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Some Qualitative Results for Nonlocal Dynamic Boundary Value Problem of Thermistor Type

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Abstract: This paper addresses the second-order nonlocal dynamic thermistor problem with two-point boundary conditions on time scales. Utilizing the fixed point theorems by Schaefer and Rus, we establish some sufficient conditions for the existence and uniqueness of solutions. Furthermore, we discuss the continuous dependence of solutions and four types of Ulam stability. We provide examples to support the applicability of our results.

Key words: Thermistor, boundary value problem, time scale, existence and uniqueness, Hyers–Ulam stability, Hyers–Ulam–Rassias stability, fixed points

1. Introduction

In 1988, Stefan Hilger introduced a new and exciting theory in his PhD thesis: the calculus on time scales. This theory unifies discrete and continuous analysis and includes quantum calculus as a special case. It has significantly transformed the study of hybrid systems, leading to numerous subsequent studies in this area. The theory of time scales allows researchers to examine both difference and differential equations within a single framework, known as dynamic equations on time scales. For further information on calculus and dynamic equations on time scales, readers can refer to excellent monographs such as [10, 11] and recent books like [1, 12, 16]. There is a great deal of research activity devoted to qualitative properties of solutions of dynamic equations on time scales. For instance, an iterative boundary value problem of second-order with mixed derivative operators is investigated for the existence and uniqueness of solutions in [4]. Iterative systems of singular multipoint boundary value problems on time scales is studied for the existence of infinitely many positive solutions in [20]. A three-species Lotka-Voltera competitive system on time scales is developed for the existence and uniform asymptotic stability of positive almost periodic solutions in [26]. The paper [27] focuses on a time-delayed SIR epidemic model with saturated treatment. In [13], the authors prove three existence results for solutions of first-order dynamic initial value problems including corresponding continuous and discrete cases. The paper [14] investigates the Ulam stability of first-order nonlinear dynamic equations on finite time scale intervals. Qualitative properties of a class of nonlinear integro-dynamic equations on time scales with local initial conditions are investigated in [15]. An iterative system of conformable fractional-order

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dynamic boundary value problems on time scales is investigated for existence of infinitely many solutions in [19]. Nonlinear dynamic equations on time scales with impulses and nonlocal initial conditions are investigated for the existence of solutions using the O'Regan fixed point theorem in the paper [30]. Qualitative and quantitative results for a class of first-order dynamic equations on time scales with nonlocal initial conditions are discussed in [31]. Two classes of dynamic initial value problems in Banach spaces are studied for existence of solutions in [32]. Even though dynamic equations on time scales is a field where advances are continuously taking place, certain topics remain less studied in the literature. The thermistor-type problem is one such topic.

A thermistor is a thermoelectric device made from a ceramic material whose electrical conductivity is highly dependent on temperature [6]. Thermistors can be used as switching devices in many electronic circuits. Applications of thermistor problems in heating processes and current flow are found in various fields of electronics and related industries [7]. In general, there are two types of thermistors. The first type has an electrical conductivity that increases with temperature, while the second type has an electrical conductivity that decreases with temperature [21, 22]. Recently, several authors have studied a thermistor type problem for fractional as well as fractional integro-differential equations. For instance, the existence, uniqueness, and stability analysis of a tempered fractional-order thermistor boundary value problems are considered in [8]. Stability results for the Hilfer fractional type thermistor problem are obtained in [33]. Qualitative analysis of fractional integro-differential equations of thermistor type is conducted in [18].

Ammi and Torres [5] utilize Guo–Krasonskiĭ fixed point theorem on cones to prove the existence of positive solutions of a nonlocal p-Laplacian thermistor problem on a time scale $\mathbb{S}$ of the type

$$-\left(\phi_p(\mathbf{w}^{\Delta}(\mathbf{p}))\right)^{\nabla} = \frac{\lambda f(\mathbf{w}(\mathbf{p}))}{\left(\displaystyle\int_0^T f(\mathbf{w}(\tau)\nabla\tau\right)^k}, \quad \mathbf{p} \in (0,T)_{\mathbb{S}},$$

subject to the boundary conditions

$$\begin{aligned} \phi_p(\mathbf{w}^{\Delta}(0)) - \beta\phi_p(\mathbf{w}^{\Delta}(\eta)) = 0, &\quad 0 < \eta < T, \\ \mathbf{w}(T) - \beta\mathbf{w}(\eta) = 0, &\quad 0 < \beta < 1, \end{aligned}$$

where $k > 1$, $\phi_p(\cdot)$ is the p-Laplacian operator defined by $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f\colon (0,T)_{\mathbb{S}} \to \mathbb{R}^+$ is a continuous function.

Song and Gao [29] used the topological degree method to obtain the result concerning the existence of solutions of nonlocal p-Laplacian thermistor problem on time scale $\mathbb{S}$ of the type

$$-(\phi_{\mathrm{p}}(\mathbf{w}^{\Delta}(\mathbf{p})))^{\nabla} = \frac{\lambda a(\mathbf{p}) f(\mathbf{w}(\mathbf{p}))}{\left(\displaystyle\int_0^T f(\mathbf{w}(\tau))\nabla\tau\right)^k}, \quad \mathbf{p} \in (0,T)_{\mathbb{S}},$$

subject to the initial conditions

$$\mathbf{w}(0) = \int_0^T g(\mathbf{q})\mathbf{w}(\mathbf{q})\nabla\mathbf{q} \text{ and } \mathbf{w}^{\Delta}(0) = A,$$

where $a\colon [0,T]_{\mathbb{S}} \to \mathbb{R}^{+}$ is ld-continuous, $g \in L^1([0,T]_{\mathbb{S}}, \mathbb{R})$, and A is a real constant. The remaining terms are as in [5].

Meinlschmidt et al. [23, 24] developed the optimal control theory of the three-dimensional thermistor problem, a fully quasilinear coupled system of a parabolic and elliptic PDEs with mixed boundary conditions. Furthermore, in [7], the authors have studied the optimal control of a nonlocal thermistor problem

$$\frac{\partial \mathbf{w}}{\partial t} - \mathcal{L}\mathbf{w} = \frac{\lambda f(\mathbf{w})}{\left(\int_{\Omega} f(\tau)d\tau\right)^2} \text{ in } Q_T = \Omega \times (0,T),$$

subject to the conditions

$$\frac{\partial \mathbf{w}}{\partial \nu} = -\beta \mathbf{w} \text{ on } S_T = \partial\Omega \times (0,T) \text{ and } \mathbf{w}(0) = \mathbf{w}_0 \text{ in } \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, $T \in \mathbb{R}^+$, $\mathcal{L}$ is the Laplacian with respect to the spacial variables, and $f\colon \Omega \to \mathbb{R}$ is a smooth function. Here ν denotes the outward unit normal and $\frac{\partial}{\partial \nu} = \nu \cdot \nabla$ is the normal derivative on $\partial\Omega$. This type of problem arises in the study of heat transfer in the resistor device where electrical conductivity is strongly dependent on the temperature.

Recently, Agarwal et al. [2], using the fixed point theorems of Banach and Schauder, established the existence and uniqueness of solutions of a fractional nonlocal thermistor problem on time scales in the conformable sense. In the literature, most of the work related to the thermistor problem has been done in the context of partial differential, fractional differential, and integral equations on $\mathbb{R}$, with very little work done in the time scale domain. Motivated by these facts, in this paper, we study the dynamic thermistor boundary value problem (DTBVP) on time scales

$$-\mathbf{w}^{\Delta^2}(\mathbf{p}) = \frac{\lambda \psi(\mathbf{w}(\mathbf{p}))}{\left[\int_{\ell}^{\mathtt{m}} \psi(\mathbf{w}(\tau))\Delta\tau\right]^2}, \quad \mathbf{p} \in [\ell, \mathtt{m}]_{\mathbb{S}}, \tag{1}$$

subject to two-point boundary conditions

$$\mathbf{w}(\ell) = \mathtt{A} \text{ and } \mathbf{w}(\sigma^2(\mathtt{m})) = \mathtt{B}, \tag{2}$$

where $\Delta^2 := \Delta\Delta$, $\lambda > 0$, $\mathtt{B} > \mathtt{A}$ are real constants, $\psi\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, and the other terms are specified in Section 2. We derive sufficient conditions for the existence and uniqueness of solutions to the problem (1)-(2) using the Schaefer fixed point theorem and the Rus fixed point theorem, respectively.

The dynamic thermistor boundary value problem (1)-(2) will provide a basic model to study the thermistor phenomena of hybrid continuous–discrete behaviour which arises in several physical processes. Our work is based on the application of the theory of fixed points, so we first obtain a corresponding integral equations and then employ suitable fixed point theorems. The main novelty of the present work is considering a new type of dynamic thermistor problem (1)-(2) and then establishing results concerning the existence of solutions.

2. Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions.

Throughout this paper, we denote by $\mathbb{S}$, a time scale which is an arbitrary nonempty closed subset of $\mathbb{R}$, with its inherited standard topology. For $\ell, m \in \mathbb{S}$, we denote by $[\ell, m]_{\mathbb{S}}$ the time scale interval which is defined as $[\ell, m]_{\mathbb{S}} := \{t \in \mathbb{S} \colon \ell \leq t \leq m\}$. For $t \in \mathbb{S}$, we define two operators, the forward jump operator $\sigma \colon \mathbb{S} \to \mathbb{S}$ by $\sigma(t) := \inf\{s \in \mathbb{S} \colon s > t\}$ and, the backward jump operator $\rho(t) := \sup\{s \in \mathbb{S} \colon s < t\}$. We make the convention that $\inf \emptyset = \sup \mathbb{S}$ and $\sup \emptyset = \inf \mathbb{S}$. The forward graininess function $\mu \colon \mathbb{S} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. With the help of jump operators, we classify points in the time scale $\mathbb{S}$ as follows. A point $t \in \mathbb{S}$ is left-dense, left-scattered, right-dense, right-scattered, dense, and isolated if $\rho(t) = t > \inf(\mathbb{S})$, $\rho(t) < t$, $\sigma(t) = t < \sup(\mathbb{S})$, $\sigma(t) > t$, $\rho(t) = t = \sigma(t)$, and $\rho(t) < t < \sigma(t)$, respectively. From a given time scale $\mathbb{S}$, we derive a new set denoted by $\mathbb{S}^{\kappa}$ as follows $\mathbb{S}^{\kappa} := \mathbb{S} \setminus \{M\}$ if M is a possible left-scattered maximum element of $\mathbb{S}$; otherwise, $\mathbb{S}^{\kappa} := \mathbb{S}$.

Definition 1 *We say a function $\mathbf{w} \colon \mathbb{S} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{S}^{\kappa}$ if there exists $\mathbf{w}^{\Delta}(t) \in \mathbb{R}$ such that for a given $\varepsilon > 0$, there is a neighborhood N of t such that*

$$\left|\mathbf{w}(\sigma(t)) - \mathbf{w}(s) - \mathbf{w}^{\Delta}(t)(\sigma(t) - s)\right| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in N.$$

Here the number $\mathbf{w}^{\Delta}(t)$ is termed as the Δ-derivative of $\mathbf{w}$ at t.

Definition 2 *We say a function $\mathbf{w} \colon \mathbb{S} \to \mathbb{R}$ is rd-continuous if it is continuous at every right-dense points in $\mathbb{S}$ and its left sided limit exists at left-dense points in $\mathbb{S}$. The set of all rd-continuous functions with domain $\mathbb{S}$ and taking values in $\mathbb{R}$ is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{S}, \mathbb{R})$.*

Definition 3 *We say $\mathbf{w} \in \mathrm{C}^2_{\mathrm{rd}}(\mathbb{S}, \mathbb{R})$ if $\mathbf{w} \colon \mathbb{S} \to \mathbb{R}$ is rd-continuous such that its first and second Δ-derivatives exist and are rd-continuous.*

Below we define the Δ-integral as a converse to the Δ-derivative.

Definition 4 *Let $\mathbf{w} \in \mathrm{C}_{\mathrm{rd}}(\mathbb{S}, \mathbb{R})$. If $\mathbf{W}^{\Delta}(t) = \mathbf{w}(t)$ for each $t \in \mathbb{S}^{\kappa}$, then we define the Δ-integral of $\mathbf{w}$ by*

$$\int_{t_0}^{t} \mathbf{w}(s) \Delta s = \mathbf{W}(t) - \mathbf{W}(t_0), \text{ where } t_0 \in \mathbb{S}.$$

Remark 1 *If $\mathbf{w}$ is Δ-differentiable at $t \in \mathbb{S}^{\kappa}$, then $\mathbf{w}$ is rd-continuous at $t \in \mathbb{S}^{\kappa}$.*

Theorem 1 (Hölder's inequality[3, Theorem 2.3.1]) *Let $\ell, m \in \mathbb{S}$. For $f, g \in \mathrm{C}_{\mathrm{rd}}([\ell, m]_{\mathbb{S}}, \mathbb{R})$, we have*

$$\int_{\ell}^{m} |f(t)g(t)| \Delta t \leq \left[\int_{\ell}^{m} |f(t)|^a \Delta t\right]^{\frac{1}{a}} \left[\int_{\ell}^{m} |g(t)|^b \Delta t\right]^{\frac{1}{b}},$$

where $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$.

Theorem 2 (Arzelà–Ascoli's theorem [34, Lemma 4]) *A subset of $\mathrm{C}(\mathbb{S}, \mathbb{R})$ which is both equicontinuous and bounded is relatively compact.*

Theorem 3 (Schaefer's fixed point theorem [25, Theorem 11.1]) *Let X be a Banach space, and let $f\colon X \to X$ be a continuous and compact mapping. Assume further that the set*

$$\Gamma = \{x \in X\colon x = \lambda f(x) \text{ for some } \lambda \in [0,1]\}$$

is bounded. Then, f has a fixed point in X.

Consider the set of real-valued functions that are defined and rd-continuous on $[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$ and denote this space by $\mathcal{X} := \mathrm{C}([\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}, \mathbb{R})$. For functions $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}$, consider the following two metrics on $\mathcal{X}$:

$$\mathrm{d}(\mathbf{w}_1, \mathbf{w}_2) := \max_{\mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}} |\mathbf{w}_1(\mathbf{p}) - \mathbf{w}_2(\mathbf{p})| \tag{3}$$

and

$$\varrho(\mathbf{w}_1, \mathbf{w}_2) := \left[\int_{\ell}^{\sigma(\mathtt{m})} |\mathbf{w}_1(\mathbf{p}) - \mathbf{w}_2(\mathbf{p})|^{\mathtt{a}} \Delta\mathbf{p}\right]^{\frac{1}{\mathtt{a}}}, \quad \mathtt{a} > 1. \tag{4}$$

For d in (3), the pair $(\mathcal{X}, \mathrm{d})$ forms a complete metric space. For ϱ in (4), the pair $(\mathcal{X}, \varrho)$ forms a metric space. The relationship between the two metrics on $\mathcal{X}$ is given by

$$\varrho(\mathbf{w}_1, \mathbf{w}_2) \le (\sigma(\mathtt{m}) - \ell)^{1/\mathtt{a}} \mathrm{d}(\mathbf{w}_1, \mathbf{w}_2) \quad \text{for all} \quad \mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}. \tag{5}$$

Theorem 4 (Rus's fixed point theorem [28, Theorem 1]) *Let $\mathcal{X}$ be a nonempty set and let d and ϱ be two metrics on $\mathcal{X}$ such that $(\mathcal{X}, \mathrm{d})$ forms a complete metric space. If the mapping $\pounds : \mathcal{X} \to \mathcal{X}$ is continuous with respect to d on $\mathcal{X}$ and*

$$\mathrm{d}(\pounds\mathbf{w}_1, \pounds\mathbf{w}_2) \le \alpha \varrho(\mathbf{w}_1, \mathbf{w}_2), \tag{6}$$

for some $\alpha > 0$ and for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}$, and

$$\varrho(\pounds\mathbf{w}_1, \pounds\mathbf{w}_2) \le \beta \varrho(\mathbf{w}_1, \mathbf{w}_2), \tag{7}$$

for some $0 < \beta < 1$ for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}$, then there is a unique $\mathbf{w}^\star \in \mathcal{X}$ such that $\pounds\mathbf{w}^\star = \mathbf{w}^\star$.

3. Existence and uniqueness

In this section, we derive sufficient conditions for the existence and uniqueness of solutions to the DTBVP (1)-(2). First, we derive following two auxiliary lemmas.

