\beta \xi,\\
\mathcal{A}_\beta \mathfrak{T}\xi + \mathcal{H}\nabla_\beta N\xi &= N\mathcal{H}\nabla_\beta \xi + n\mathcal{A}_\beta \xi,\\
\hat{\nabla}_\beta tX + \mathcal{A}_\beta nX &= \mathfrak{T}\mathcal{A}_\beta X + t\hat{\nabla}_\beta X,\\
\mathcal{A}_\beta tX + \mathcal{H}\nabla_\beta nX &= N\mathcal{A}_\beta X + n\hat{\nabla}_\beta X,\\
\hat{\nabla}_X \mathfrak{T}\xi + \mathcal{T}_X N\xi &= \mathfrak{T}\mathcal{A}_\xi X + t\mathcal{T}_X \xi,\\
\mathcal{T}_X \mathfrak{T}\xi + \mathcal{A}_{N\xi} X &= N\mathcal{A}_\xi X + n\mathcal{T}_X \xi,
\end{aligned}
$$

where $X,Y \in \Gamma(\ker \pi_*)$ *and* $\beta,\xi \in \Gamma(\ker {\pi_*}^\perp)$*.*

Proof The proof follows from (2.5) $\sim$ (2.8), (2.12), (3.2), and (3.3). □

Remark 3.7 *From now on, we will use the abbreviation l.p.R. for locally product Riemannian.*

If we consider the pointwise distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ with the previous Lemma 3.6, we obtain the following results.

Lemma 3.8 *Let* π *be a pointwise bislant submersion from an l.p.R. manifold* (M,g,φ) *onto a Riemannian manifold* $(N,\bar{g})$*. Then, for any* $V \in \Gamma(\mathcal{D}^{\theta_i})$*,* $i = 1,2$*,*

- $tV \in \Gamma(\mathcal{D}^{\theta_i})$,
- $t^2 V = \cos^2(\theta_i) V$,
- $ntU = -NnU$,
- $\mathfrak{T}nV = \sin^2(\theta_i) V$,
- $g(tV,tV) = \cos^2(\theta_i) g(V,V)$,
- $g(nV,nV) = \sin^2(\theta_i) g(V,V)$.

Now, we give a lemma, which is useful and used throughout our paper.

Lemma 3.9 *Let* π *be a pointwise bislant submersion from an l.p.R. manifold* (M,g,φ) *onto a Riemannian manifold* $(N,\bar{g})$*. Then,* $g(\nabla_X Y, U)$ *is equivalent to the followings*

$$-\csc^2\theta_j \Big[g(X, \mathcal{T}_Y ntU + \mathcal{T}_{tY} nU + \mathcal{A}_{nY} nU)\Big], \tag{3.8}$$

$$\sec^2\theta_j \left[g(\hat{\nabla}_X tY, tU) + g(X, \mathcal{T}_{tU} nY + \mathcal{T}_Y ntU)\right], \tag{3.9}$$

where $X,Y \in \Gamma(\mathcal{D}^{\theta_i})$ *and* $U \in \Gamma(\mathcal{D}^{\theta_j})$*,* $i \neq j$*,* $i,j = 1,2$*.*

Proof Let $X,Y\in\Gamma(\mathcal{D}^{\theta_i})$ and $U\in\Gamma(\mathcal{D}^{\theta_j})$, $i\neq j,\ i,j=1,2$. By using (2.11) and (3.2), we obtain

$$g(\nabla_X Y,U)=g(\varphi\nabla_X Y,tU)+g(\varphi\nabla_X Y,nU). \tag{3.10}$$

The first expression on the right side of (3.10) with Lemma 3.8 gives

$$\begin{aligned} g(\varphi\nabla_X Y,tU) &= g(\nabla_X Y,t^2U)+g(\nabla_X Y,ntU)\\ &= \cos^2\theta_j g(\nabla_X Y,U)+g(T_X Y,ntU), \end{aligned}$$

which yields with (3.10), (2.5), and the symmetry of the product structure φ

$$\sin^2\theta_j g(\nabla_X Y,U)=g(T_X Y,ntU)+g(T_X tY,nU)+g(A_{nY}X,nU)$$

shows (3.8).

