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EMİNAĞA M. MAMEDOV

ŞEYMA ÇETİN

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Interior Schauder-type estimates for $m-th$ order elliptic operators in rearrangement-invariant Sobolev spaces

Eminaga M. MAMEDOV[1], **Şeyma ÇETİN**[2,3,*]
[1]Ministry of Science and Education Institute of Mathematics and Mechanics, Baku, Azerbaijan
[2]Department of Mathematics, Faculty of Arts and Sciences, Yıldız Technical University, İstanbul, Turkiye
[3]Faculty of Engineering and Natural Sciences, İstinye University, İstanbul, Turkiye

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Abstract: In this study, we investigate the m-th order elliptic operators on n-dimensional bounded domain $\Omega \subset R^n$ with discontinuous coefficients in the rearrangement-invariant Sobolev space $W_X^m(\Omega)$. In general, the considered rearrangement-invariant spaces are not separable, so the use of classical methods in these spaces requires substantial modification of classical methods and a lot of preparation, concerning correctness of substitution operator, problems related to the extension operator in such spaces, etc. For this purpose, the corresponding separable subspaces of these spaces, in which the set of compact supported infinitely differentiable functions is dense, are introduced based on the shift operator. We establish interior Schauder-type estimates in the above subspaces. Note that Lebesgue spaces $L_p(\Omega)$, grand-Lebesgue spaces, Marcinkiewicz spaces, weak-type L_p^w spaces, etc. are also covered by such spaces.

Key words: Banach function space, rearrangement-invariant spaces, Sobolev spaces, elliptic operator, Schauder type estimate

1. Introduction

Over the last years, so-called nonstandard spaces of functions have been actively used in many problems of pure mathematics, mechanics, and mathematical physics. The emergence of new functional spaces, such as Morrey space and grand-Lebesgue space, naturally requires the development of appropriate theory. That is why various problems in such spaces began to be intensively studied (see [1, 4–23, 25–33]). The methods of harmonic analysis in such spaces are well developed. At the same time, the various problems of differential equations in nonstandard Sobolev spaces, generated by the norms of these spaces, have also begun to be studied. It should be noted that, according to Luxemburg's classification (see [2]), all these spaces are Banach function spaces (b.f.s). The first research of this kind about Morrey type spaces dates back to 2000 to continue up to the present. (Note that in case of $|\Omega| = +\infty$, the Morrey space $L_{p,\lambda}(\Omega)$ is not Banach function space. In this work, we consider only the case $|\Omega| < +\infty$.) Similar studies have been also carried out for grand-Lebesgue spaces(see [7, 9, 10, 16, 28, 29, 33]). Most of these spaces are rearrangement-invariant (see [2, 27]). These circumstances require the study of solvability problems of elliptic equations in rearrangement-invariant Sobolev spaces, generated by rearrangement-invariant Banach function spaces.

*Correspondence: seyymacetin@gmail.com
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This work is a continuation and generalization of the studies [8–10, 31], which deal with the elliptic operators in grand Sobolev spaces and rearrangement-invariant Sobolev spaces. In [31], the boundedness of substitution operator and extension operators from $W^m_{X_s}(\Omega)$ to $W^m_{X_s}(\Omega_1)$ (for some $\Omega_1 \supset \Omega$) has been studied. Some basic aspects of these works, which we will use in this work, are described in Section 2.

In this paper, we study the m-th order elliptic operators on n-dimensional bounded domain $\Omega \subset R^n$ with discontinuous coefficients in the rearrangement-invariant Sobolev space $W^m_X(\Omega)$, generated by norm of some rearrangement-invariant Banach function space $X(\Omega)$. In general, the considered rearrangement-invariant spaces are not separable; therefore, using classical methods in these spaces requires the essential modification of classical methods and a lot of preparation, concerning correctness of substitution operator, problems related to the extension operator in such spaces, etc. To this aim, based on the shift operator, corresponding separable subspaces of these spaces are introduced, in which the set of compact supported infinitely differentiable functions is dense. In the classical case, Schauder type estimates play a very important role in the establishment of the Fredholmness of elliptic operators. For this purpose, we establish interior Schauder type estimates in the above subspaces. Note that Lebesgue spaces $L_p(\Omega)$, grand Lebesgue spaces, Marcinkiewicz spaces, weak-type L^w_p spaces, etc. are also covered by such spaces.

2. Essential information and notations

We will use the following standard notations: Z_+- set of nonnegative integers, $R_+ = [0, +\infty)$, $|x| = \sqrt{x_1^2 + ... + x_n^2}$ - the norm of $x = (x_1, ..., x_n) \in R^n$, $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ - the open ball in R^n, and $\partial\Omega$ will be the boundary of the domain Ω, $\overline{\Omega} = \Omega \bigcup \partial\Omega$ will stand for the closure of Ω. $\Omega_1 \subset\subset \Omega_2$ means that $\overline{\Omega_1} \subset \Omega_2$. $|\Omega|$ is Lebesgue measure of the set Ω. The diameter of the set Ω will be denoted by $d(\Omega) = d_\Omega = diam\,\Omega$, $\rho(x, M) = dist(x, M)$ - the distance between x and the set M. $M_1 \Delta M_2$ will be the symmetric difference of the sets M_1 and M_2. Accept

$$\Omega_r(x_0) = \Omega \bigcap B_r(x_0),\ B_r = B_r(0), \Omega - \delta = \{x : x + \delta \in \Omega\}\ (\forall \delta \in R^n),$$

$$\Omega_\varepsilon = \{x : dist(x, \Omega) < \varepsilon\},\ (\forall \varepsilon > 0).$$

$\Im(\Omega)$ will denote the set of measurable functions on $\Omega \subset R^n$, $[X, Y]$ - Banach space of bounded operators acting from X to Y, $\|T\|_{X \to Y}$. $\|T\|_{[X]}$ -the norm of the operator $T \in [X]$. Unit balls in Banach function space X and associate space X' will be denoted by B_X and $B_{X'}$, respectively. $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ will be a multiindex with the coordinates $\alpha_k \in Z_+$, $\forall k = \overline{1, n};$ $\partial_i = \frac{\partial}{\partial x_i}$ will denote the differentiation operator and $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}$. For every $\xi = (\xi_1, \xi_2, ..., \xi_n)$, we assume $\xi^\alpha = (\xi_1^{\alpha_1}, \xi_2^{\alpha_2}, ..., \xi_n^{\alpha_n})$. By the m-th order diffeomorphism of two domains in R^n with sufficiently smooth boundaries, we will mean the homeomorphism of these domains, i.e. an invertible function that maps one domain into another, such that both the function and its inverse are m-time differentiable. By $C^m(\overline{\Omega})$, we denote Banach space of $m - th$ order continuous differentiable on $\overline{\Omega}$ functions with norm

$$\|f\|_{C^m(\overline{\Omega})} = \sum_{|\alpha| \le m} \|\partial^\alpha f\|_{C^m(\overline{\Omega})}, \tag{2.1}$$

where $\|g\|_{C(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |g(x)|$.

2.1. Banach function spaces

For more details on Banach function spaces, related notions, and main properties of these spaces, we refer the readers to [2, 20, 27]. Here we give some necessary information.

Let X be a Banach function space, X' be an associated space, $\rho(f)$ be a function norm of $f \in X$. Denote the corresponding associate norm by $\rho'(f)$. Also, denote by X_b the closure of the set of all simple functions in X, and by X_a the set of all functions from X with an absolutely continuous norm. The theorem below is true.

Theorem 2.1 *a) The inclusions $X_a \subset X_b \subset X$, hold.*

b) Subspaces X_a and X_b coincide if and only if for every set E of finite measure, χ_E has an absolutely continuous norm.

Let $X = X(\rho)$ be a rearrangement-invariant Banach function space over an infinite, nonatomic, totally σ-finite measure space (M, μ).

Definition 2.2 *For each $t > 0$, let E_t denote the dilation operator defined on $\Im_0(R^+, m)$ by*

$$(E_t f)(s) = f(ts), \qquad (0 < t < \infty).$$

Let

$$h_X(t) = \left\|E_{1/t}\right\|_{[\tilde{X}]}, \quad (0 < t < \infty),$$

where $\tilde{X}$ is Luxemburg presentation of X (about this concept see f.e. [2]).

Definition 2.3 *Let $X = X(\rho)$ be a rearrangement-invariant Banach function space over an infinite, nonatomic, totally σ-finite measure space (M, μ). The Boyd indices of X are the numbers α_X and β_X defined by*

$$\alpha_X = \sup_{0<t<1} \frac{\log h_X(t)}{\log t}, \ \beta_X = \sup_{1<t<\infty} \frac{\log h_X(t)}{\log t}.$$

2.2. Some assumptions

Hereinafter, we will assume the following: let $\mathbf{K} = \left\{(x_1, ..., x_n) : |x_i| < \frac{d}{2}\right\} \subset R^n$ be a cube, $X(\mathbf{K})$ be a rearrangement-invariant Banach function space defined on $\mathbf{K}$ with Lebesgue measure and the function $\|.\|_{X(K)}$. If $\Omega \subset \mathbf{K} : \overline{\Omega} \subset \mathbf{K}$ is a connected domain, by $X(\Omega)$ we will mean the space of restrictions of all functions from $X(\mathbf{K})$ to Ω with corresponding norm, i.e.

$$X(\Omega) = \left\{f \in \Im(\mathbf{K}) : \ \|f\|_{X(\Omega)} = \|f\chi_\Omega\|_{X(\mathbf{K})} < +\infty\right\},$$

with the norm $\|\cdot\|_{X(\Omega)}$.