Lemma 1 *The function $\mathbf{w}$ is a solution of DTBVP (1)-(2) if and only if $\mathbf{w}$ solves the integral equation*

$$\mathbf{w}(\mathbf{p}) = \mathtt{A} + \frac{\mathtt{B} - \mathtt{A}}{\sigma^2(\mathtt{m}) - \ell}(\mathbf{p} - \ell) + \lambda \int_{\ell}^{\sigma(\mathtt{m})} \mho(\mathbf{p}, \mathbf{q}) \frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_{\ell}^{\mathtt{m}} \psi(\mathbf{w}(\tau))\Delta\tau\right]^2} \Delta\mathbf{q} \text{ for } \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}, \tag{8}$$

where

$$\mho(\mathbf{p}, \mathbf{q}) := \frac{1}{\sigma^2(\mathtt{m}) - \ell} \begin{cases} (\mathbf{p} - \ell)(\sigma^2(\mathtt{m}) - \sigma(\mathbf{q})), & \mathbf{p} \le \mathbf{q}, \\ (\sigma(\mathbf{q}) - \ell)(\sigma^2(\mathtt{m}) - \mathbf{p}), & \sigma(\mathbf{q}) \le \mathbf{p}. \end{cases} \tag{9}$$

Proof From [9], the Green function for the boundary value problem

$$\begin{cases} -\mathbf{u}^{\Delta^2}(\mathbf{p}) = 0, \\ \mathbf{w}(\ell) = 0,\ \mathbf{w}(\sigma^2(\mathtt{m})) = 0, \end{cases} \tag{10}$$

is given by (9). With the help of (9) and (10), the equivalent integral equation for (1)-(2) can be written as

$$\mathbf{w}(\mathbf{p}) = \mathrm{C}_1 + \mathrm{C}_2(\mathbf{p}-\ell) + \lambda \int_{\ell}^{\sigma(\mathtt{m})} \mho(\mathbf{p},\mathbf{q}) \frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_{\ell}^{\mathtt{m}} \psi(\mathbf{w}(\tau))\Delta\tau\right]^2} \Delta\mathbf{q}, \tag{11}$$

where C_1 and C_2 are real constants. Using the boundary conditions (2), we obtain

$$\mathrm{C}_1 = \mathtt{A} \text{ and } \mathrm{C}_2 = \frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}.$$

Plugging C_1 and C_2 into (11), we get (8). □

Lemma 2 *The function* $\mho(\mathbf{p},\mathbf{q})$ *has the following properties:*

(i) $\mho(\mathbf{p},\mathbf{q}) \geq 0$ *on* $[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}} \times [\ell,\mathtt{m}]_{\mathbb{S}}$.

(ii) $\mho(\mathbf{p},\mathbf{q}) \leq \mho(\sigma(\mathbf{q}),\mathbf{q}),\ (\mathbf{p},\mathbf{q}) \in [\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}} \times [\ell,\mathtt{m}]_{\mathbb{S}}$.

(iii) $\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\mathbf{p}_1,\mathbf{q}) - \mho(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q} \leq 4(\sigma^2(\mathtt{m})-\ell)|\mathbf{p}_1-\mathbf{p}_2|$ *for every* $\mathbf{p}_1,\mathbf{p}_2 \in [\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$.

Proof Here (i) and (ii) are evident from (9). To prove (iii), we set $\mho_1(\mathbf{p},\mathbf{q}) = \frac{(\mathbf{p}-\ell)(\sigma^2(\mathtt{m})-\sigma(\mathbf{q}))}{\sigma^2(\mathtt{m})-\ell}$ and $\mho_2(\mathbf{p},\mathbf{q}) = \frac{(\sigma(\mathbf{q})-\ell)(\sigma^2(\mathtt{m})-\mathbf{p})}{\sigma^2(\mathtt{m})-\ell}$. Let $\mathbf{p}_1,\mathbf{p}_2 \in [\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$ with $\mathbf{p}_2 \leq \mathbf{p}_1$. Then,

$$\begin{aligned} \int_{\ell}^{\mathbf{p}_2} |\mho_1(\mathbf{p}_1,\mathbf{q}) - \mho_1(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q} &= \frac{1}{\sigma^2(\mathtt{m})-\ell} \int_{\ell}^{\mathbf{p}_2} |(\mathbf{p}_1-\ell)(\sigma^2(\mathtt{m})-\sigma(\mathbf{q})) - (\mathbf{p}_2-\ell)(\sigma^2(\mathtt{m})-\sigma(\mathbf{q}))|\Delta\mathbf{q} \\ &\leq \frac{1}{\sigma^2(\mathtt{m})-\ell}|\mathbf{p}_1-\mathbf{p}_2| \int_{\ell}^{\mathbf{p}_2} |(\sigma^2(\mathtt{m})-\sigma(\mathbf{q}))|\Delta\mathbf{q} \\ &\leq \frac{1}{\sigma^2(\mathtt{m})-\ell}|\mathbf{p}_1-\mathbf{p}_2| \int_{\ell}^{\mathbf{p}_2} |(\sigma^2(\mathtt{m})-\ell)|\Delta\mathbf{q} \\ &\leq (\sigma^2(\mathtt{m})-\ell)|\mathbf{p}_1-\mathbf{p}_2|, \end{aligned}$$

$$\int_{\mathbf{p}_2}^{\mathbf{p}_1} |\mho_1(\mathbf{p}_1,\mathbf{q})-\mho_2(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q} = \frac{1}{\sigma^2(\mathtt{m})-\ell}\int_{\mathbf{p}_2}^{\mathbf{p}_1} |(\mathbf{p}_1-\ell)(\sigma^2(\mathtt{m})-\sigma(\mathbf{q}))-(\sigma(\mathbf{q})-\ell)(\sigma^2(\mathtt{m})-\mathbf{p}_2)|\Delta\mathbf{q}$$

$$\leq \frac{1}{\sigma^2(\mathtt{m})-\ell}\int_{\mathbf{p}_2}^{\mathbf{p}_1} (\mathbf{p}_1-\ell)(\sigma^2(\mathtt{m})-\sigma(\mathbf{q}))\Delta\mathbf{q}$$

$$+\frac{1}{\sigma^2(\mathtt{m})-\ell}\int_{\mathbf{p}_2}^{\mathbf{p}_1} (\sigma(\mathbf{q})-\ell)(\sigma^2(\mathtt{m})-\mathbf{p}_2)\Delta\mathbf{q}$$

$$\leq \int_{\mathbf{p}_2}^{\mathbf{p}_1}(\mathbf{p}_1-\ell)\Delta\mathbf{q}+\int_{\mathbf{p}_2}^{\mathbf{p}_1}(\sigma(\mathbf{q})-\ell)\Delta\mathbf{q}$$

$$\leq 2(\sigma^2(\mathtt{m})-\ell)|\mathbf{p}_1-\mathbf{p}_2|,$$

and

$$\int_{\mathbf{p}_1}^{\sigma(\mathtt{m})} |\mho_2(\mathbf{p}_1,\mathbf{q})-\mho_2(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q} = \frac{1}{\sigma^2(\mathtt{m})-\ell}\int_{\mathbf{p}_1}^{\sigma(\mathtt{m})} |(\sigma(\mathbf{q})-\ell)(\sigma^2(\mathtt{m})-\mathbf{p}_1)-(\sigma(\mathbf{q})-\ell)(\sigma^2(\mathtt{m})-\mathbf{p}_2)|\Delta\mathbf{q}$$

$$\leq \frac{1}{\sigma^2(\mathtt{m})-\ell}\int_{\ell}^{\sigma(\mathtt{m})} (\sigma(\mathbf{q})-\ell)|\mathbf{p}_1-\mathbf{p}_2|\Delta\mathbf{q}$$

$$\leq (\sigma^2(\mathtt{m})-\ell)|\mathbf{p}_1-\mathbf{p}_2|,$$

Thus,

$$\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\mathbf{p}_1,\mathbf{q})-\mho(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q} = \int_{\ell}^{\mathbf{p}_2} |\mho_1(\mathbf{p}_1,\mathbf{q})-\mho_1(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q}+\int_{\mathbf{p}_2}^{\mathbf{p}_1} |\mho_1(\mathbf{p}_1,\mathbf{q})-\mho_2(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q}$$

$$+\int_{\mathbf{p}_1}^{\sigma(\mathtt{m})} |\mho_2(\mathbf{p}_1,\mathbf{q})-\mho_2(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q}$$

$$\leq 4(\sigma^2(\mathtt{m})-\ell)|\mathbf{p}_1-\mathbf{p}_2|.$$

□

Throughout the rest of the paper, we use the following conditions:

($\mathtt{H}_1$) The function $\psi\colon \mathbb{R}\to\mathbb{R}$ is continuous.

($\mathtt{H}_2$) There exist positive constants $\mathtt{c}_1$ and $\mathtt{c}_2$ such that $\mathtt{c}_1 \leq \psi(\mathbf{w}) \leq \mathtt{c}_2$, $\mathbf{w}\in\mathbb{R}$.

($\mathtt{H}_3$) There exists $\mathtt{M}>0$ such that

$$\left|\frac{\psi(\mathbf{w}_1)}{\left[\int_{\ell}^{\sigma(\mathtt{m})}\psi(\mathbf{w}_1(\tau))\Delta\tau\right]^2}-\frac{\psi(\mathbf{w}_2)}{\left[\int_{\ell}^{\sigma(\mathtt{m})}\psi(\mathbf{w}_2(\tau))\Delta\tau\right]^2}\right| \leq \mathtt{M}|\mathbf{w}_1-\mathbf{w}_2|, \quad \mathbf{w}_1,\mathbf{w}_2\in\mathcal{X}.$$

($\mathtt{H}_4$) For any $\mathtt{A}>0, \mathtt{B}>0$ with $\mathtt{B}>\mathtt{A}$, (1)-(2) has a solution $\mathbf{w}$ satisfying $\mathbf{w}(\ell)=\mathtt{A}$, and $\mathbf{w}(\sigma^2(\mathtt{m}))=\mathtt{B}$.