On the other side, the second expression on the right side of (3.10) with the symmetry of the product structure φ, (3.3), and Lemma 3.8 gives

$$\begin{aligned} g(\varphi\nabla_X Y,nU) &= g(\nabla_X Y,\mathfrak{T}nU)+g(\nabla_X Y,NnU)\\ &= \sin^2\theta_j g(\nabla_X Y,U)+g(T_X Y,NnU), \end{aligned}$$

which yields with (3.10), (3.2), (2.5), and the symmetry of the product structure φ

$$\cos^2\theta_j g(\nabla_X Y,U)=g(\hat{\nabla}_X tY,tU)+g(T_X nY,tU)+g(T_X Y,NnU)$$

completes the proof. □

3.1. Integrability

In the case of studying a submersion, a natural question would be *integrability conditions.* In this section, we work the integrability conditions for the pointwise slant distributions $\mathcal{D}^{\theta_i}, i=1,2$ and horizontal distribution $\ker\pi_*^{\perp}$, respectively.

The following theorem gives some conditions for the integrability of pointwise distributions D^{θ_i}.

Theorem 3.10 *Let π be a pointwise bislant submersion from an l.p.R. manifold (M,g,φ) onto a Riemannian manifold $(N,\bar{g})$. Then, the following conditions are equivalent to each other*

i) *the pointwise distribution D^{θ_i} is integrable,*

ii) $g(T_{tX}Y-T_{tY}X,nU)=g(A_{nY}X-A_{nX}Y)$,

iii) $g(tU,\hat{\nabla}_X tY-\hat{\nabla}_Y tX)=g(T_Y nX-T_X nY,tU)$,

where $X,Y\in\Gamma(\mathcal{D}^{\theta_i})$ and $U\in\Gamma(\mathcal{D}^{\theta_j})$, $i\neq j,\ i,j=1,2$.

Proof Let $X,Y\in\Gamma(\mathcal{D}^{\theta_i})$ and $U\in\Gamma(\mathcal{D}^{\theta_j})$, $i\neq j$, $i,j=1,2$. The pointwise slant distribution $\mathcal{D}^{\theta_i}$ is integrable if and only if $[X,Y]\in\mathcal{D}^{\theta_i}$, i.e. $[X,Y]\perp\mathcal{D}^{\theta_j}$. The equation (3.8) yields

$$\begin{aligned} g([X,Y],U) &= -\csc^2\theta_j(g(T_{tY}nU,X)-g(T_{tX}nU,Y)+g(A_{nY}nU,X) \\ &\quad -g(A_{nX}nU,Y)), \end{aligned}$$

which proves **i)** $\Leftrightarrow$ **ii)**. On the other side, (3.9) gives

$$\begin{aligned} g([X,Y],U) &= \sec^2\theta_j(g(\hat{\nabla}_X tY,tU)+g(X,T_{tU}nY)-g(\hat{\nabla}_Y tX,tU) \\ &\quad -g(Y,T_{tU}nX)), \end{aligned}$$

which helps to prove **i)** $\Leftrightarrow$ **iii)**, and completes the proof. □

Remark 3.11 *Since for any* $X,Y\in\Gamma(ker\pi_*)$,

$$\begin{aligned} [X,Y] &= \nabla_X Y-\nabla_Y X \\ &= T_X Y+\hat{\nabla}_X Y-T_Y X-\hat{\nabla}_Y X \\ &= \hat{\nabla}_X Y-\hat{\nabla}_Y X \quad \Rightarrow [X,Y]\in ker\pi_*, \end{aligned}$$

it is known that $ker\pi_*$ *is always integrable.*
On the other side, for any $\xi,\beta\in\Gamma(ker\pi_*^{\perp})$, *the relation* (2.4)

$$\mathcal{A}_\xi\beta=-\mathcal{A}_\beta\xi=\frac{1}{2}\mathcal{V}[\xi,\beta]$$

gives that $ker\pi_*^{\perp}$ *is integrable if and only if it defines totally geodesic foliations on* M.