Suppose $\Omega + \Omega \subset \mathbf{K}$. In case of relation $\Omega + \delta = \{t + \delta : t \in \Omega\}$, we consider such $\delta \in R^n$: $\overline{\Omega + \delta} \subset \mathbf{K}$. When we consider the function $\forall f \in X(\Omega)$ as a function from $X(\mathbf{K})$, we assume that $f\Big|_{K\setminus\overline{\Omega}} \equiv 0$.

By T_δ, for $\delta \in R^n : |\delta| < dist\,(\partial\Omega, \partial\mathbf{K})$, we denote the additive shift operator, defined in the following way: $(T_\delta f)(x) = f(x+\delta)$, for every $f \in X(\Omega)$. By $X_s(\Omega)$, we denote the subspace of all functions from $X(\Omega)$, which have the following property:

$$\alpha) \qquad \|T_\delta (f) - f\|_X \to 0,\ \delta \to 0, \tag{2.2}$$

where $\delta \in R^n$ is a shift vector and $T_\delta f(x) = f(x+\delta)$ is a corresponding shift operator.

Let us accept the following condition:

$$\beta) \qquad \forall E_n \to \emptyset \Rightarrow \|\chi_{E_n}\|_X \to 0. \tag{2.3}$$

Lemmas 2.4 and 2.5 below have been proved in [8, 31].

Lemma 2.4 *If $\beta)$ holds, then $X_s = X_a = X_b = \overline{C_0^\infty(\Omega)}$ (the closure is taken in topology of $X(\Omega)$).*

Lemma 2.5 *Let $X(\Omega)$ be a rearrangement-invariant Banach function space defined on the domain $\Omega \subset R^n$ and $\|\chi_E\|_E \to 0,\ |E| \to 0$. Then $\forall\varphi \in L_\infty(\Omega)$, $\forall f \in X_s(\Omega)$ implies $\varphi f \in X_s(\Omega)$.*

2.3. Convolution operator

For the function f defined on $\Omega \subset \mathbf{K}$, we define a new function f_d on R^n as follows: Firstly, we continue f by zero on the whole of $\mathbf{K}$, and then periodically on the whole of R^n, and denote

$$\|f_d(\cdot + kd)\|_{X(\mathbf{K})} = \|f_d(\cdot)\|_{X(\mathbf{K})} = \|f\|,\ \ \forall k \in Z^n.$$

Since $X(\mathbf{K})$ is a rearrangement invariant space, it follows that $f_d(\cdot)$ and $f_d(\cdot + y)$, $(\forall y \in R^n)$ are equimeasurable functions. Then we have

$$\|f_d(\cdot + y)\|_{X(\Omega)} = \|f_d\|_{X(\Omega)} = \|f\|_{X(\Omega)}, \forall y \in R^n.$$

By the convolution of the functions $f,\ g$ defined on $\Omega \subset \mathbf{K}$, $f \in X(\Omega)$, $g \in X'(\Omega)$, we mean

$$(f * g)(x) = \int f_d(x-y)\, g_d(y)\, dy, \tag{2.4}$$

denoted as $f * g$.

2.4. The singular operator

Let $\omega : [0,\infty) \to R_+$ be an infinitely differentiable function, which is equal to zero for $t \geq 1$ and takes positive values for arbitrary $t < 1$. Then the cap function is defined as follows

$$\omega_r(x) = c r^{-n} \omega\left(\frac{|x|^2}{r^2}\right), \tag{2.5}$$

where c is chosen in such a way that $\int_{R^n} \omega_r(x)\, dx = 1$.

Let f be any integrable function defined on $\Omega : \overline{\Omega} \subset \mathbf{K}$. We introduce

$$f_r(x) = (\omega_r * f)(x) = \int_\Omega \omega_r(x-y) f(y)\, dy. \tag{2.6}$$

Note that f is equal to zero on $\mathbf{K} \backslash \overline{\Omega}$, and we always consider such $r > 0$ that $\operatorname{supp} f_r \subset \mathbf{K}$.

The following statements have been proved in [8].

Lemma 2.6 *For $\forall f \in X, g \in X'$, the relations*

$$T_\delta (f * g)(x) = (T_\delta f * g)(x) = (f * T_\delta g)(x),$$

hold.

The following lemma shows that the convolution operator can be defined for arbitrary $f, g \in X$ (see [8]).

Lemma 2.7 *Let $X(\Omega)$ be a rearrangement-invariant Banach function space defined on the domain Ω. Then for arbitrary $f, g \in X(\Omega)$ there is a convolution $f * g$ and the following inequality holds:*

$$\|f * g\|_{X(\Omega)} \le \|f\|_{X(\Omega)} \|g\|_{L_1(\Omega)}.$$

This lemma directly implies the following

Theorem 2.8 *Let $X(\Omega)$ be a rearrangement-invariant Banach function space defined on the domain $\Omega \subset R^n$. Then for arbitrary $f, g \in X(\Omega)$ there is a convolution $f * g \in X(\Omega)$ and*

$$\|f * g\|_{X(\Omega)} \le C \|f\|_{X(\Omega)} \|g\|_{X(\Omega)},$$

where C is independent of f and g, i.e. the convolution operator acts continuously from X to X.

Let us consider the following singular kernel

$$k(x) = \frac{\omega(x)}{|x|^n},$$

where $\omega(x)$ is infinitely differentiable positive homogeneous function of degree zero, which satisfies

$$\int_{|x|=1} \omega(x)\, d\sigma = 0,$$

$d\sigma$ being a surface element on the unit sphere. Denote by S the corresponding singular operator

$$(Sf)(x) = (k * f)(x) = \int_\Omega k(x-y) f(y)\, dy. \tag{2.7}$$

The following theorem is true.

Theorem 2.9 *(see, [24]) For $\forall p \in (1, \infty)$, singular operator acts boundedly in $L_p(\Omega)$, i.e. $S \in [L_p(\Omega)]$.*

The following Boyd's theorem plays a very important role in obtaining many results.

Theorem 2.10 *(see, [2]) Let* $1 < p < q < \infty$, $T \in [L_p]$, $T \in [L_q]$ *and* X *be a rearrangement-invariant Banach function space with Boyd indices* α_X, β_X : $\frac{1}{q} < \alpha_X \le \beta_X < \frac{1}{p}$. *Then* $T \in [X]$.

The above two theorems have the following:

Corollary 2.11 *If* X *is a rearrangement-invariant Banach function space with Boyd indices* α_X, β_X : $0 < \alpha_X \le \beta_X < 1$, *then the singular operator* S *is bounded in* X: $S \in [X]$.

Moreover, the following statement holds.

Problem 2.12 *(see, [8]) If* X *is a rearrangement-invariant Banach function space with Boyd indices* α_X, β_X : $0 < \alpha_X \le \beta_X < 1$, *then the subspace* X_s *is invariant subspace of* S.

2.5. Substitution operator

Let $D; \Omega$ be domains in R^n and $\varphi : D \to \Omega$ be an invertible mapping preserving measurable sets. Then the substitution operator is defined as follows $\varphi : f \to f \circ \varphi$.

The following theorem has been proved in [31].

Theorem 2.13 *Let* $D; \Omega \subset \mathbf{K}$. *Then: a) Let* $\varphi : D \to \Omega$ *be a one-to-one mapping from* D *onto* Ω, *itself and its inverse transforms measurable sets to measurable sets and satisfies*

$$\exists \delta > 0 : \forall E \in (D, \mu) \Rightarrow \delta \mu(E) \le \mu(\varphi(E)) \le \delta^{-1} \mu(E). \tag{2.8}$$

Then the substitution operator φ *is an isomorphism between* $X(\Omega)$ *and* $X(D)$. *Furthermore,*

$$\delta \le \|\varphi\| \le \delta^{-1}.$$

b) If $X(D)$ *and* $X(\Omega)$ *have the property* β), *then the operator* φ *is an isomorphism between* $X(D)$ *and* $X(\Omega)$ *if and only if the relation* (2.8) *holds.*

2.6. Sobolev spaces and extension theorems

We will denote by $W_X^m(\Omega)$ and $W_{X_s}^m(\Omega)$ the following spaces of functions

$$W_X^m(\Omega) = \left\{ f \in X(\Omega) : \ \partial^p f \in X(\Omega), \forall p \in Z_+^n, \ |p| \le m \right\},$$

$$W_{X_s}^m(\Omega) = \left\{ f \in W_{X(\Omega)}^m : \ \|T_\delta f - f\|_{W_X^m(\Omega)} \to 0, \, \delta \to 0 \right\},$$

with the corresponding norm

$$\|f\|_{W_X^m(\Omega)} = \sum_{|p| \le m} \|\partial^p f\|_{X(\Omega)}. \tag{2.9}$$

The shift operator is continuous on $W_{X_s}^m(\Omega)$; therefore, $W_{X_s}^m(\Omega)$ is a closed subspace of $W_{X(\Omega)}^m$.