Theorem 5 *Assume that* $(\mathtt{H}_1)$ *and* $(\mathtt{H}_2)$ *hold. Then the DTBVP* (1)-(2) *has at least one solution in* $\mathrm{C}[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$.

Proof Define a mapping $\pounds\colon \mathrm{C}[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}} \to \mathrm{C}[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$ by

$$(\pounds\mathbf{w})(\mathbf{p}) := \mathtt{A} + \frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}(\mathbf{p}-\ell) + \lambda\int_\ell^{\sigma(\mathtt{m})} \mho(\mathbf{p},\mathbf{q})\frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\Delta\mathbf{q}, \tag{12}$$

for $\mathbf{p}\in[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$. Clearly, this mapping $\pounds$ is well-defined. Now, we divide the proof into four steps.

I. $\pounds$ **is continuous:**

Let $\mathbf{w}_n$ be a sequence in $\mathrm{C}[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$ such that $\mathbf{w}_n \to \mathbf{w}$ in $\mathrm{C}[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$. Then for $\mathbf{p}\in[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$, we have

$$\begin{aligned}
|\pounds\mathbf{w}_n(\mathbf{p}) - \pounds\mathbf{w}(\mathbf{p})| &\le \lambda\int_\ell^{\sigma(\mathtt{m})} |\mho(\mathbf{p},\mathbf{q})|\left|\frac{\psi(\mathbf{w}_n(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}_n(\tau))\Delta\tau\right]^2} - \frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\right|\Delta\mathbf{q} \\
&= \lambda\int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|\left|\frac{1}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}_n(\tau))\Delta\tau\right]^2}(\psi(\mathbf{w}_n(\mathbf{q}))-\psi(\mathbf{w}(\mathbf{q})))\right. \\
&\quad \left. +\psi(\mathbf{w}(\tau))\left\{\frac{1}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}_n(\tau))\Delta\tau\right]^2} - \frac{1}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\right\}\right|\Delta\mathbf{q} \\
&\le \lambda\int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|\mathtt{I}(\mathbf{q})|\Delta\mathbf{q}.
\end{aligned}$$

Here,

$$|\mathtt{I}(\mathbf{q})| = \Bigg| \frac{1}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}_n(\tau))\Delta\tau\right]^2}(\psi(\mathbf{w}_n(\mathbf{q})) - \psi(\mathbf{w}(\mathbf{q})))$$

$$+\psi(\mathbf{w}(\tau))\left\{\frac{1}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}_n(\tau))\Delta\tau\right]^2} - \frac{1}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\right\}\Bigg|$$

$$\leq \frac{1}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}_n(\tau))\Delta\tau\right]^2}|\psi(\mathbf{w}_n(\mathbf{q})) - \psi(\mathbf{w}(\mathbf{q}))| + \frac{|\psi(\mathbf{w}(\tau))|}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}_n(\tau))\Delta\tau\right]^2\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}(\tau))\Delta\tau\right]^2}$$

$$\times \int_\ell^{\mathtt{m}} |\psi(\mathbf{w}_n(\tau)) - \psi(\mathbf{w}(\tau))|\Delta\tau \int_\ell^{\mathtt{m}} |\psi(\mathbf{w}_n(\tau)) + \psi(\mathbf{w}(\tau))|\Delta\tau$$

$$\leq \frac{1}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2}|\psi(\mathbf{w}_n(\mathbf{q})) - \psi(\mathbf{w}(\mathbf{q}))| + \frac{2\mathtt{c}_2^2}{\mathtt{c}_1^4(\mathtt{m}-\ell)^3}\int_\ell^{\mathtt{m}} |\psi(\mathbf{w}_n(\tau)) - \psi(\mathbf{w}(\tau))|\Delta\tau$$

$$\leq \left[\frac{1}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2} + \frac{2\mathtt{c}_2^2}{\mathtt{c}_1^4(\mathtt{m}-\ell)^2}\right]\|\psi(\mathbf{w}_n) - \psi(\mathbf{w})\|$$

$$= \mathtt{c}_3\|\psi(\mathbf{w}_n) - \psi(\mathbf{w})\|,$$

where $\mathtt{c}_3 := \frac{1}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2} + \frac{2\mathtt{c}_2^2}{\mathtt{c}_1^4(\mathtt{m}-\ell)^2}$. Thus,

$$\|\mathcal{L}\mathbf{w}_n(\mathbf{p}) - \mathcal{L}\mathbf{w}(\mathbf{p})\| \leq \lambda\mathtt{c}_3\int_\ell^{\sigma(\mathtt{m})} \mho(\sigma(\mathbf{q},\mathbf{q}))\|\psi(\mathbf{w}_n) - \psi(\mathbf{w})\|\Delta\mathbf{q}. \tag{13}$$

Since ψ is continuous, it follows that $\|\mathcal{L}\mathbf{w}_n(\mathbf{p}) - \mathcal{L}\mathbf{w}(\mathbf{p})\| \to 0$ as $n \to \infty$.

II. $\mathcal{L}$ maps bounded sets to bounded sets in $\mathrm{C}[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$:

Let us define $\mathbb{B}_\delta := \{\mathbf{w} \in \mathrm{C}[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}} : \|\mathbf{w}\| \leq \delta,\ \delta > 0\}$. Now, we prove that for any $\mathbf{w} \in \mathbb{B}_\delta$, $\|\mathcal{L}\mathbf{w}\| < \infty$. For $\mathbf{p} \in [\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$, we get from (12) that

$$|\mathcal{L}\mathbf{w}(\mathbf{p})| \leq \mathtt{A} + \frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}|\mathbf{p}-\ell| + \lambda\int_\ell^{\sigma(\mathtt{m})} |\mho(\mathbf{p},\mathbf{q})|\left|\frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\right|\Delta\mathbf{q}$$

$$\leq \mathtt{B} + \lambda\int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|\frac{\mathtt{c}_2}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2}\Delta\mathbf{q}.$$

Thus,

$$\|\mathcal{L}\mathbf{w}\| \leq \mathtt{B} + \frac{\lambda\mathtt{c}_2}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2}\int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|\Delta\mathbf{q} < \infty.$$

III. $\mathcal{L}$ **maps bounded sets into equicontinuous sets of** $C[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$**:**
Let $\mathbf{p}_1,\mathbf{p}_2\in[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$ such that $\mathbf{p}_2<\mathbf{p}_1$. Let $\mathbf{w}\in\mathbb{B}_\delta$. Then,

$$\begin{aligned}|(\mathcal{L}\mathbf{w})(\mathbf{p}_1)-(\mathcal{L}\mathbf{w})(\mathbf{p}_2)| &\le \frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}|\mathbf{p}_1-\mathbf{p}_2|+\lambda\int_\ell^{\sigma(\mathtt{m})}|\mho(\mathbf{p}_1,\mathbf{q})-\mho(\mathbf{p}_2,\mathbf{q})|\left|\frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\right|\Delta\mathbf{q}\\
&\le \frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}|\mathbf{p}_1-\mathbf{p}_2|+\frac{\lambda \mathtt{c}_2}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2}\int_\ell^{\sigma(\mathtt{m})}|\mho(\mathbf{p}_1,\mathbf{q})-\mho(\mathbf{p}_2,\mathbf{q})|\Delta\mathbf{q}\\
&\le \frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}|\mathbf{p}_1-\mathbf{p}_2|+\frac{4\lambda \mathtt{c}_2}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2(\sigma^2(\mathtt{m})-\ell)}|\mathbf{p}_1-\mathbf{p}_2|\\
&=\left(\frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}+\frac{4\lambda \mathtt{c}_2}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2(\sigma^2(\mathtt{m})-\ell)}\right)|\mathbf{p}_1-\mathbf{p}_2|\\
&\to 0\quad\text{as}\quad \mathbf{p}_2\to\mathbf{p}_1.\end{aligned}$$