Theorem 3.12 *Let* π *be a pointwise bislant submersion from an l.p.R. manifold* (M,g,φ) *onto a Riemannian manifold* $(N,\bar{g})$. *Then, the horizontal distribution* $ker\pi_*^{\perp}$ *is integrable and totally geodesic if and only if*

$$\mathfrak{T}(A_\alpha\mathfrak{T}\xi+\mathcal{H}\nabla_\alpha N\xi)+t(\mathcal{V}\nabla_\alpha\mathfrak{T}\xi+A_\alpha N\xi)=0,$$

where $\alpha,\xi\in\Gamma(ker\pi_*^{\perp})$.

Proof Let $\alpha,\xi\in\Gamma(ker\pi_*^{\perp})$. The parallelism of the almost product structure φ, (2.6), (2.8), (3.2), and (3.3) give

$$\begin{aligned} \nabla_\alpha\xi &= \varphi\nabla_\alpha\varphi\xi \\ &= \mathfrak{T}(A_\alpha\mathfrak{T}\xi+\mathcal{H}\nabla_\alpha N\xi)+N(A_\alpha\mathfrak{T}\xi+\mathcal{H}\nabla_\alpha N\xi) \\ &\quad +t(\mathcal{V}\nabla_\alpha\mathfrak{T}\xi+A_\alpha N\xi)+n(\mathcal{V}\nabla_\alpha\mathfrak{T}\xi+A_\alpha N\xi), \end{aligned}$$

which completes the proof. □

3.2. Totally geodesics

The concept of totally geodesic fibers is essential in understanding the geometric and topological properties of the submersions, providing insights into the relationship between the base space and the total space and revealing the presence of symmetries and isometries. This section is devoted for the geodesics of a pointwise bislant Riemannian submersion.

The first result is for the geodesics of the vertical distribution $ker\pi_*$.

Theorem 3.13 *Let π be a pointwise bislant submersion from an l.p.R. manifold (M, g, φ) onto a Riemannian manifold $(N, \bar{g})$. Then, the following are equivalent to each other*

i) *the vertical distribution $ker\pi_*$ defines totally geodesic fibers,*

ii) *for any $X, Y \in \Gamma(ker\pi_*)$,*

$$N(T_X tY + A_{nY} X) + n(\hat{\nabla}_X tY + T_X nY) = 0,$$

iii) *for any $X, Y \in \Gamma(ker\pi_*)$, $\xi \in \Gamma(ker\pi_*^\perp)$,*

$$\begin{aligned} \bar{g}((\nabla\pi_*)(X, \mathfrak{T}\xi), \pi_*(nY)) &= \bar{g}((\nabla\pi_*)(X, tY), \pi_*(N\xi)) + g(\hat{\nabla}_X tY, \mathfrak{T}\xi) \\ &+ g(\mathcal{H}\nabla_X nY, N\xi). \end{aligned}$$

Proof To show **i)** $\Leftrightarrow$ **ii)**, for any $X, Y \in \Gamma(ker\pi_*)$, we need to prove $T_X Y = 0$, i.e. $\hat{\nabla}_X Y \in ker\pi_*$. By (2.5), (2.6), (2.12), (3.2), and (3.3), we obtain

$$\begin{aligned} \nabla_X Y &= \varphi\nabla_X \varphi Y \\ &= \mathfrak{T}(T_X tY + A_{nY} X) + N(T_X tY + A_{nY} X) \\ &\quad + t(\hat{\nabla}_X tY + T_X nY) + n(\hat{\nabla}_X tY + T_X nY). \end{aligned}$$

Since $\mathcal{V}\nabla_X Y = \hat{\nabla}_X Y \in ker\pi_*$, i.e. $\mathcal{H}\nabla_X Y = 0$, considering the horizontal part of the last equation, we get **i)** $\Leftrightarrow$ **ii)**.