Subspace $\overset{0}{W}{}_{X_s}^m(\Omega)$ is defined as $\overset{0}{W}{}_{X_s}^m(\Omega) = \overline{C_0^\infty}(\Omega)$ (closure is taken in the space $W_X^m(\Omega)$).

Remark 2.14 *It is clear that every function* $u \in \overset{0}{W}{}^{m}_{X_s}(\Omega)$ *can be extended by zero on the whole of* $\mathbf{K}$.

Lemma 2.15 *(Minkowski-type inequality) (see, [8, 31]) Let* $\Omega_1 \subset \mathbf{K}$, $\Omega_2 \subset R^k$ *be domains,* $X(\Omega_1)$ *be a functional Banach space, and* $f : \Omega_1 \times \Omega_2 \to R$ *be a measurable function. If* $f(\cdot, y) \in X(\Omega_1)$ *for* $m-$ *a.e.* $y \in \Omega_2$ *and* $\|f(\cdot, y)\|_{X(\Omega_1)} \in L_1(\Omega_2)$, *then the following inequality holds:*

$$\left\| \int_{\Omega_2} f(x,y)dy \right\|_{X(\Omega_1)} \leq \int_{\Omega_2} \|f(\cdot, y)\|_{X(\Omega_1)}\, dy. \tag{2.10}$$

Remark 2.16 *Let* $f \in X(\Omega), \forall h > 0$. *Let us consider the function defined as follows:*

$$f_{i,h}(x) = \int_{x_i}^{x_i+h} f(x_1, ..., x_{i-1}, \tau, x_{i+1}, ..., x_n)\, d\tau =$$

$$= \int_0^h f(x_1, ..., x_{i-1}, x_i + \tau, x_{i+1}, ..., x_n)\, d\tau.$$

In case of rearrangement-invariant space, the following relation has been proved in [24]:

$$\|f_{i,h}\|_X \leq h \|f\|_X. \tag{2.11}$$

When $\Omega = B_r$, we will use the notations $X(r)$, $X_s(r)$, $W_X^m(r)$, $W_{X_s}^m(r)$, and in case of space $W_{X_s}^m(\Omega)$, we can introduce the equivalence norm

$$\|f\|_{W_{X_s(\Omega)}, d_\Omega} = \sum_{|p| \leq m} d_\Omega^{|p|} \|\partial^p f\|_{X(\overline{\Omega})}.$$

Theorems and corollaries below have been proved in [31].

Theorem 2.17 *Let* D; $\Omega : \overline{D}, \overline{\Omega} \subset K$ *and* $\varphi : \overline{D} \to \overline{\Omega}$ *be a* $C^{(m)}$*-class diffeomorphism. If* $u \in WX_s^m(\Omega)$, *then* $v = u \circ \varphi \in WX_s^m(D)$ *and the following inequality holds:*

$$c_1 \|u\|_{WX_s^1(\Omega)} \leq \|v\|_{WX_s^1(D)} \leq c_2 \|u\|_{WX_s^1(\Omega)}, \tag{2.12}$$

where the constants depend only on the norms of φ *and* φ^{-1}.

Theorem 2.18 *Let* Ω *be a* $C^{(m)}$*-class bounded domain and* $\overline{\Omega} \subset \Omega_1, \overline{\Omega_1} \subset \mathbf{K}$. *Then there exists a bounded extension operator* θ *acting from* $W_{X_s}^m(\Omega)$ *to* $\overset{0}{W}{}^{m}_{X_s}(\Omega_1)$ *such that the relations*

$$u \in W_{X_s}^m(\Omega),\ v = \theta u \Rightarrow (\forall x \in \Omega \Rightarrow v(x) = u(x)), \forall x \in \Omega \to (\theta v)(x) = u(x),$$

hold and

$$\exists c > 0 : \|v\|_{W_{X_s}^m(\Omega_1)} \leq c \|u\|_{W_{X_s}^m(\Omega)},\ \forall u \in W_{X_s}^m(\Omega), \tag{2.13}$$

where c *is independent of* $u(\cdot)$.

Corollary 2.19 *If there exists a bounded extension operator from* $W^m_{X_s}(\Omega)$ *to* $\overset{0}{W}{}^m_{X_s}(\Omega)$, *then* $\overline{C^\infty(\overline{\Omega})} = WX^m_s(\Omega)$ *in topology of* $W^m_{X_s}(\Omega)$.

Corollary 2.20 *If there exists a bounded extension operator from* $W^m_{X_s}(\Omega)$ *to* $W^m_{X_s}(\Omega_1)$, *then there exists a bounded extension operator from* $W^m_{X_s}(\Omega)$ *to* $\overset{0}{W}{}^m_{X_s}(\Omega_1)$.

3. Main results

Let $\Omega : \overline{\Omega} \subset \mathbf{K}$ be an arbitrary domain and all assumptions made at the beginning of Section 2.1 hold. $X(\mathbf{K})$ is a rearrangement-invariant Banach function space (with Lebesgue measure), which has the property β). Without loss of generality, we can assume $B_1 \subset \mathbf{K}$ (remember that B_1 is a unit ball in R^n), $\Omega + \Omega \subset \mathbf{K}$.

3.1. Elliptic operator of m-th order

Let L be an elliptic differential operator of m-th order

$$L = \sum_{|p| \le m} a_p(x)\, \partial^p, \tag{3.1}$$

where $p = (p_1, p_2, ..., p_n), p_k \in Z_+, \ \forall k = \overline{1,n}, a_p(\cdot) \in L_\infty(\Omega)$ are real functions. Consider the elliptic operator L_0 :

$$L_0 = \sum_{|p| = m} a^0_p \partial^p, \tag{3.2}$$

with the constant coefficients a^0_p.

By solution of the equation $Lu = f$, we will mean a strong solution, i.e. a function u which belongs to the corresponding space and satisfies a.e. the equality $Lu = f$. By $J(x)$ we denote the fundamental solution of the equation $L_0\varphi = 0$, with constant coefficients.

We will use the following classical result (see, e.g., [3, p.222]).

Theorem 3.1 *For an arbitrary elliptic operator* L_0 *of* m*-th order of the form* (3.2) *with constant coefficient, the function* $J(x)$ *with following properties can be constructed:*

i) If n *is odd or if* n *is even and* $n > m$, *then*

$$J(x) = \frac{\omega(x)}{|x|^{n-m}},$$

where $\omega(x)$ *is a homogeneous function of degree zero (i.e.* $\omega(tx) = \omega(x)$, $\forall t > 0$).

If n *is even and* $n \le m$, *then* $J(x) = q(x)\log|x| + \frac{\omega(x)}{|x|^{n-m}}$, *where* q *is a homogeneous polynomial of degree* $m - n$.

ii) The function $J(x)$ *satisfies (in a generalized sense) the equation*

$$L_0 J(x) = \delta(x),$$

where δ *is a Dirac function, i.e. for every infinitely differentiable compactly supported function* $\varphi(\cdot)$*, the following equation is true:*

$$\varphi(x) = \int [L_0\varphi(y)]\, J(x-y)\, dy = L_0 \int \varphi(y)\, J(x-y)\, dy.$$

Let us consider the elliptic operator (3.1) and assign a "tangential operator"

$$L_{x_0} = \sum_{|p|=m} a_p(x_0)\, \partial^p, \tag{3.3}$$

to it at every point $x_0 \in \Omega$. Denote by $J_{x_0}(.)$ the fundamental solution of the equation $L_{x_0}\varphi = 0$ in accordance with Theorem 3.1. The function J_{x_0} is called a parametrix for the equation $L\varphi = 0$ with singularity at the point x_0. Let

$$(S_{x_0}\varphi)(x) = \psi(x) = \int J_{x_0}(x-y)\, \varphi(y)\, dy,$$

and

$$T_{x_0} = S_{x_0}(L_{x_0} - L). \tag{3.4}$$

For every infinitely differentiable compactly supported function φ the relation

$$S_{x_0}L_{x_0} = L_{x_0}S_{x_0} = I, \text{ i.e. } S_{x_0}L_{x_0}\varphi = L_{x_0}S_{x_0}\varphi = \varphi,$$

holds (see, e.g., [3, pp. 224-225]). Some properties of these operators are established in the following.