Thus, $\mathcal{L}(\mathbb{B}_\delta)$ is equicontinuous. Now, since $\mathcal{L}(\mathbb{B}_\delta)$ is bounded and equicontinuous, in view of Theorem 2, we find that $\mathcal{L}(\mathbb{B}_\delta)$ is relatively compact. Hence, $\mathcal{L}\colon C[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}\to C[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$ is a compact mapping.
IV. A priori bound:
Define $\mathbb{P}:=\{\mathbf{w}\in C[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}\colon \mathbf{w}=\gamma\mathcal{L}\mathbf{w} \text{ for some } 0<\gamma<1\}$.
Let $\mathbf{w}\in\mathbb{P}$. Then,

$$\mathbf{w}(\mathbf{p})=\mathtt{A}\gamma+\frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}(\mathbf{p}-\ell)\gamma+\lambda\gamma\int_\ell^{\sigma(\mathtt{m})}\mho(\mathbf{p},\mathbf{q})\frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\Delta\mathbf{q},\quad \mathbf{p}\in[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}.$$

From $(\mathtt{H}_2)$, we have

$$\begin{aligned}|\mathbf{w}(\mathbf{p})| &= \left|\mathtt{A}\gamma+\frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}(\mathbf{p}-\ell)\gamma\right|+\lambda\gamma\int_\ell^{\sigma(\mathtt{m})}|\mho(\mathbf{p},\mathbf{q})|\left|\frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}}\psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\right|\Delta\mathbf{q}\\
&\le \mathtt{B}\gamma+\frac{\lambda \mathtt{c}_2\gamma}{\mathtt{c}_1^2(\mathtt{m}-\ell)^2}\int_\ell^{\sigma(\mathtt{m})}|\mho(\sigma(\mathbf{q}),\mathbf{q})|\Delta\mathbf{q}<\infty.\end{aligned}$$

This shows that the set $\mathbb{P}$ is bounded. Now, by employing Theorem 3, we find that $\mathcal{L}$ has a fixed point which is solution of the problem (1)-(2). □

Example 1 *Let $\mathbb{S}$ be a time scale and consider the following DTBVP*

$$-\mathbf{w}^{\Delta^2}(\mathbf{p})=\frac{\lambda\,(1+3|\cos(\mathbf{w})|)}{\left[\int_0^1(1+3|\cos(\mathbf{w}(\tau))|)\,\Delta\tau\right]^2},\ \lambda>0,\ \mathbf{p}\in[0,\sigma^2(1)]_{\mathbb{S}},\tag{14}$$

subject to the conditions

$$\mathbf{w}(0)=\mathtt{A}\quad \textit{and}\quad \mathbf{w}(\sigma^2(1))=\mathtt{B}.\tag{15}$$

Here, $\psi(\mathbf{w}) = 1 + 3|\cos(\mathbf{w})|$, *which is a continuous function. Therefore,* $(\mathtt{H}_1)$ *holds. Furthermore, by taking* $\mathtt{c}_1 = 1$ *and* $\mathtt{c}_2 = 4$, *we see that* $(\mathtt{H}_1)$ *is satisfied. Therefore, by Theorem 5, DTBVP (14)-(15) has at least one solution in* $\mathrm{C}[0, \sigma^2(1)]_{\mathbb{S}}$.

Next, we derive sufficient conditions for the existence of unique solution to DTBVP (1)-(2), where we employ Theorem 4.

Theorem 6 *Assume that* $(\mathtt{H}_1) - (\mathtt{H}_3)$ *hold. Furthermore, assume that there are constants* $\mathtt{a} > 1$ *and* $\mathtt{b} > 1$ *such that* $1/\mathtt{a} + 1/\mathtt{b} = 1$ *with*

$$\lambda \mathtt{M}(\sigma(\mathtt{m}) - \ell)^{1/\mathtt{a}} \left[\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}), \mathbf{q})|^{\mathtt{b}} \Delta \mathbf{q} \right]^{\frac{1}{\mathtt{b}}} < 1. \tag{16}$$

Then the DTBVP (1)-(2) has a unique nontrivial solution in $\mathcal{X}$.

Proof Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}$ and $\mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$. Then, in view of Theorem 1 and (13), we have

$$\begin{aligned} |(\mathcal{L}\mathbf{w}_1)(\mathbf{p}) - (\mathcal{L}\mathbf{w}_2)(\mathbf{p})| &= \lambda \int_{\ell}^{\sigma(\mathtt{m})} |\mho(\mathbf{p}, \mathbf{q})| \left| \frac{\psi(\mathbf{w}_1(\mathbf{q}))}{\left[\int_{\ell}^{\mathtt{m}} \psi(\mathbf{w}_1(\tau)) \Delta \tau \right]^2} - \frac{\psi(\mathbf{w}_2(\mathbf{q}))}{\left[\int_{\ell}^{\mathtt{m}} \psi(\mathbf{w}_2(\tau)) \Delta \tau \right]^2} \right| \Delta \mathbf{q} \\ &\leq \lambda \mathtt{M} \int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}), \mathbf{q})| |\mathbf{w}_1(\mathbf{q}) - \mathbf{w}_2(\mathbf{q})| \Delta \mathbf{q} \\ &\leq \lambda \mathtt{M} \left[\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}), \mathbf{q})|^{\mathtt{b}} \Delta \mathbf{q} \right]^{\frac{1}{\mathtt{b}}} \left[\int_{\ell}^{\sigma(\mathtt{m})} |\mathbf{w}_1(\mathbf{q}) - \mathbf{w}_2(\mathbf{q})|^{\mathtt{a}} \Delta \mathbf{q} \right]^{\frac{1}{\mathtt{a}}} \\ &\leq \lambda \mathtt{M} \left[\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}), \mathbf{q})|^{\mathtt{b}} \Delta \mathbf{q} \right]^{\frac{1}{\mathtt{b}}} \varrho(\mathbf{w}_1, \mathbf{w}_2). \end{aligned} \tag{17}$$

Thus, defining

$$\alpha := \lambda \mathtt{M} \left[\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}), \mathbf{q})|^{\mathtt{b}} \Delta \mathbf{q} \right]^{\frac{1}{\mathtt{b}}},$$

we see that

$$\mathrm{d}(\mathcal{L}\mathbf{w}_1, \mathcal{L}\mathbf{w}_2) \leq \alpha \varrho(\mathbf{w}_1, \mathbf{w}_2), \tag{18}$$

for some $\alpha > 0$, for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}$, and so the inequality (6) of Theorem 4 holds. Now, for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}$, we may apply (5) to (18) to obtain

$$\mathrm{d}(\mathcal{L}\mathbf{w}_1, \mathcal{L}\mathbf{w}_2) \leq \alpha \varrho(\mathbf{w}_1, \mathbf{w}_2) \leq \alpha (\sigma(\mathtt{m}) - \ell)^{1/\mathtt{a}} \mathrm{d}(\mathbf{w}_1, \mathbf{w}_2).$$

Thus, for any given $\varepsilon > 0$, we can choose $\delta := \varepsilon[\alpha(\sigma(\mathtt{m}) - \ell)^{1/\mathtt{a}}]^{-1}$ so that $\mathrm{d}(\mathcal{L}\mathbf{w}_1, \mathcal{L}\mathbf{w}_2) < \varepsilon$, whenever $\mathrm{d}(\mathbf{w}_1, \mathbf{w}_2) < \delta$. Hence, $\mathcal{L}$ is continuous on $\mathcal{X}$ with respect to metric d. Finally, we show that $\mathcal{L}$ is contractive on $\mathcal{X}$ with respect to metric ϱ. From (17), for each $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}$ consider

$$\left[\int_{\ell}^{\sigma(\mathtt{m})} |(\pounds\mathbf{w}_1)(\mathbf{p}) - (\pounds\mathbf{w}_2)(\mathbf{p})|^{\mathtt{a}} \Delta\mathbf{p}\right]^{\frac{1}{\mathtt{a}}} \leq \left[\int_{\ell}^{\sigma(\mathtt{m})} \left|\lambda\mathtt{M}\left[\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|^{\mathtt{b}}\Delta\mathbf{q}\right]^{\frac{1}{\mathtt{b}}} \varrho(\mathbf{w}_1,\mathbf{w}_2)\right|^{\mathtt{a}} \Delta\mathbf{p}\right]^{\frac{1}{\mathtt{a}}}$$

$$\leq \lambda\mathtt{M}(\sigma(\mathtt{m}) - \ell)^{1/\mathtt{a}} \left[\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|^{\mathtt{b}}\Delta\mathbf{q}\right]^{\frac{1}{\mathtt{b}}} \varrho(\mathbf{w}_1,\mathbf{w}_2).$$

That is,

$$\varrho(\pounds\mathbf{w}_1, \pounds\mathbf{w}_2) \leq \lambda\mathtt{M}(\sigma(\mathtt{m}) - \ell)^{1/\mathtt{a}} \left[\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|^{\mathtt{b}}\Delta\mathbf{q}\right]^{\frac{1}{\mathtt{b}}} \varrho(\mathbf{w}_1,\mathbf{w}_2).$$

From the assumption (16), we have

$$\varrho(\pounds\mathbf{w}_1, \pounds\mathbf{w}_2) \leq \beta\varrho(\mathbf{w}_1,\mathbf{w}_2)$$

for some $\beta < 1$ and all $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}$. Employing Theorem 4, we see that $\pounds$ has a unique fixed point in $\mathcal{X}$. Thus, the DTBVP (1)-(2) has a unique nontrivial solution. □

For the choice $\mathtt{a} = 2$ and $\mathtt{b} = 2$, we get the following Corollary from Theorem 6.