Now, another approach to show the vertical distribution $ker\pi_*$ defines totally geodesic fibers is for any $X, Y \in \Gamma(ker\pi_*)$, and $\xi \in \Gamma(ker\pi_*^\perp)$, $\hat{\nabla}_X Y \perp \xi$, i.e. $g(\hat{\nabla}_X Y, \xi) = 0$. Using (2.5), (2.6), (2.9), (3.2), and (3.3), we obtain

$$\begin{aligned} g(\hat{\nabla}_X Y, \xi) &= g(\nabla_X \varphi Y, \varphi\xi) \\ &= g(\hat{\nabla}_X tY, \mathfrak{T}\xi) + g(\mathcal{H}\nabla_X tY, N\xi) - g(\nabla_X \mathfrak{T}\xi, nY) \\ &\quad + g(\mathcal{H}\nabla_X nY, N\xi) \\ &= g(\hat{\nabla}_X tY, \mathfrak{T}\xi) - \bar{g}((\nabla\pi_*)(X, tY), \pi_*(N\xi)) \\ &\quad + \bar{g}((\nabla\pi_*)(X, \mathfrak{T}\xi), \pi_*(nY)) + g(\mathcal{H}\nabla_X nY, N\xi), \end{aligned}$$

which shows **i)** $\Leftrightarrow$ **iii)**, completes the proof. □

By Remark 3.11 and Theorem 3.13, we give the following result.

Corollary 3.14 *Let π be a pointwise bislant submersion from an l.p.R. manifold (M, g, φ) onto a Riemannian manifold $(N, \bar{g})$. Then, M is a locally product*

$$M_{ker\pi_*} \times M_{ker\pi_*^\perp}$$

if and only if Theorem 3.12 and one of the conditions in Theorem 3.13 are satisfied, where $M_{ker\pi_*}$ and $M_{ker\pi_*^\perp}$ are integral manifolds of the distributions $ker\pi_*$, $ker\pi_*{}^\perp$, respectively.

The following result is another point of view on Theorem 3.12 and Theorem 3.13.

Theorem 3.15 *Let π be a pointwise bislant submersion from an l.p.R. manifold (M, g, φ) onto a Riemannian manifold $(N, \bar{g})$. Then, π is a totally geodesic map if and only if Theorem 3.12 and at least one of the conditions in Theorem 3.13 are satisfied, where $M_{ker\pi_*}$ and $M_{ker\pi_*^\perp}$ are integral manifolds of the distributions $ker\pi_*$, $ker\pi_*{}^\perp$, respectively.*

Our subsequent interest is the geodesics of the components of the fibers, in other words, the geodesics of the pointwise slant distributions D^{θ_i}, $i = 1, 2$.

Theorem 3.16 *Let π be a pointwise bislant submersion from an l.p.R. manifold (M, g, φ) onto a Riemannian manifold $(N, \bar{g})$. Then, the followings are equivalent to each other*

i) *the pointwise slant distribution D^{θ_i} defines totally geodesic fibers on the vertical distribution $ker\pi_*$, $(i = 1, 2)$,*

ii) $g(X, T_Y ntU + T_{tY} nU + A_{nY} nU) = 0,$

iii) $g(\hat{\nabla}_X tY, tU) + g(X, \mathcal{T}_{tU} nY + \mathcal{T}_Y ntU) = 0,$

iv) $g(\hat{\nabla}_X tY, tU) + \bar{g}((\nabla\pi_*)(X, tU), \pi_*(nY)) = \bar{g}((\nabla\pi_*)(X, \varphi Y), \pi_*(nU)),$

where $X, Y \in \Gamma(D^{\theta_i})$, $U \in \Gamma(D^{\theta_j})$, $i \neq j$, $i, j = 1, 2$.

Proof The relation **i)** $\Leftrightarrow$ **ii)** and **i)** $\Leftrightarrow$ **iii)** follow from (3.8) and (3.9), respectively.
For the relation **i)** $\Leftrightarrow$ **iv)**, let $X, Y \in \Gamma(D^{\theta_i})$, $U \in \Gamma(D^{\theta_j})$, $i \neq j$, $i, j = 1, 2$. By using (2.5), (2.6), (2.9), (2.11), and (2.12), we obtain

$$\begin{aligned} g(\hat{\nabla}_X Y, U) &= g(\nabla_X tY, tU) + g(\nabla_X tY, nU) \\ &\quad + g(\nabla_X nY, tU) + g(\nabla_X nY, nU) \\ &= g(\hat{\nabla}_X tY, tU) + \bar{g}((\nabla\pi_*)(X, tU), \pi_*(nY)) \\ &\quad - \bar{g}((\nabla\pi_*)(X, \varphi Y), \pi_*(nU)), \end{aligned}$$

which gives **i)** $\Leftrightarrow$ **iv)**, and completes the proof. □

As a result of Theorem 3.16, we give the following corollary.