Lemma 3.2 *Let* $X(\boldsymbol{K})$ *be a rearrangement-invariant Banach function space with Boyd indices* $\alpha_X, \beta_X \in (0;1)$ *and* L *be the* m*-th order elliptic operator on the domain* $\Omega \subset\subset K$*. Then:*

i) $L, L_{x_0} \in \left[W^m_{X(\Omega)}, X(\Omega)\right]$*. Furthermore, if* $\beta)$ *holds, then* $L, L_{x_0} \in \left[W^m_{X_s(\Omega)}, X_s(\Omega)\right]$*.*

ii) Let the property $\beta)$ *holds,* $r > 0$: $B_{2r}(x_0) \subset \Omega$ *and* $\varphi \in \overset{0}{W}{}^m_{X_s}(B_r(x_0))$*. Then there exists* $C = C(r, m, L) > 0$*, such that*

$$\|S_{x_0}\varphi\|_{W^m_X(r)} \le C\, \|\varphi\|_{X(r)}\,. \tag{3.5}$$

iii) $S_{x_0} \in [X(\Omega)]$*.*

iv) If $\beta)$ *holds, then* $S_{x_0} \in [X_s(\Omega)]$*.*

v) If $\varepsilon \in C^m\left(\overline{\Omega}\right)$*, then the multiplication operator defined as* $M_\varepsilon(u) = \varepsilon u,\ u \in W^m_X(\Omega)$*, is a continuous operator on* $W^m_{X_s}(\Omega)$*.*

Proof *i)* If $u \in W^m_X(\Omega)$, then taking into account that $a_p(x) \in L_\infty(\Omega)$, we have

$$\left\| a_p(\cdot)\frac{\partial^p u}{\partial x^p}\right\|_{X(\Omega)} \le \|a_p(\cdot)\|_{L_\infty} \left\|\frac{\partial^p u}{\partial x^p}\right\|_{X(\Omega)}, \forall\, |p| = \overline{0,m}.$$

If $\beta)$ holds, then by Lemma 2.5 the relation

$$\partial^p u \in X_s(\Omega) \Rightarrow a_p(x)\frac{\partial^p u}{\partial x^p} \in X_s(\Omega),$$

is true. Consequently, in both cases, we obtain

$$\|Lu\|_{X(\Omega)} = \left\| \sum_{|p|\leq m} a_p(x) \frac{\partial^p u}{\partial x^p} \right\|_{X(\Omega)} \leq C \|u\|_{W_X^m(\Omega)},$$

where the constant C is independent of u.

ii) Let $\varphi \in C_0^\infty(B_r(x_0))$. Since

$$(S_{x_0}\varphi)(x) = \int_{B_r} J_{x_0}(x-y)\varphi(y)\,dy, \quad (x \in B_r(x_0)),$$

for $|p| = m$ the formula

$$(\partial^p S_{x_0})\varphi(x) = \int_{B_r} \partial_x^p J_{x_0}(x-y)\varphi(y)\,dy + C'\varphi(x), \tag{3.6}$$

is true, where $C' \neq 0$ is a constant (see, [3, p. 235]). The kernel $\partial^p J_{x_0}(x)$ is singular for $|p| = m$. Applying the continuity of singular operator in $X(\mathbf{K})$ to (3.6), we have

$$\|\partial^p S_{x_0}\varphi\|_{X(r)} \leq C\|\varphi\|_{X(r)}. \tag{3.7}$$

Consider the case $|p| < m$. In this case, the kernel has a weak singularity and the relation

$$\partial^p S_{x_0}\varphi = \int_{B_r} \partial_x^p J_{x_0}(x-y)\varphi(y)\,dy,$$

holds. For $\partial^p J_{x_0}$, the estimate

$$|\partial^p J_{x_0}(x)| \leq C|x|^{m-n-|p|},$$

is true. Therefore,

$$|\partial^p S_{x_0}\varphi| \leq C\int_{B_r} |x-y|^{m-n-|p|} |\varphi(y)|\,dy = CI(x), \tag{3.8}$$

where $I(x) = \int_{B_r} |x-y|^{m-n-|p|} |\varphi(y)|\,dy$. Let

$$f(x) = \begin{cases} |x|^{m-n-|p|}, & |x| < r, \\ 0, & |x| \geq r \end{cases} \quad ; \quad g(x) = \begin{cases} |\varphi(x)|, & |x| < r, \\ 0, & |x| \geq r. \end{cases}$$

It is clear that $\operatorname{supp}(f*g) \subset B_{2r}$. Using Lemma 2.7, we have

$$\|I(\cdot)\|_{X(r)} \leq \|f*g\|_{X(\Omega)} \leq \|f\|_{L_1(R^n)} \|g\|_{X(r)}.$$

Since

$$\|f\|_{L_1(R^n)} = \|f\|_{L_1(r)} = \int_{B_r} \frac{dx}{|x|^{n-m+|p|}} = \frac{|B_1|\,2^{m-|p|}}{m-|p|} r^{m-|p|},$$

the estimate

$$||\partial^p S_{x_0}\varphi||_{X(r)} \leq C\|I(\cdot)\|_{X(r)} \leq Cr^{m-|p|}\|\varphi\|_{X(r)},$$

holds, where $C > 0$ is a constant independent of r. Hence, for $\forall p: \ |p| < m$ the relation

$$||\partial^p S_{x_0}\varphi||_{X(r)} \leq Cr^{m-|p|} \|\varphi\|_{X(r)}, \tag{3.9}$$

holds. It is clear that $Cr^{m-|p|}$ is independent of φ. From $\overline{C_0^\infty (B_r(x_0))} = \overset{0}{W}{}^m_{X_s}(\Omega)$ and the continuity of the operator $\partial^p J_{x_0}(x)\varphi(\cdot)$, it immediately follows that (3.5) is true for every $\varphi \in \overset{0}{W}{}^m_{X_s}(\Omega)$.

iii) Let $u \in X(\Omega)$ be an arbitrary function. Without loss of a generality, we consider the case $n > m$. Using Lemma 2.7, we obtain

$$\|S_{x_0}u(x)\|_{X(\Omega)} = \left\|\int J_{x_0}(x-y)\,u(y)\,dy\right\|_{X(\Omega)} \leq \left\|\int_{B_\varepsilon(x_0)} J_{x_0}(x-y)\,u(y)\,dy\right\|_{X(\Omega)} +$$

$$+\left\|\int_{\Omega\backslash B_\varepsilon(x_0)} J_{x_0}(x-y)\,u(y)\,dy\right\|_{X(\Omega)} \leq C\varepsilon^m \|u\|_{X(\varepsilon)} + C_1\|u\|_{X(\Omega\backslash B_\varepsilon(x))} \leq C'\|u\|_{X(\Omega)}.$$

iv) We will use the following evident relation

$$T_\delta((S_{x_0}u)(x)) = (S_{x_0}u)(x+\delta) = \int J_{x_0}(x+\delta-y)\,u(y)\,dy =$$

$$= \int J_{x_0}(x-(y-\delta))\,u((y-\delta)+\delta)\,dy = \int J_{x_0}(x-z)\,u(z+\delta)\,dz = S_{x_0}(T_\delta u(x)), \tag{3.10}$$

(by the convolution $f * g$ we mean (2.4)). Let $u \in X_s(\Omega)$. Then, by statement *iii*) and (3.10), we have

$$\|T_\delta(S_{x_0}u)(.) - (S_{x_0}u)(\cdot)\|_{X(\Omega)} \leq \|S_{x_0}T_\delta u - S_{x_0}u\|_{X(\Omega)} =$$

$$= \|S_{x_0}(T_\delta u - u)\| \leq C'\|T_\delta u - u\|_{X(\Omega)} \to 0,\ \delta \to 0.$$

v) Let $\varphi \in W^m_{X_s}(\Omega)$ and $\psi = M_\varepsilon\varphi$. For $k \leq m$, it is clear that

$$\frac{\partial^k}{\partial x^k}\psi(x) = \sum_{|p|\leq|k|} C_p \partial^{\tilde{p}}\varepsilon\partial^p u,$$

where $|\tilde{p}| = |k| - |p|$ and C_p are some constants. Consequently,

$$\left\|\frac{\partial^k}{\partial x^k}\psi\right\|_{X(\Omega)} \leq \max_{|p|<|k|}|C_p| \sum_{|p|\leq|k|}\left\|\partial^{\tilde{p}}\varepsilon\right\|_{C^m(\overline{\Omega})} \sum_{|p|\leq|k|}\|\partial^p u\|_{X(\Omega)} =$$

$$= const\ \|\varepsilon\|_{C^m(\overline{\Omega})}\|u\|_{W^{m-1}_X(\Omega)}.$$

The lemma is proved. □

Lemma below plays a special role in establishing the existence of the solution to the equation $Lu = f$.

Lemma 3.3 *(see [3, p. 216]) If $\varphi \in W_p^m(\Omega)$ and $supp\varphi \subset\subset \Omega$ has a compact support, then*

$$\varphi = T_{x_0}\varphi + S_{x_0}L\varphi,$$

and if $\varphi = T_{x_0}\varphi + S_{x_0}f$, then $L\varphi = f$.