Corollary 1 *Assume that* $(\mathtt{H}_1) - (\mathtt{H}_3)$ *hold. Also, assume that*

$$\lambda\mathtt{M}\sqrt{(\sigma(\mathtt{m}) - \ell)\int_{\ell}^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|^2\Delta\mathbf{q}} < 1. \tag{19}$$

Then the DTBVP (1)-(2) has a unique nontrivial solution in $\mathcal{X}$.

Example 2 *Let*

$$\mathbb{S} := \{10^k : k \in \mathbb{Z}\} \cup \{0\}$$

and consider the following DTBVP

$$-\mathbf{w}^{\Delta^2}(\mathbf{p}) = \frac{10^{-6}(0.1 + |\sin(\mathbf{w})|)}{\left[\int_0^1 (0.1 + |\sin(\mathbf{w}(\tau))|)\Delta\tau\right]^2}, \quad \mathbf{p} \in [0, \sigma^2(1)]_{\mathbb{S}}, \tag{20}$$

subject to the conditions

$$\mathbf{w}(0) = \mathtt{A}, \quad \mathbf{w}(\sigma^2(1)) = \mathtt{B}. \tag{21}$$

Take $\ell = 0$, $\mathtt{m} = 1$, $\lambda = 10^{-6}$. *Here* $\psi(\mathbf{w}) = 0.1 + |\sin(\mathbf{w})|$. *So,* $\psi\colon \mathbb{R} \to \mathbb{R}$ *is continuous and* $(\mathtt{H}_1)$ *holds. Also,*

$$\left| \frac{\psi(\mathbf{w}_1(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}_1(\tau))\Delta\tau\right]^2} - \frac{\psi(\mathbf{w}_2(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}_2(\tau))\Delta\tau\right]^2} \right|$$

$$= \left| \frac{0.1 + |\sin(\mathbf{w}_1(\mathbf{q}))|}{\left[\int_0^1 (0.1 + |\sin(\mathbf{w}_1(\tau))|)\Delta\tau\right]^2} - \frac{0.1 + |\sin(\mathbf{w}_2)(\mathbf{q})|}{\left[\int_0^1 (0.1 + |\sin(\mathbf{w}_2(\tau))|)\Delta\tau\right]^2} \right|$$

$$\leq \frac{\left| \left(|\sin(\mathbf{w}_1(\mathbf{q}))| - |\sin(\mathbf{w}_2(\mathbf{q}))|\right) \left[\int_0^1 (0.1 + |\sin(\mathbf{w}_2(\tau))|)\Delta\tau\right]^2 \right|}{\left|\int_0^1 (0.1 + |\sin(\mathbf{w}_1(\tau))|)\Delta\tau\right|^2 \left|\int_0^1 (0.1 + |\sin(\mathbf{w}_2(\tau))|)\Delta\tau\right|^2}$$

$$+ \frac{\left| (0.1 + |\sin(\mathbf{w}_2(\mathbf{q}))|) \left(\left[\int_0^1 (0.1 + |\sin(\mathbf{w}_2(\tau))|)\Delta\tau\right]^2 - \left[\int_0^1 (0.1 + |\sin(\mathbf{w}_1(\tau))|)\Delta\tau\right]^2 \right) \right|}{\left|\int_0^1 (0.1 + |\sin(\mathbf{w}_1(\tau))|)\Delta\tau\right|^2 \left|\int_0^1 (0.1 + |\sin(\mathbf{w}_2(\tau))|)\Delta\tau\right|^2}$$

$$\leq 4\,|\sin(\mathbf{w}_1) - \sin(\mathbf{w}_2)| + 8\,|\sin(\mathbf{w}_2) - \sin(\mathbf{w}_1)|$$

$$\leq 12|\mathbf{w}_1 - \mathbf{w}_2|.$$

Hence, $(\mathtt{H}_3)$ *is satisfied with* $\mathtt{M} = 12$. *Moreover, for* $\mathbf{p} = \frac{1}{10^k}$, $k = -1, 0, 1, \ldots$, *we see that*

$$\max_{\mathbf{p} \in [0,100]_{\mathbb{S}}} \int_0^{10} |\mho(\mathbf{p}, \mathbf{q})|^2 \Delta\mathbf{q}$$

$$= \max_k \left[\lim_{n\to\infty} \int_{\frac{1}{10^n}}^{\frac{1}{10^k}} 10^2 \mathbf{q}^2 \left(10^2 - \frac{1}{10^k}\right)^2 \Delta\mathbf{q} + \int_{\frac{1}{10^k}}^{10} \frac{1}{10^{2k}} \left(10^2 - 10\mathbf{q}\right)^2 \Delta\mathbf{q} \right]$$

$$= \max_k \Big[10^3 - 10^{2-k} + \frac{2}{11}\left(10^4 - 10^{2-2k}\right)$$

$$+ \frac{1}{111}\left(10^5 - 10^{2-3k} + 11 \times 10^{5-3k} - 11 \times 10^{1-5k} - 22 \times 10^{3-4k}\right) \Big]$$

$$\approx \frac{869}{111} \times 10^6.$$

and

$$\lambda\mathtt{M}\sqrt{(\sigma(\mathtt{m}) - \ell) \int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}), \mathbf{q})|^2 \Delta\mathbf{q}} \approx 0.1061768031 < 1.$$

Therefore, all the conditions of Corollary 1 are satisfied. Hence, DTBVP (20)-(21) has a unique solution in $\mathrm{C}[0,\sigma^2(1)]_{\mathbb{S}}$*. In fact, by Lemma 1, this unique solution is given by*

$$\mathbf{w}(\mathbf{p}) = \mathtt{A} + \frac{\mathtt{B}-\mathtt{A}}{100}\mathbf{p} + 10^{-6}\int_0^{10} \mho(\mathbf{p},\mathbf{q})\frac{(0.1+|\sin(\mathbf{w})|)}{\left[\int_0^1 (0.1+|\sin(\mathbf{w}(\tau))|)\Delta\tau\right]^2}\Delta\mathbf{q},$$

where

$$\mho(\mathbf{p},\mathbf{q}) = \frac{1}{100}\begin{cases}\mathbf{p}\,(100-\sigma(\mathbf{q}))\,, & \mathbf{p}\le\mathbf{q},\\ \sigma(\mathbf{q})\,(100-\mathbf{p})\,, & \sigma(\mathbf{q})\le\mathbf{p}.\end{cases}$$

Below, we establish continuous dependence of the unique solution on ψ.

Theorem 7 *Suppose* $(\mathtt{H}_2)$ *and* $(\mathtt{H}_3)$ *hold. Then the unique solution of the DTBVP (1)-(2) obtained in Theorem 6 depends continuously on* ψ *provided*

$$\lambda < \frac{1}{\mho^\star \mathtt{M}},$$

where

$$\mho^* := \int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})|\Delta\mathbf{q}. \tag{22}$$

Proof Let ψ and $\psi^\star$ be two given functions and consider $\pounds, \pounds^\star \colon \mathrm{C}[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}} \to \mathrm{C}[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$, defined respectively, as follows.

$$(\pounds\mathbf{w})(\mathbf{p}) := \mathtt{A} + \frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}(\mathbf{p}-\ell) + \lambda\int_\ell^{\sigma(\mathtt{m})} \mho(\mathbf{p},\mathbf{q})\frac{\psi(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}(\tau))\Delta\tau\right]^2}\Delta\mathbf{q}$$

and

$$(\pounds^\star\mathbf{w})(\mathbf{p}) := \mathtt{A} + \frac{\mathtt{B}-\mathtt{A}}{\sigma^2(\mathtt{m})-\ell}(\mathbf{p}-\ell) + \lambda\int_\ell^{\sigma(\mathtt{m})} \mho(\mathbf{p},\mathbf{q})\frac{\psi^\star(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi^\star(\mathbf{w}(\tau))\Delta\tau\right]^2}\Delta\mathbf{q}.$$

Next, in view of Theorem 6, we see that there exist two unique functions $\mathbf{w},\mathbf{w}^\star \in \mathrm{C}[\ell,\sigma^2(\mathtt{m})]_{\mathbb{S}}$ such that