Corollary 3.17 *Let π be a pointwise bislant submersion from an l.p.R. manifold (M, g, φ) onto a Riemannian manifold $(N, \bar{g})$. Then, the fibers are a locally product*

$$M_{D^{\theta_1}} \times M_{D^{\theta_2}}$$

if and only if one of the conditions in Theorem 3.16 is satisfied for each pointwise distribution D^{θ_1} and D^{θ_2}, where $M_{D^{\theta_1}}$ and $M_{D^{\theta_2}}$ are integral manifolds of the distributions D^{θ_1}, D^{θ_2}, respectively.

3.3. $\varphi-$pluriharmonicity of π

The intriguing concept of pluriharmonicity plays a crucial role in understanding the behavior of functions over higher-dimensional domains and has significant applications in various branches of mathematics and physics. In this section, we investigate the $\varphi-$pluriharmonicity of a pointwise bislant submersion π.

First, we give the following definition.

Definition 3.18 *[3] Let π be a pointwise bislant Riemannian submersion from an l.p.R. manifold (M, φ, g) onto a Riemannian manifold $(N, \bar{g})$.*
π is called

- *$D^{\theta_i} - \varphi-$pluriharmonic if for any $X, Y \in \Gamma(D^{\theta_i})$, $i = 1, 2$,*
- *$(D^{\theta_i} - D^{\theta_j}) - \varphi-$pluriharmonic if for any $X \in \Gamma(D^{\theta_i})$, $Y \in \Gamma(D^{\theta_j})$, $i, j = 1, 2$, $i \neq j$,*
- *$(ker\pi_* - ker\pi_*^{\perp}) - \varphi-$pluriharmonic if for any $X \in \Gamma(ker\pi_*)$, $Y \in \Gamma(ker\pi_*^{\perp})$,*

$$(\nabla\pi_*)(X, Y) + (\nabla\pi_*)(\varphi X, \varphi Y) = 0. \tag{3.11}$$

The following theorem gives a result in the case of π is $D^{\theta_i} - \varphi-$pluriharmonic.

Theorem 3.19 *Let π be a $D^{\theta_i} - \varphi-$pluriharmonic pointwise bislant Riemannian submersion from an l.p.R. manifold (M, φ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the submersion π is a $nD^{\theta_i}-$geodesic map if and only if*

$$N(\cos^2(\theta_i)T_{tX}Y + T_{tX}ntY) + T_XY + A_{nY}tX + A_{nX}tY$$
$$+n(-\sin(2\theta_i)tX(\theta_i)Y + A_{ntY}tX + \cos^2(\theta_i)\hat{\nabla}_{tX}Y) = 0,$$

where $X, Y \in \Gamma(D^{\theta_i})$, $i \neq j$, $i, j = 1, 2$.

Proof Assumption gives, for any $X, Y \in \Gamma(D^{\theta_i})$, $i \neq j$, $i, j = 1, 2$,

$$(\nabla\pi_*)(X, Y) + (\nabla\pi_*)(\varphi X, \varphi Y) = 0.$$

By using (2.5), (2.6), (2.9), (2.10), (2.12), (3.2), (3.3), and Lemma 3.8, we have

$$
\begin{aligned}
0 \quad = \quad & -\pi_*(T_XY + A_{nY}tX + A_{nX}tY) + ((\nabla\pi_*)(nX, nY)) \\
& -\pi_*(\varphi(\nabla_{tX}t^2Y + \nabla_{tX}ntY)) \\
\Rightarrow \quad & ((\nabla\pi_*)(nX, nY)) = \pi_*\Big(T_XY + A_{nY}tX + A_{nX}tY - \sin(2\theta_i)tX(\theta_i)nY \\
& \qquad + \cos^2(\theta_i)NT_{tX}Y + \cos^2(\theta_i)n\hat{\nabla}_{tX}Y + NT_{tX}ntY + nA_{ntY}tX\Big),
\end{aligned}
$$

which completes the proof. □

We recall the definition of *mixed geodesic;* given two distributions D^1 and D^2 defined on the fibers of a Riemannian submersion π, the fibers are called $(D^1 - D^2)-$ *mixed geodesic* if $T_{D^1}D^2 = 0$.
$(D^{\theta_1} - D^{\theta_2}) - \varphi-$pluriharmonicity of the pointwise bislant submersion gives the following result.