Applicability of this lemma is based on the boundedness of T_{x_0}. In the sequel, this condition is fulfilled every time. It is a consequence of Main Lemma 3.5 given below.

Definition 3.4 *We will say that the operator L has the property $P_{x_0})$ if its coefficients satisfy the conditions: i) $a_p \in L_\infty(B_r(x_0))$, $|p| \le m$, for some $r > 0$; ii) $\exists r > 0$: for $|p| = m$ the coefficients $a_p(\cdot)$ coincide a.e. in $B_r(x_0)$ with some bounded and continuous function at the point x_0.*

If the condition $P_{x_0})$ is fulfilled, then the operator T_{x_0} has the property stated in the following Main Lemma 3.5, proved in [8].

Lemma 3.5 *(Main Lemma) Let $X(\boldsymbol{K})$ be a rearrangement-invariant Banach function space with Boyd indices α_X, $\beta_X \in (0;1)$ and L be the m-th order elliptic operator on domain $\Omega \subset K$, which has the property $P_{x_0})$ at the point x_0. Let $\varphi \in W^m_{X_s}(B_r(x_0))$ and φ vanish in some neighborhood of $|x - x_0| = r_0$. Then*

$$\|T_{x_0}\varphi\|_{W^m_X(B_r(x_0))} \le \sigma(r)\|\varphi\|_{W^m_X(B_r(x_0))},$$

where the function $\sigma(r) \to 0, r \to 0$, depends only on the coefficients of L and their modulus of continuity.

Before stating the next lemma, let us make some remark concerning the domain Ω.

Property c). We say that the domain Ω admits the extension of functions of the space $W^m_{X_s}(\Omega)$, if there exists a domain $\Omega' \supset \overline{\Omega}$ and a linear mapping θ of the space $W^m_{X_s}(\Omega)$ into $W^m_{X_s}(\Omega')$ such that

$$\forall x \in \Omega \Rightarrow (\theta u)(x) = u(x),$$

$$\|\theta u\|_{W^m_{X_s}(\Omega')} \le const\,\|u\|_{W^m_{X_s}(\Omega)}, \ \forall u \in W^m_{X_s}(\Omega).$$

Remark 3.6 *Theorem 2.18 shows that the domains with sufficiently smooth boundary have Property c). Moreover, if so, then a bounded extension operator from $W^m_{X_s}(\Omega)$ to $\overset{0}{W}{}^m_{X_s}(\Omega')$ exists.*

To establish our main result, we need some local estimates. For this, we introduce the following function. Let $\omega(\cdot)$ be a function defined on $[0, 1]$ such that

$$0 \le t < \frac{1}{3},\ \omega(t) = 1,\ \frac{2}{3} < t \le 1,\ \omega(t) = 0.$$

For $0 < R_1 < R_2$, the function $\xi(x)$ is defined as

$$\xi(x) = \xi(R_1, R_2, x) = \begin{cases} 1, & |x| \le R_1, \\ \omega\left(\frac{|x| - R_1}{R_2 - R_1}\right), & R_1 < |x| \le R_2. \end{cases}$$

The following lemma was proved in [9].

Lemma 3.7 *$\forall R_1 :\ 0 < R_1 < R_2$, the inequality*

$$\|\xi\|_{C^m(R_2)} \leq C\left(1 - \frac{R_1}{R_2}\right)^{-m}, \tag{3.11}$$

holds.

The following lemma is true.

Lemma 3.8 *Let $X(\boldsymbol{K})$ be a rearrangement-invariant Banach function space with Boyd indices $\alpha_X,\ \beta_X \in (0;1)$ and L be an $m-$th order elliptic operator on domain $\Omega \subset\subset K$, whose coefficients satisfy the conditions:*

$$\exists R_2 :\ a_p(\cdot) \in C\left(\overline{B(R_2)}\right),\ \forall p:\ |p| = m;\ a_p(\cdot) \in L_\infty(B(R_2)),\ \forall p:\ |p| < m.$$

Then there exists $C = (R_2, L) > 0$, depending only on R_2 and the coefficients of L, such that for $\forall u \in WX_s^m(R_2)$ the following inequality holds

$$\|u\|_{W^m_{X_s}(R_1)} \leq C\left(1 - \frac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{X(R_2)} + \|u\|_{W^{m-1}_{X_s}(R_2)}\right),\ \forall R_1 :\ 0 < R_1 < R_2. \tag{3.12}$$

Proof Consider the function $\varphi(x) = \xi(R_1,\ R_2,\ x)\ u(x)$. It is clear that

$$\forall x \in B_{R_1} \Rightarrow \varphi(x) = u(x) \Rightarrow \|u\|_{W^m_{X_s}(R_1)} \leq \|\varphi\|_{W^m_{X_s}(R_2)},$$

and $\varphi \in W^m_{X_s}(\Omega)$. Moreover, φ vanishes in some neighborhood of $|x| = R_2$. Therefore, we can apply Lemmas 3.2-*iii*), 3.3 and 3.5. Consequently, it suffices to prove the following inequality

$$\|\varphi\|_{W^m_{X_s}(R_1)} \leq C\left(1 - \frac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{X(R_2)} + \|u\|_{W^{m-1}_{X_s}(R_2)}\right),\ \forall R_1 :\ 0 < R_1 < R_2.$$

Since $\operatorname{supp}\varphi \subset B_{R_2}$, we have $\varphi = T_0\varphi + S_0 L\varphi$. By Main Lemma 3.5, $\exists R' > 0$ such that the inequality

$$\|T_0\varphi\|_{W^m_X(R_2)} \leq \frac{1}{2}\|\varphi\|_{W^m_X(R_2)},$$

holds for $\forall R_2 < R'$. We assume that R_2 is selected from this condition and $B_{2R_2} \subset \Omega$. So we immediately obtain the following inequality:

$$\|\varphi\|_{W^m_X(R_2)} \leq 2\|S_0 L\varphi\|_{W^m_X(R_2)}.$$

By Lemma 3.3 - *ii*), the inequality

$$\|S_0 L\varphi\|_{W^m_X(R_2)} \leq C'\|L\varphi\|_{X(R_2)},$$

holds for some $C' > 0$ depending only on R_2. On the other hand,

$$L\varphi = \xi Lu + M(u, \xi), \tag{3.13}$$

where

$$M(u,\xi)=\sum_{|p|<m} C_p(x)\,\partial^{\tilde{p}}\xi\partial^p u,$$

$|\tilde{p}|=m-|p|$ and $C_p(\cdot)$ is some linear combination of coefficients of the operator L. Consequently,

$$\|M(u,\xi)\|_{X(R_2)}\le \max_{|p|<m}\|C_p(x)\|\sum_{|p|<m}\left\|\partial^{\tilde{p}}\xi\right\|_{C^m(R_2)}\sum_{|p|<m}\|\partial^p u\|_{X(R_2)}=$$

$$=C\,\|\xi\|_{C^m(R_2)}\,\|u\|_{W_X^{m-1}(R_2)}.$$

As a result, we obtain

$$\|L\varphi\|_{X(R_2)}\le\|\xi\|_{C^m(R_2)}\,\|Lu\|_{X(R_2)}+\|M(u,\xi)\|_{X(R_2)}\le$$

$$\le C\,\|\xi\|_{C^m(R_2)}\left(\|Lu\|_{X(R_2)}+\|u\|_{W_{X_s}^{m-1}(R_2)}\right)\le$$

$$\le C\left(1-\frac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{X(R_2)}+\|u\|_{W_{X_s}^{m-1}(R_2)}\right), \tag{3.14}$$

where C is a constant depending only on R_2 and the coefficients $a_p(\cdot)$.

Lemma is proved. □

Remark 3.9 *The relation (3.12) holds for arbitrary domain Ω, which admits extension to the domain Ω_1 : $\overline{\Omega}\subset\Omega_1$. It can be similarly proved as the estimate (3.14):*

$$\|Lu\|_{X(\Omega)}\le\|L\varphi\|_{X(\Omega_1)}\le C\left(\|Lu\|_{X(\Omega_1)}+\|u\|_{W_{X_s}^{m-1}(\Omega_1)}\right),$$

holds, where $\varphi=u\varepsilon,\ \varepsilon\in C^\infty(\mathbf{K}),\ 0\le\varepsilon\le1,\ \varepsilon|_{\Omega}=1,\ \varepsilon|_{C\setminus\overline{\Omega_1}}=0.$

Lemma 3.10 *Let $\overline{\Omega}\subset\Omega_1:\overline{\Omega_1}\subset\Omega'\subset\mathbf{K}$ be domains in R^n, $\omega\in C_0^\infty(\mathbf{K}):\ \mathrm{supp}\omega\subset\Omega_1$, and $\forall u:\Omega'\to R$: $u|_{\overline{\Omega_1}}\in C^\infty\left(\overline{\Omega_1}\right)$. Then the function defined as*

$$\varphi=\begin{cases} u\,\omega, & \text{on } \Omega',\\ 0 & \text{on } \mathbf{K}\setminus\overline{\Omega},\end{cases}$$

satisfies:

a) belongs to $C_0^\infty(\Omega')$*;*

b) $\forall m\in N,\exists c=c(\omega):\ \|\varphi\|_{W_{X_s}^m(\Omega_1)}\le c\,\|u\|_{W_{X_s}^m(\Omega_1)}.$

Proof *a)* This statement is obvious.

b) The statement is a consequence of

$$\frac{\partial^p\varphi}{\partial x^p}=\sum_{|j|\le p}C_j(\omega)\frac{\partial^j u}{\partial x^j},$$

where $C_j(\omega)=\sum_{i=0}^{j}C_{ij}\frac{\partial^i\omega}{\partial x_i}$.