$\mathbf{w} = \mathcal{L}\mathbf{w}$ and $\mathbf{w}^\star = \mathcal{L}^\star\mathbf{w}^\star$. Then,

$$\begin{aligned}
\|\mathcal{L}\mathbf{w}^\star - \mathcal{L}^\star\mathbf{w}^\star\| &\leq \lambda \int_\ell^{\sigma(\mathtt{m})} |\mho(\mathbf{p},\mathbf{q})| \left| \frac{\psi(\mathbf{w}^\star(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}^\star(\tau))\Delta\tau\right]^2} - \frac{\psi^\star(\mathbf{w}^\star(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi^\star(\mathbf{w}^\star(\tau))\Delta\tau\right]^2} \right| \Delta\mathbf{q} \\
&\leq \lambda \int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})| \left\{ \left| \frac{\psi(\mathbf{w}^\star(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}^\star(\tau))\Delta\tau\right]^2} - \frac{\psi^\star(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi^\star(\mathbf{w}(\tau))\Delta\tau\right]^2} \right| \right. \\
&\quad \left. + \left| \frac{\psi^\star(\mathbf{w}(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi^\star(\mathbf{w}(\tau))\Delta\tau\right]^2} - \frac{\psi^\star(\mathbf{w}^\star(\mathbf{q}))}{\left[\int_\ell^{\mathtt{m}} \psi^\star(\mathbf{w}^\star(\tau))\Delta\tau\right]^2} \right| \right\} \Delta\mathbf{q} \\
&\leq \lambda \int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})| \left\{ \left| \left[\int_\ell^{\mathtt{m}} \psi^\star(\mathbf{w}^\star(\tau))\Delta\tau\right]^2 [\psi(\mathbf{w}^\star(\mathbf{q})) - \psi^\star(\mathbf{w}^\star(\mathbf{q}))] \right| \right. \\
&\quad \left. + \left| \psi^\star(\mathbf{w}^\star) \left[\left[\int_\ell^{\mathtt{m}} \psi^\star(\mathbf{w}^\star(\tau))\Delta\tau\right]^2 - \left[\int_\ell^{\mathtt{m}} \psi(\mathbf{w}^\star(\tau))\Delta\tau\right]^2 \right] \right| + \mathtt{M}\|\mathbf{w} - \mathbf{w}^\star\| \right\} \Delta\mathbf{q} \\
&\leq \lambda \int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q})| \left\{ \left[\mathtt{c}_2^2(\mathtt{m}-\ell)^2 |\psi(\mathbf{w}^\star) - \psi^\star(\mathbf{w}^\star)| \right.\right. \\
&\quad \left.\left. + 2\mathtt{c}_2(\mathtt{m}-\ell) \int_\ell^{\mathtt{m}} |\psi(\mathbf{w}^\star(\tau)) - \psi^\star(\mathbf{w}^\star(\tau))|\Delta\tau \right] + \mathtt{M}\|\mathbf{w} - \mathbf{w}^\star\| \right\} \Delta\mathbf{q} \\
&\leq \lambda \int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}),\mathbf{q}) \left\{ \left[\mathtt{c}_2^2(\mathtt{m}-\ell)^2 + 2\mathtt{c}_2(\mathtt{m}-\ell)^2 \right] \|\psi - \psi^\star\| + \mathtt{M}\|\mathbf{w} - \mathbf{w}^\star\| \right\} \Delta\mathbf{q} \\
&= \lambda \mho^\star [\mathtt{c}_2^2(\mathtt{m}-\ell)^2 + 2\mathtt{c}_2(\mathtt{m}-\ell)^2] \|\psi - \psi^\star\| + \lambda \mho^\star \mathtt{M}\|\mathbf{w} - \mathbf{w}^\star\|.
\end{aligned}$$

Therefore, from (13) and the above inequality, we get

$$\begin{aligned}
\|\mathbf{w} - \mathbf{w}^\star\| &\leq \|\mathcal{L}\mathbf{w} - \mathcal{L}\mathbf{w}^\star\| + \|\mathcal{L}\mathbf{w} - \mathcal{L}^\star\mathbf{w}^\star\| \\
&\leq \lambda \mho^\star \mathtt{c}_3 \|\psi - \psi^\star\| + \lambda \mho^\star [\mathtt{c}_2^2(\mathtt{m}-\ell) + 2\mathtt{c}_2(\mathtt{m}-\ell)^2] \|\psi - \psi^\star\| + \lambda \mho^\star \mathtt{M}\|\mathbf{w} - \mathbf{w}^\star\|.
\end{aligned}$$

That is,

$$\|\mathbf{w} - \mathbf{w}^\star\| \leq \frac{\lambda \mho^\star [\mathtt{c}_2^2(\mathtt{m}-\ell)^2 + 2\mathtt{c}_2(\mathtt{m}-\ell)^2 + \mathtt{c}_3]}{1 - \lambda \mho^\star \mathtt{M}} \|\psi - \psi^\star\|.$$

This completes the proof. $\square$

4. The Ulam stability

In this section, we shall define and investigate four types of Ulam stability of (1). For a function $\mathbf{z} \in \mathrm{C}^2_{\mathrm{rd}}([\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}, \mathbb{R})$, we denote

$$\mathcal{U}_{\mathbf{z}}(\mathbf{p}) = \frac{\psi(\mathbf{z}(\mathbf{p}))}{\left[\int_{\ell}^{\mathtt{m}} \psi(\mathbf{z}(\tau))\Delta\tau\right]^2}$$

and

$$\mathrm{h}_{\mathbf{z}}(\mathbf{p}) := \mathbf{z}^{\Delta^2}(\mathbf{p}) + \lambda\mathcal{U}_{\mathbf{z}}(\mathbf{p}).$$

First, we recall the concept of Hyers–Ulam stability, see [17].

Definition 5 *We say that DTBVP (1)-(2) has the Hyers–Ulam stability provided there exists $\mathfrak{N} > 0$ with the following property. For any $\varepsilon > 0$, if $\mathbf{z} \in \mathrm{C}^2_{\mathrm{rd}}([\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}, \mathbb{R})$ is such that*

$$|\mathrm{h}_{\mathbf{z}}(\mathbf{p})| \le \varepsilon, \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]^{\kappa^2}_{\mathbb{S}}, \tag{23}$$

then there exists a solution $\mathbf{w} \colon [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}} \to \mathbb{R}$ of (1)-(2) such that

$$|\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| \le \mathfrak{N}\varepsilon, \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}.$$

Here, the constant $\mathfrak{N} > 0$ is called HUS constant.

Definition 6 *We say that DTBVP (1)-(2) has the generalized Hyers–Ulam stability provided there exists $\theta_\psi \in \mathrm{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_\psi(0) = 0$ with the following property. For any $\varepsilon > 0$, if $\mathbf{z} \in \mathrm{C}^2_{\mathrm{rd}}([\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}, \mathbb{R})$ is such that*

$$|\mathrm{h}_{\mathbf{z}}(\mathbf{p})| \le \varepsilon, \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]^{\kappa^2}_{\mathbb{S}}, \tag{24}$$

then there exists a solution $\mathbf{w} \colon [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}} \to \mathbb{R}$ of (1)-(2) such that

$$|\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| \le \theta_\psi(\varepsilon), \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}.$$

Here, the function θ_ψ is called GHUS function.

Definition 7 *Let $\mathcal{P}$ be a family of positive rd-continuous functions defined on $[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$. We say that DTBVP (1)-(2) has the Hyers–Ulam–Rassias stability of type $\mathcal{P}$ provided there exists $\mathfrak{R} > 0$ with the following property. For any $\mathbf{f} \in \mathcal{P}$ and $\varepsilon > 0$, if $\mathbf{z} \in \mathrm{C}^2_{\mathrm{rd}}([\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}, \mathbb{R})$ is such that*

$$|\mathrm{h}_{\mathbf{z}}(\mathbf{p})| \le \varepsilon\mathbf{f}(\mathbf{p}), \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]^{\kappa^2}_{\mathbb{S}}, \tag{25}$$

then there exists a solution $\mathbf{w} \colon [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}} \to \mathbb{R}$ of (1)–(2) such that

$$|\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| \le \mathfrak{R}\varepsilon\mathbf{f}(\mathbf{p}), \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}.$$

Here, the constant $\mathfrak{R} > 0$ is called as HURS constant.

Definition 8 *Let $\mathcal{P}$ be a family of positive rd-continuous functions defined on $[\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$. We say that DTBVP* (1)-(2) *has the generalized Hyers–Ulam–Rassias stability of type $\mathcal{P}$ provided there exists $\mathfrak{R} > 0$ with the following property. For any $\mathbf{f} \in \mathcal{P}$, if $\mathbf{z} \in \mathrm{C}^2_{\mathrm{rd}}([\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}, \mathbb{R})$ is such that*

$$|\mathrm{h}_{\mathbf{z}}(\mathbf{p})| \leq \mathbf{f}(\mathbf{p}), \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]^{\kappa^2}_{\mathbb{S}}, \tag{26}$$

then there exists a solution $\mathbf{w}\colon [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}} \to \mathbb{R}$ of (1)–(2) *such that*

$$|\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| \leq \mathfrak{R}\mathbf{f}(\mathbf{p}), \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}.$$

Here, the constant $\mathfrak{R} > 0$ is called GHURS constant.