Theorem 3.20 *Let π be a $(D^{\theta_1} - D^{\theta_2}) - \varphi-$pluriharmonic pointwise bislant Riemannian submersion from an l.p.R. manifold (M, φ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the followings are equivalent to each other*

- ***i)*** *the fibers are $(D^{\theta_1} - D^{\theta_2})-$mixed geodesic,*
- ***ii)*** $\nabla^N_{\pi_*(\varphi X)}\pi_*(\varphi U) = \pi_*(T_{tX}tU + A_{nU}tX + A_{nX}tU + \mathcal{H}\nabla_{nX}nU)$,
- ***iii)***

$$
\begin{aligned}
\nabla^N_{\pi_*(\varphi X)}\pi_*(\varphi U) \quad = \quad & \pi_*\Big(n\big(-\sin(2\theta_2)tX(\theta_2)U + \cos^2(\theta_2)\hat{\nabla}_{tX}U + T_{tX}ntU\big) \\
& + N\big(\cos^2(\theta_2)T_{tX}U + A_{ntU}tX\big) + T_{tX}tU + A_{nU}tX\Big),
\end{aligned}
$$

where $X \in \Gamma(D^{\theta_1})$ and $U \in \Gamma(D^{\theta_2})$.

Proof The $(D^{\theta_1} - D^{\theta_2}) - \varphi-$pluriharmonicity yields, for any $X \in \Gamma(D^{\theta_1})$, $U \in \Gamma(D^{\theta_2})$,

$$(\nabla\pi_*)(X, U) + (\nabla\pi_*)(\varphi X, \varphi U) = 0,$$

which gives with (2.9), (3.2), and (3.3)

$$
\begin{aligned}
-\pi_*(\nabla_XU) \quad - \quad & \pi_*(\nabla_{tX}tU) - \pi_*(\nabla_{tX}nU) - \pi_*(\nabla_{nX}tU) \\
& + \nabla^N_{\pi_*(\varphi X)}\pi_*(\varphi U) - \pi_*(\nabla_{nX}nU) = 0. \qquad (3.12)
\end{aligned}
$$

By using (2.5) $\sim$ (2.8), we prove the relation **i)** $\Leftrightarrow$ **ii)**.
To show **i)** $\Leftrightarrow$ **iii)**, we consider the second term in (3.12), which can be expressed with the help of (2.12), (3.2), (3.3), and Lemma 3.8 as

$$\begin{aligned}
-\pi_*(\nabla_{tX}tU) &= -\pi_*(\varphi(\nabla_{tX}t^2U) + \nabla_{tX}ntU) \\
&= -\pi_*(\varphi(\nabla_{tX}(\cos^2(\theta_2)U) + \nabla_{tX}ntU)) \qquad (3.13)\\
&= -\pi_*(\varphi(-\sin(2\theta_2)tX(\theta_2)U + \cos^2(\theta_2)T_{tX}U + \cos^2(\theta_2)\hat{\nabla}_{tX}U) \\
&\quad +\varphi(T_{tX}ntU + A_{ntU}tX)).
\end{aligned}$$

Thus, if we consider (3.12) and (3.13) together, we prove the relation **i)** $\Leftrightarrow$ **iii)**, which completes the proof. □

The last result is in the case of $(ker\pi_* - ker\pi_*^{\perp}) - \varphi-$ pluriharmonicity of a pointwise bislant submersion.

Theorem 3.21 *Let π be a $(ker\pi_* - ker\pi_*^{\perp}) - \varphi-$ pluriharmonic pointwise bislant Riemannian submersion from an l.p.R. manifold (M, φ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the fibers are $(ker\pi_* - ker\pi_*^{\perp})-$ mixed geodesic if and only if*

$$\nabla^N_{\pi_*(\varphi X)}\pi_*(\varphi\xi) = \pi_*(T_{tX}\mathfrak{T}\xi + A_{N\xi}tX + A_{nX}\mathfrak{T}\xi + \mathcal{H}\nabla_{nX}N\xi),$$

where $X \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$.