The lemma is proved. □

Lemma 3.11 *Let $X(\boldsymbol{K})$ be a rearrangement-invariant Banach function space with Boyd indices α_X, $\beta_X \in (0;1)$, which has Property c) and $\Omega \subset\subset \boldsymbol{K}$ some domain. Then $\exists C > 0$, depending only n and a constant from (3.12), and $\exists \delta > 0$, for $\forall k = \overline{1, m-1}$, and $\forall \varepsilon : 0 < \varepsilon < \delta$, the inequality*

$$\|u\|_{W^k_{X_s}(\Omega)} \leq \varepsilon \|u\|_{W^{k+1}_{X_s}(\Omega)} + C\varepsilon^{-k} \|u\|_{X(\Omega)}, \forall u \in \overset{0}{W}{}^m_{X_s}(\Omega), \tag{3.15}$$

holds.

Proof *i*) Let us first consider one-dimensional case. We will use the following formula:

$$f(t+h) - f(t) = f'(t)h + \tfrac{f''(t)}{2!}h^2 + ... + \tfrac{f^{(k)}(t)}{k!}h^k + R_k,$$

$$R_k = \tfrac{1}{k!}\int_t^{t+h} f^{(k+1)}(x)(t+h-x)\,dx,$$

where $f \in C^{(2)}(t-\delta, t+\delta)$, $t > 0$, $h : |h| < \delta$. In particular, for $k = 1$, we have

$$f(t+h) - f(t) = f'(t)h + \int_t^{t+h} (t+h-\tau)f''(\tau)\,d\tau.$$

Since

$$\left|\int_t^{t+h} (t+h-\tau) f''(\tau)\,d\tau\right| \leq h\int_t^{t+h} |f''(\tau)|\,d\tau,$$

it follows that

$$|f'(t)| \leq \int_t^{\xi} |f''(\tau)|\,d\tau + \frac{1}{h}|f(t+h) - f(t)| \leq$$

$$\leq \int_t^{t+h} |f''(\tau)|\,d\tau + \frac{1}{h}|f(t+h) - f(t)|, \tag{3.16}$$

ii) $k > 1$. Let $\varphi \in C_0^\infty(\Omega)$. Fix some x_i. Taking into account the inequalities (3.12) and (3.16), we obtain

$$\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{X(\Omega)} \leq h\left\|\frac{\partial^2\varphi}{\partial x_i^2}\right\|_{X(\Omega)} + \frac{2}{h}\|\varphi\|_{X(\Omega)}, \tag{3.17}$$

In the general case, for $p = (p_1, p_2, ..., p_n)$, if $p_i \neq 0$, the following inequalities hold

$$\|\partial^{p_1 p_2 \dots p_i \dots p_n}\varphi\|_{X(\Omega)} \leq h\left\|\partial^{p_1 \dots (p_i+1) \dots p_n}\varphi\right\|_{X(\Omega)} + \frac{2}{h}\left\|\partial^{p_1 \dots (p_i-1) \dots p_n}\varphi\right\|_{X(\Omega)}. \tag{3.18}$$

Taking into account that for the fixed multiindices $p = (p_1, p_{2,}, ..., p_n)$ the number of difference chains of the form

$$(p_1, ..., p_i - 1, ..., p_n) \to (p_1, ..., p_i, ..., p_n) \to (p_1, ..., p_i + 1, ..., p_n),$$

is equal to n, we get the validity of the following relations

$$n\|\partial^{p_1 p_2 \dots p_i \dots p_n}\varphi\|_{X(\Omega)} \leq \sum_{i=1}^n \left(h\left\|\partial^{p_1 \dots (p_i+1) \dots p_n}\varphi\right\|_{X(\Omega)} + \frac{2}{h}\left\|\partial^{p_1 \dots (p_i-1) \dots p_n}\varphi\right\|_{X(\Omega)}\right).$$

Consequently,

$$\sum_{|p|=k} \|\partial^p \varphi\|_{X(\Omega)} \leq \frac{1}{n}\left(h \sum_{|p|=k+1} \|\partial^p \varphi\|_{X(\Omega)} + \frac{2}{h} \sum_{|p|=k-1} \|\partial^p \varphi\|_{X(\Omega)} \right). \tag{3.19}$$

Assuming $\varepsilon = \frac{h}{n}$, we have

$$\sum_{|p|=k} \|\partial^p \varphi\|_{X(\Omega)} \leq \varepsilon \sum_{|p|=k+1} \|\partial^p \varphi\|_{X(\Omega)} + \frac{2}{n\varepsilon} \sum_{|p|=k-1} \|\partial^p \varphi\|_{X(\Omega)}.$$

Taking into account that

$$\|\varphi\|_{X(\Omega)} \leq \varepsilon \|\varphi\|_{X(\Omega)} + \frac{1}{4\varepsilon} \|\varphi\|_{X(\Omega)},$$

for $\forall k = \overline{1,\, m-1}$, we have

$$\|\varphi\|_{W^k_{X_s}(\Omega)} \leq \varepsilon \|\varphi\|_{W^{k+1}_{X_s}(\Omega)} + \max\left(\frac{1}{4},\, \frac{2}{n}\right) \varepsilon^{-1} \|\varphi\|_{W^{k-1}_{X_s}(\Omega)}.$$

Assume $A_k = \|\varphi\|_{W^k_{X_s}(\Omega)}$. Consequently,

$$A_1 \leq \varepsilon_1 A_2 + C{\varepsilon_1}^{-1} A_0, \quad A_2 \leq \varepsilon_2 A_3 + C\varepsilon_2^{-1} A_1,$$

where $\varepsilon_1,\, \varepsilon_2 \in (0,\, \delta)$ are sufficiently small numbers. Therefore,

$$A_2 \leq \varepsilon_2 A_3 + C\varepsilon_1 {\varepsilon_2}^{-1} A_2 + C^2 \varepsilon_1^{-1} \varepsilon_2^{-1} A_0.$$

Taking $\varepsilon_1 = \frac{\varepsilon_2}{2C}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$, we obtain

$$A_2 \leq \varepsilon A_3 + C_{2,3} \varepsilon^{-2} A_0,$$

where $\varepsilon > 0$ is a sufficiently small number, and $C_{2;3}$ is a constant depending only on n. Continuing this process, we obtain the validity of the estimate

$$A_k \leq \varepsilon A_{k+1} + C_{k;k+1} \varepsilon^{-k} A_0,$$

for $\forall k = \overline{1,\, m-1}$ and arbitrarily small $\varepsilon > 0$, where $C_{k;k+1}$ is a constant depending only n and m. Taking $C = \max\limits_k C_{k;k+1}$, we finally obtain

$$\|\varphi\|_{W^k_{X_s}(\Omega)} \leq \varepsilon \|\varphi\|_{W^{k+1}_{X_s}(\Omega)} + C\varepsilon^{-k} \|\varphi\|_{X(\Omega)}.$$

The lemma is proved. □

Lemma 3.12 *Let all conditions of Lemma 3.11 hold and* $\Omega \subset\subset \Omega_1 \subset\subset K$ *some domains. Then:*

i) $\exists C > 0$, *depending only on* n *and a constant from* (3.12), *and* $\exists \delta > 0$, *such that for* $\forall k = \overline{1, m-1}$, *and* $\forall \varepsilon :\ 0 < \varepsilon < \delta$, $\forall u \in W^m_{X_s}(\Omega)$ *the following inequality holds*

$$\|u\|_{W^k_{X_s}(\Omega)} \leq \varepsilon \|u\|_{W^{k+1}_{X_s}(\Omega_1)} + C\varepsilon^{-k} \|u\|_{X(\Omega_1)}. \tag{3.20}$$

ii) If $\Omega_0 : \overline{\Omega_0} \subset \Omega$, then the following relation holds

$$\|u\|_{W^k_{X_s}(\Omega_0)} \leq \varepsilon \|u\|_{W^{k+1}_{X_s}(\Omega)} + C\varepsilon^{-k} \|u\|_{X(\Omega_1)}. \tag{3.21}$$

Proof *i*) Let $\Omega \subset \Omega_2 \subset \Omega_1 : \overline{\Omega} \subset \Omega_2, \overline{\Omega_2} \subset \Omega_1$ and Ω_2 have Property c) with respect to the domain Ω_1. Then, by the Corollaries 2.19 and 2.20 the sets of restrictions of functions from $C_0^\infty(\Omega_1)$ on the domains $\overline{\Omega}$ and $\overline{\Omega_2}$ are dense in the spaces $W^m_{X_s}(\Omega)$ and $W^m_{X_s}(\Omega_2)$ correspondingly. For this reason it is sufficient to prove (3.20) for $\forall u \in W^m_{X_s}(\Omega_1) : u|_{\Omega_2} \in C^\infty\left(\overline{\Omega_2}\right)$.

Let $\omega \in C_0^\infty(\mathbf{K}) : 0 \leq \omega \leq 1, \omega|_\Omega \equiv 1, \mathrm{supp}\omega \subset \Omega_2$. Consider the function $\varphi = \omega u,\ u|_{\Omega_2} \in C^\infty\left(\overline{\Omega_2}\right)$. It is clear that $\varphi \in C_0^\infty(\Omega_1)$. Then, by Lemmas 3.10 and 3.11, we have

$$\|u\|_{W^k_{X_s}(\Omega)} \leq \|\varphi\|_{W^m_{X_s}(\Omega_1)} \leq \varepsilon \|\varphi\|_{W^{k+1}_s(\Omega_1)} + C\varepsilon^{-k} \|\varphi\|_{X(\Omega_1)} \leq$$

$$\leq C_1\varepsilon \|u\|_{W^{k+1}_{X_s}(\Omega_1)} + CC_2\varepsilon^{-k} \|u\|_{X(\Omega_1)},$$

where the constants are independent of u. It suffices to choose $\varepsilon := C_1\varepsilon$.

ii) In this case, it suffices to consider

$$\omega \in C_0^\infty(\mathbf{K}) :\ 0 \leq \omega \leq 1,\ \omega|_{\Omega_0} = 1,\ \omega|_{C\setminus\overline{\Omega}} = 0,$$

and $\varphi = \omega u,\ u \in C^\infty\left(\overline{\Omega}\right)$.

This statement can be proved by another way: consider $\Omega_0 \subset \Omega' \subset \Omega : \overline{\Omega_0} \subset \Omega', \overline{\Omega'} \subset \Omega$ and Ω' have the property c) with respect to the domain Ω. Consequently, the statement i) can be applied.

Lemma is proved. □

Remark 3.13 *In the last step of the proof, the inequality*

$$\exists C > 0 :\ \|\varphi\|_{W^k_X(\Omega)} \leq C \|u\|_{W^k_X(\Omega)},\ \forall k = \overline{0, m},$$

was used.

Remark 3.14 *a) The inequality (3.15) holds for arbitrary $\varepsilon > 0$ and arbitrary $u \in \overset{0}{W}{}^m_{X_s}(\Omega)$. In fact, u can be extended on R^n in the following way*

$$f_{\mathbf{K}}(x_1 + p_1 d, ..., x_n + p_n d) = f(x_1, ..., x_n),\ \ p_1, ..., p_n \in Z,$$

$\|f_{\mathbf{K}}\|_{X(\Omega)}$ is equal to the sum of the norms of the restriction $f_{\mathbf{K}}$ on the cubes which intersect Ω. Then, the inequalities (3.17), (3.18), (3.19) hold for arbitrary $h > 0$.

b) From the proof of Lemma 3.11, it follows that the inequality (3.19) holds for arbitrary $u \in C^\infty(\Omega)$ which can be extended by zero on $\mathbf{K}$.

The main result of this work is the following

Theorem 3.15 *Let $X(\boldsymbol{K})$ be a rearrangement-invariant Banach function space with Boyd indices α_X, $\beta_X \in (0;1)$, L is elliptic operator on domain $\Omega \subset\subset K$ with coefficients $a_\alpha(\cdot)$, which satisfy*

$$i)\, a_p(\cdot) \in C\left(\overline{\Omega}\right), \forall p: |p| = m; \;\; ii)\, a_p(\cdot) \in L_\infty(\Omega), \forall p: |p| < m.$$

Then for arbitrary domain $\Omega_0 \subset\subset \Omega$, there is a constant $C > 0$, which depends only on the ellipticity constant of L, of domains $\Omega_0; \Omega$, such that for $\forall u \in W^m_{X_s}(\Omega)$ the following a priori estimate holds:

$$\|u\|_{WX_s^m(\Omega_0)} \le C\left(\|Lu\|_{X(\Omega)} + \|u\|_{X(\Omega)}\right). \tag{3.22}$$

Proof We will carry out the proof in accordance with the scheme presented in the monograph [3, p. 243]. Ω_0 can be covered by a finite number of open balls B_R, for which the estimates of Lemmas 3.8 and 3.10 hold. Therefore, it suffices to prove the theorem for the case where Ω_0 and Ω_0 are concentric balls of small radius centered at the point $x_0 = 0$.

Therefore, let $R > 0$ be a sufficiently small number. We are going to prove that for $\forall r: 0 < r < R$ the following estimate holds:

$$\|u\|_{W^m_{X_s}(r)} \le C\left(1 - \frac{r}{R}\right)^{-m^2}\left(\|Lu\|_{X(R)} + \|u\|_{X(R)}\right), \tag{3.23}$$

where $C > 0$ is a constant depending on R, but independent of r and u. Denote

$$A = \sup_{0\le r\le R}\left(1 - \frac{r}{R}\right)^{m^2}\|u\|_{W^m_{X_s}(r)} \le \|u\|_{W^m_{X_s}(R)}.$$

If $u = 0$, there is nothing to prove. For this reason, suppose $u \ne 0$. Then it is clear that there exists $R_1: R/2 < R_1 < R$, such that

$$A \le 2\left(1 - \frac{R_1}{R}\right)^{m^2}\|u\|_{W^m_{X_s}(R_1)}.$$

Then for $R_2: R_1 < R_2 < R$, by Lemma 3.8, the corresponding inequality (3.12) holds, so we have

$$A \le 2\left(1 - \tfrac{R_1}{R}\right)^{m^2} C_1\left(1 - \tfrac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{X(R_2)} + \|u\|_{W^{m-1}_{X_s}(R_2)}\right) \le$$

$$\le 2C_1\left(1 - \tfrac{R_1}{R}\right)^{m^2}\left(1 - \tfrac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{X(R)} + \|u\|_{W^{m-1}_{X_s}(R_2)}\right).$$

By Lemma 3.12, for $R_3: R_2 < R_3 < R$, the relation

$$\|u\|_{W^{m-1}_{X_s}(R_2)} \le \varepsilon\|u\|_{W^m_{X_s}(R_3)} + C\varepsilon^{-m}\|u\|_{X(R_3)},$$

holds. Therefore,

$$A \le 2C_1\left(1 - \frac{R_1}{R}\right)^{m^2}\left(1 - \frac{R_1}{R_2}\right)^{-m}\left(\|Lu\|_{X(R)} + \varepsilon\|u\|_{W^m_{X_s}(R_3)} + C_2\varepsilon^{-m+1}\|u\|_{X(R_3)}\right).$$

Paying attention to the fact that

$$\left(1-\frac{R_3}{R}\right)^{m^2}\|u\|_{W^m_{X_s}(R_3)}\leq A,$$

we have

$$A\leq 2\left(1-\tfrac{R_1}{R}\right)^{m^2}C_1\left(1-\tfrac{R_1}{R_2}\right)^{-m}\|Lu\|_{X(R)}+$$

$$+2\varepsilon\left(1-\tfrac{R_1}{R_2}\right)^{m^2}C_1\left(1-\tfrac{R_1}{R_2}\right)^{-m}\left(1-\tfrac{R_3}{R}\right)^{-m^2}A+$$

$$+2C_1C_2\varepsilon^{-m+1}\left(1-\tfrac{R_1}{R}\right)^{m^2}\left(1-\tfrac{R_1}{R_2}\right)^{-m}\|u\|_{X(R)},$$

where $\varepsilon>0$ is an arbitrary small number. Let us choose ε from the relation

$$2\varepsilon C_1\left(1-\frac{R_1}{R}\right)^{m}\left(1-\frac{R_1}{R_2}\right)^{-m}\left(1-\frac{R_3}{R}\right)^{-m^2}<\frac{1}{2}.$$

Then we have

$$\frac{1}{2}A\leq 2C_1\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m}\|Lu\|_{X(R)}+$$

$$+2C_1C_2\varepsilon^{-m+1}\left(1-\frac{R_1}{R}\right)^{m^2}\left(1-\frac{R_1}{R_2}\right)^{-m}\|u\|_{X(R_3)}\leq$$

$$\leq C\left(\|Lu\|_{X(R)}+\|u\|_{X(R)}\right).$$

Taking into account the expression for A, we finally have

$$\|u\|_{W^m_{X_s}(r)}\leq C\left(1-\frac{r}{R}\right)^{-m^2}\left(\|Lu\|_{X(R)}+\|u\|_{X(R)}\right),$$

for $\forall r:\ 0<r<R$, where $C>0$ is a constant independent of r.