Proof Since the submersion π is $(ker\pi_* - ker\pi_*^{\perp}) - \varphi-$ pluriharmonic, for any $X \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we have

$$(\nabla\pi_*)(X, \xi) + (\nabla\pi_*)(\varphi X, \varphi\xi) = 0.$$

By using (2.5) $\sim$ (2.8), (2.9), (3.2), and (3.3), we obtain

$$\begin{aligned}
\pi_*(\nabla_X\xi) &= \pi_*(\nabla_{tX}\mathfrak{T}\xi) + \pi_*(\nabla_{tX}N\xi) + \pi_*(\nabla_{nX}\mathfrak{T}\xi) + \pi_*(\nabla_{nX}N\xi) \\
&\quad -\nabla^N_{\pi_*(\varphi X)}\pi_*(\varphi\xi),
\end{aligned}$$

which completes the theorem. □

Remark 3.22 *The notion of a pluriharmonic map is a generalization of the idea of a harmonic map. Here it helps us to understand when the fibers are either geodesic or mixed geodesic. This is a new approach that makes our section important.*

3.4. $\varphi-$ invariant and totally geodesics

In this section, we find some conditions for a pointwise bislant submersion to be the $\varphi-$ invariant of the distributions on the total space.

We give the following concept, which helps to provide new conditions for some other concepts studied before.

Definition 3.23 *[3] Let π be a pointwise bislant Riemannian submersion from an l.p.R. manifold (M, φ, g) onto a Riemannian manifold $(N, \bar{g})$.*
Then, π is called

- $D^{\theta_i} - \varphi-$ *invariant if for any* $X, Y \in \Gamma(D^{\theta_i})$,
- $(D^{\theta_i} - D^{\theta_j}) - \varphi-$ *invariant if for any* $X \in \Gamma(D^{\theta_i})$, $Y \in \Gamma(D^{\theta_j})$, $i, j = 1, 2$, $i \neq j$,
- $(ker\pi_* - ker\pi_*^{\perp}) - \varphi-$ *invariant if for any* $X \in \Gamma(ker\pi_*)$, $Y \in \Gamma(ker\pi_*^{\perp})$,

$$(\nabla\pi_*)(X, Y) = (\nabla\pi_*)(\varphi X, \varphi Y). \tag{3.14}$$

Now, we give a result of π being $D^{\theta_i} - \varphi-$invariant, $i, j = 1, 2$, $i \neq j$.

Theorem 3.24 *Let* π *be a* $D^{\theta_i} - \varphi-$ *invariant pointwise bislant Riemannian submersion from an l.p.R. manifold* (M, φ, g) *onto a Riemannian manifold* $(N, \bar{g})$*. Then, the fibers are* $D^{\theta_i}-$*geodesic if and only if*

$$\nabla^N_{\pi_*(\varphi X)}\pi_*(\varphi Y) = \pi_*(T_{tX}tY + A_{nY}tX + A_{nX}tY + \mathcal{H}\nabla_{nX}nY),$$

where $X, Y \in \Gamma(D^{\theta_i})$, $i = 1, 2$, $i \neq j$.

Proof The $\varphi-$invariance of the submersion π yields, for any $X, Y \in \Gamma(D^{\theta_i})$, $i = 1, 2$, $i \neq j$,

$$(\nabla\pi_*)(X, Y) = (\nabla\pi_*)(\varphi X, \varphi Y),$$

with the help of (2.5) $\sim$ (2.8), (2.9), (3.2), and (3.3), we see

$$\begin{aligned} -\pi_*(\nabla_X Y) &= -\pi_*(\nabla_{tX}tY) - \pi_*(\nabla_{tX}nY) - \pi_*(\nabla_{nX}tY) - \pi_*(\nabla_{nX}nY) \\ &= -\pi_*(T_{tX}tY + A_{nY}tX + A_{nX}tY + \mathcal{H}\nabla_{nX}nY) + \nabla^N_{\pi_*(\varphi X)}\pi_*(\varphi Y) \end{aligned}$$

and completes the proof. □

Next result gives a relation between $(D^{\theta_1} - D^{\theta_2}) - \varphi-$invariance and $(D^{\theta_1} - D^{\theta_2})-$mixed geodesics of a pointwise bislant submersion.