Theorem is proved. □

4. Some applications

In this section, we apply the above obtained theorems to some rearrangement-invariant spaces. Let $\Omega\subset R^n$ be some measurable bounded domain. Throughout this section, it is assumed that the coefficients of the elliptic operator L satisfy the following conditions

$$i)\ a_p(\cdot)\in C\left(\overline{\Omega}\right),\ \forall p:\ |p|=m;\ ii)\ a_p(\cdot)\in L_\infty(\Omega),\ \forall p:\ |p|<m.$$

4.1. The Lebesgue spaces $X=L_p(\Omega)$ $(1<p<\infty)$

The corresponding norm is

$$\|f\|_p=\left(\int_\Omega|f|^p\,dx\right)^{\frac{1}{p}}.$$

It is clear that these spaces are rearrangement-invariant Banach function spaces, and the property $\beta)$ holds. Indeed

$$|E| \to 0 \Rightarrow \|\chi_E\|_p = \left(\int_E dx\right)^{\frac{1}{p}} = (mesE)^{\frac{1}{p}} \to 0.$$

In this case, $X_s = L_p(\Omega)$. Consequently, $W^m_{X_s} = W^m_p(\Omega)$, where $W^m_p(\Omega)$ is a classical Sobolev space of m times differentiable functions. It is well known that the Boyd indices of these spaces are equal to (see, [27]) $0 < \alpha_{L_p} = \beta_{L_p} = \frac{1}{p} < 1$. Therefore, the following classical result is true.

Corollary 4.1 *Let $\Omega \subset R^n$ be a bounded domain and $\Omega_0 : \overline{\Omega_0} \subset \Omega$. Then for $\forall u \in W^m_p(\Omega)$, the a priori estimate*

$$\|u\|_{W^m_p(\Omega_0)} \leq C\left(\|Lu\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}\right),$$

holds, where the constant C depends only on the ellipticity constant of L, m, Ω, Ω_0 and the coefficients of the operator L.

4.2. The grand-Lebesgue spaces $X = L_{p)}(\Omega)$, $(1 < p < +\infty)$

The norm in these spaces is defined as follows:

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_\Omega |f|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}},\ f \in L_{p)}(\Omega).$$

It is well known that the space $L_{p)}(\Omega)$ is a nonseparable rearrangement-invariant Banach function space, and from the inclusion $L_p \subset L_{p)}$ it follows that the property $\beta)$ holds. Therefore, in this case, the relation $X_s = X_a = X_b = \overline{C_0^\infty(\Omega)}$, holds (the closure is taken in topology of $L_{p)}(\Omega)$).

The following lemma was proved in [8].

Lemma 4.2 *The Boyd indices of grand Lebesgue spaces $X = L_{p)}(\Omega)$, $1 < p < \infty$, are $\alpha_X = \beta_X = \frac{1}{p}$.*

Corresponding result for these spaces takes the following form.

Corollary 4.3 *Let $\Omega \subset R^n$ be a bounded domain and $\Omega_0 \subset \Omega$ be an arbitrary compact. Then, for $\forall u \in W^m_{p)_s}(\Omega)$, the following a priori estimate holds:*

$$\|u\|_{W^m_{\left(L_{p)}\right)_s}(\Omega_0)} \leq C\left(\|Lu\|_{L_{p)}(\Omega)} + \|u\|_{L_{p)}(\Omega)}\right),$$

where the constant C depends only on the ellipticity constant of L, m, Ω, Ω_0 and the coefficients of the operator L.

This corollary is established in the [9]. It also should be noted that the Boyd indices of $L_{p)}(\Omega)$ have been first calculated in [25] directly from the definition of these indices.

4.3. Marcinkiewicz space $X = SL_{p,\lambda}(\Omega)$

This is a Banach function space of measurable functions (in Lebesgue sense) on Ω $(1 < p < +\infty,\ 0 \le \lambda < 1)$ with the norm

$$\|f\|_{p,\lambda} = \sup_{E\subset\Omega}\left(\frac{1}{|E|^{1-\lambda}}\int_E |f|^p\,dt\right)^{\frac{1}{p}},$$

where $E \subset \Omega$ is an arbitrary measurable subset. This space is a rearrangement-invariant Banach function space. Recall that in the classical Morrey space $L_{p,\lambda}(\Omega)$ sup is taken over $B\cap\Omega$, where $B \subset R^n$ is an arbitrary ball. Unlike Marcinkiewicz space, $L_{p,\lambda}(\Omega)$ is not a rearrangement-invariant space. It is clear that the inclusion $SM_{p,\lambda}(\Omega) \subset L_{p,\lambda}(\Omega)$ is true. Let us prove that the property $\beta)$ holds in $SM_{p,\lambda}(\Omega)$. Indeed

$$\forall E \subset \Omega, \Rightarrow$$

$$\Rightarrow \left(\tfrac{1}{|E|^{1-\lambda}}\int_\Omega \chi_E^p dt\right)^{\frac{1}{p}} = \left(\tfrac{|\Omega\cap E|}{|E|^{1-\lambda}}\right)^{\frac{1}{p}} \le \left(|\Omega \bigcap E|^{\lambda}\right)^{\frac{1}{p}} \le |E|^{\frac{\lambda}{p}} \Rightarrow$$

$$\Rightarrow \|\chi_E\|_{SL_{p.\lambda}(\Omega)} \le |E|^{\frac{\lambda}{p}} \to 0,\ E \to 0\,.$$

Under the condition $0 < \lambda < 1$, $SL_{p,\lambda}(\Omega)$ is nonseparable.

Using the results of monograph [27], it can easily be proved as follows:

Lemma 4.4 *The indices of rearrangement-invariant Marcinkiewicz space* $X = SL_{p,\lambda}(\Omega)$, $1 < p < +\infty$, $0 < \lambda \le 1$, *are equal to* $\alpha_X = \beta_X = \frac{1-\lambda}{p}$.

Consequently, the following corollary is true.

Corollary 4.5 *Let* $\Omega \subset (-\pi,\pi) \subset R^1$ *and* $\Omega_0 \subset \Omega$ *be an arbitrary compact. Then, for* $\forall u \in W^m_{X_s}(\Omega)$ *with* $X = SL_{p,\lambda}(\Omega)$, *the following a priori estimate holds:*

$$\|u\|_{W^m_{\left(SL_{p,\lambda}\right)_s}(\Omega)} \le C\left(\|Lu\|_{SL_{p,\lambda}(\Omega)} + \|u\|_{SL_{p,\lambda}(\Omega)}\right),$$

where the constant C *depends only on the* L, m, Ω, Ω_0 *and the coefficients of the operator* L.

4.4. Weak-type $L^w_p(\Omega)$ **space**

$L^w_p(\Omega)$, $1 \le p < \infty$, is a space of functions

$$L^w_p(\Omega) = \left\{f \in \Im(\Omega) : \sup_{0<\lambda<+\infty} \lambda^p m_f(\lambda) < +\infty\right\},$$

where $\Im(\Omega)$ is a set of measurable functions on Ω. In [30], the space $M_r(\Omega)$, $r > 1$, of measurable functions was introduced with the norm

$$\|f\|_{M_r} = \sup_{E\subset\Omega}\frac{1}{|E|^{1-\frac{1}{r}}}\int_E |f|\,dx,$$

where sup is taken over all measurable subsets $E \subset \Omega$. The following lemma was also proved in [17, 30].

Lemma 4.6 *For arbitrary* $r > 1$*, the spaces* $L_r^w(\Omega)$ *and* $M_r(\Omega)$ *coincide* $L_r^w(\Omega) = M_r(\Omega)$, $r > 1$.

In line with our notations, $SL_{1,\lambda}(\Omega) = M_{\frac{1}{\lambda}}(\Omega)$, $0 < \lambda < 1$. Consequently, $L_{\frac{1}{\lambda}}^w(\Omega) = SL_{1,\lambda}(\Omega)$ and the following corollary is true.

Corollary 4.7 *Let* $\Omega \subset R^n$ *be a bounded domain and* $\Omega_0 \subset \Omega$ *be an arbitrary compact. Then for* $\forall u \in W_{X_s}^m(\Omega)$, $X = L_{\frac{1}{\lambda}}^w(\Omega)$, $0 < \lambda < 1$*, the following a priori estimate holds:*

$$\|u\|_{W_{X_s}^m(\Omega_0)} \leq C\left(\|Lu\|_{L_{\frac{1}{\lambda}}^w(\Omega)} + \|u\|_{L_{\frac{1}{\lambda}}^w(\Omega)}\right),$$

where the constant C *depends only of the ellipticity constant of* L*,* m, Ω, Ω_0*, and the coefficients of the operator* L.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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