Theorem 3.25 *Let* π *be a* $(D^{\theta_1} - D^{\theta_2}) - \varphi-$ *invariant pointwise bislant Riemannian submersion from an l.p.R. manifold* (M, φ, g) *onto a Riemannian manifold* $(N, \bar{g})$*. Then, the fibers are* $(D^{\theta_1} - D^{\theta_2})-$*mixed geodesics if and only if*

$$(\nabla\pi_*)(nX, nU) = \pi_*(T_{tX}tU + A_{nU}tX + A_{nX}tU),$$

where $X \in \Gamma(D^{\theta_1})$ *and* $U \in \Gamma(D^{\theta_2})$.

Proof The $(D^{\theta_1} - D^{\theta_2}) - \varphi-$invariance of the submersion π gives

$$(\nabla\pi_*)(X, U) = (\nabla\pi_*)(\varphi X, \varphi U),$$

from which we obtain with (2.5) $\sim$ (2.7), (2.9), (3.2), and (3.3)

$$-\pi_*(\nabla_X U) = -\pi_*(T_{tX}tU + A_{nU}tX + A_{nX}tU) + (\nabla\pi_*)(X, U).$$

This completes the proof. □

The last result for $\varphi-$invariance is a relation between $(ker\pi_* - ker\pi_*^{\perp}) - \varphi-$invariance and $(ker\pi_* - ker\pi_*^{\perp})-$mixed geodesics.

Theorem 3.26 *Let π be a $(ker\pi_* - ker\pi_*^{\perp}) - \varphi-$ invariant pointwise bislant Riemannian submersion from an l.p.R. manifold (M, φ, g) onto a Riemannian manifold $(N, \bar{g})$. Then, the fibres are $(ker\pi_* - ker\pi_*^{\perp})-$ mixed geodesics if and only if*

$$(\nabla\pi_*)(nX, N\xi) = \pi_*(T_{tX}\mathfrak{T}\xi + A_{N\xi}tX + A_{nX}\mathfrak{T}\xi),$$

where $X \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$.

Proof Since π is a $(ker\pi_* - ker\pi_*^{\perp}) - \varphi-$invariant pointwise bislant Riemannian submersion, for any $X \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we have

$$(\nabla\pi_*)(X, \xi) = (\nabla\pi_*)(\varphi X, \varphi\xi).$$

By using (2.5) $\sim$ (2.7), (3.2), and (3.3), we have

$$-\pi_*(\nabla_X\xi) = (\nabla\pi_*)(nX, N\xi) - \pi_*(T_{tX}\mathfrak{T}\xi + A_{N\xi}tX + A_{nX}\mathfrak{T}\xi),$$

which completes the proof. □

Remark 3.27 *In this last section, we give another new approach for the mixed geodesics of the fibers by considering the notion of $\varphi-$ invariance.*

4. Conclusion

In this study, we have investigated the properties of a pointwise bislant Riemannian submersion originating from an almost product Riemannian manifold. Our analysis focused on the integrability and geodesic conditions of the fibers, shedding light on the intricate interplay between these structures. Additionally, we introduced a novel concept of $\varphi-$ pluriharmonicity and explored its implications in the context of the submersion. This new notion brings forth intriguing possibilities for understanding the behavior of the submersion in a broader context. Furthermore, we examined the $\varphi-$ invariance of the submersion, which allows us to identify particular symmetries and transformations that leave the submersion unchanged. Our findings contribute to understanding pointwise bislant Riemannian submersions and extend the knowledge of Riemannian geometry. This work opens up new avenues for research and may have implications in various applications across mathematics and physics.

Acknowledgment

The research of this project **Pointwise Bi-Slant Riemannian Submersions** was done during the Summer 2023 REU program "**AMRPU @ FIU**" that took place at the Department of Mathematics and Statistics, Florida International University, and Grayson LIGHT and Cem SAYAR were supported by the NSF (REU Site) grant DMS-2050971, Mehmet Akif AKYOL was supported by the Scientific and Technological Council of Türkiye (TÜBİTAK) with the project number 121F277. In particular, the support of Jose Medel and Diego Castro came from the above grant. The most considerable appreciation is for the excellent person, Prof. Svetlana Roudenko, who guided and supported us during that research.

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