The next theorem establishes sufficient conditions for the Hyers–Ulam–Rassias stability of DTBVP (1)-(2).

Theorem 8 *Assume that $(\mathtt{H}_1) - (\mathtt{H}_4)$ hold. If $\lambda\mathtt{M}\mho^\star < 1$, then the DTBVP* (1)-(2) *has the Hyers–Ulam–Rassias stability of type $\mathcal{P}$ with HURS constant $\dfrac{\varepsilon\mho^\star}{1 - \lambda\mathtt{M}\mho^\star}$, where $\mho^\star$ is defined in* (22).

Proof Let $\varepsilon > 0$ and $\mathbf{z} \in \mathrm{C}^2_{\mathrm{rd}}([\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}, \mathbb{R})$ be such that (25) holds. Then, $\mathbf{z}$ satisfies

$$-\mathbf{z}^{\Delta^2}(\mathbf{p}) = \lambda\mathcal{U}_{\mathbf{z}}(\mathbf{p}) - \mathrm{h}_{\mathbf{z}}(\mathbf{p}), \quad \mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}.$$

In view of Lemma 1, this $\mathbf{z}$ satisfies

$$\mathbf{z}(\mathbf{p}) = \mathtt{A} + \frac{\mathtt{B} - \mathtt{A}}{\sigma^2(\mathtt{m}) - \ell}(\mathbf{p} - \ell) + \int_\ell^{\sigma(\mathtt{m})} \mho(\mathbf{p}, \mathbf{q}) \big[\lambda\mathcal{U}_{\mathbf{z}}(\mathbf{q}) - \mathrm{h}_{\mathbf{z}}(\mathbf{q})\big] \Delta\mathbf{q}.$$

Also, by hypothesis $(\mathtt{H}_4)$, we see that there exists a solution $\mathbf{w}$ of (1)-(2) with $\mathbf{w}(\ell) = \mathtt{A}$ and $\mathbf{w}(\sigma^2(\mathtt{m})) = \mathtt{B}$. Again, in view of Lemma 1, we obtain

$$\mathbf{w}(\mathbf{p}) = \mathtt{A} + \frac{\mathtt{B} - \mathtt{A}}{\sigma^2(\mathtt{m}) - \ell}(\mathbf{p} - \ell) + \lambda \int_\ell^{\sigma(\mathtt{m})} \mho(\mathbf{p}, \mathbf{q}) \mathcal{U}_{\mathbf{w}}(\mathbf{q}) \Delta\mathbf{q}.$$

Then, for $\mathbf{p} \in [\ell, \sigma^2(\mathtt{m})]_{\mathbb{S}}$,

$$\begin{aligned}
|\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| &\leq \left| - \int_\ell^{\sigma(\mathtt{m})} \mho(\mathbf{p}, \mathbf{q}) \mathrm{h}_{\mathbf{z}}(\mathbf{q}) \Delta\mathbf{q} \right| + \left| \lambda \int_\ell^{\sigma(\mathtt{m})} \mho(\mathbf{p}, \mathbf{q}) \big[\mathcal{U}_{\mathbf{z}}(\mathbf{q}) - \mathcal{U}_{\mathbf{w}}(\mathbf{q})\big] \Delta\mathbf{q} \right| \\
&\overset{(\mathtt{H}_3)}{\leq} \int_\ell^{\sigma(\mathtt{m})} |\mho(\mathbf{p}, \mathbf{q})| |\mathrm{h}_{\mathbf{z}}(\mathbf{q})| \Delta\mathbf{q} + \lambda \int_\ell^{\sigma(\mathtt{m})} |\mho(\mathbf{p}, \mathbf{q})| |\mathcal{U}_{\mathbf{z}}(\mathbf{q}) - \mathcal{U}_{\mathbf{w}}(\mathbf{q})| \Delta\mathbf{q} \\
&\overset{(25)}{\leq} \varepsilon\mathbf{f}(\mathbf{P}) \int_\ell^{\sigma(\mathtt{m})} |\mho(\mathbf{p}, \mathbf{q})| \Delta\mathbf{q} + \lambda\mathtt{M} \int_\ell^{\sigma(\mathtt{m})} |\mho(\mathbf{p}, \mathbf{q})| |\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| \Delta\mathbf{q} \\
&\leq \varepsilon\mathbf{f}(\mathbf{P}) \int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}), \mathbf{q})| \Delta\mathbf{q} + \lambda\mathtt{M} \int_\ell^{\sigma(\mathtt{m})} |\mho(\sigma(\mathbf{q}), \mathbf{q})| |\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| \Delta\mathbf{q} \\
&\overset{(22)}{\leq} \varepsilon\mathbf{f}(\mathbf{P})\mho^\star + \lambda\mathtt{M}\mho^\star |\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})|.
\end{aligned}$$

This yields

$$|\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| \leq \varepsilon \mathbf{f}(\mathbf{P}) \frac{\mho^\star}{1 - \lambda \mathtt{M} \mho^\star}.$$

Thus, (1)-(2) has the Hyers–Ulam–Rassias stability with HURS constant $\mathfrak{N} := \dfrac{\mho^\star}{1 - \lambda \mathtt{M} \mho^\star}$. □

Based on the above theorem, we now obtain the results for other Ulam stability of DTBVP (1)-(2).

Corollary 2 *Assume that* $(\mathtt{H}_1) - (\mathtt{H}_4)$ *hold. If* $\lambda \mathtt{M} \mho^\star < 1$*, then the DTBVP* (1)-(2) *has generalized Hyers–Ulam–Rassias stability of type* $\mathcal{P}$ *with GHURS constant* $\dfrac{\mho^\star}{1 - \lambda \mathtt{M} \mho^\star}$.

Proof In the proof of Theorem 8, if we take $\varepsilon = 1$, then we obtain

$$|\mathbf{z}(\mathbf{p}) - \mathbf{w}(\mathbf{p})| \leq \mathbf{f}(\mathbf{P}) \frac{\mho^\star}{1 - \lambda \mathtt{M} \mho^\star}.$$

This shows that the DTBVP (1)-(2) has the generalized Hyers–Ulam–Rassias stability of type $\mathcal{P}$ with GHURS constant $\dfrac{\mho^\star}{1 - \lambda \mathtt{M} \mho^\star}$. □

Corollary 3 *Assume that* $(\mathtt{H}_1) - (\mathtt{H}_4)$ *hold. If* $\lambda \mathtt{M} \mho^\star < 1$*, then the DTBVP* (1)-(2) *has the Hyers–Ulam stability with HUS constant* $\dfrac{\mho^\star}{1 - \lambda \mathtt{M} \mho^\star}$.

Proof The proof follows easily by taking $\mathbf{f}(\mathbf{p}) \equiv 1$ in the proof of Theorem 8. □

Corollary 4 *Assume that* $(\mathtt{H}_1) - (\mathtt{H}_4)$ *hold. If* $\lambda \mathtt{M} \mho^\star < 1$*, then the DTBVP* (1)-(2) *has the generalized Hyers–Ulam stability.*

Proof The proof follows easily by taking $\theta_\psi(\varepsilon) = \dfrac{\varepsilon \mho^\star}{1 - \lambda \mathtt{M} \mho^\star}$ in the proof of Theorem 8. □

Example 3 *Let*

$$\mathbb{S} := \{10^k : k \in \mathbb{Z}\} \cup \{0\}$$

and consider the DTBVP

$$-\mathbf{w}^{\Delta^2}(\mathbf{p}) = \frac{10^{-6}(0.1 + |\sin(\mathbf{w})|)}{\left[\int_0^1 (0.1 + |\sin(\mathbf{w}(\tau))|)\Delta\tau\right]^2}, \quad \mathbf{p} \in [0, \sigma^2(1)]_{\mathbb{S}}, \tag{27}$$

subject to the conditions

$$\mathbf{w}(0) = \mathtt{A} \text{ and } \mathbf{w}(\sigma^2(1)) = \mathtt{B}. \tag{28}$$

Here, $\ell = 0$*,* $\mathtt{m} = 1$*, and* $\lambda = 10^{-6}$*. Then, as in Example 2, we find that* $(\mathtt{H}_1) - (\mathtt{H}_3)$ *hold. Therefore, by Corollary 1, DTBVP* (27)–(28) *has a unique solution in* $\mathrm{C}[0, \sigma^2(1)]_{\mathbb{S}}$*. In fact, by Lemma 1, this unique solution* *is given by*

$$\mathbf{w}(\mathbf{p}) = \mathtt{A} + \frac{\mathtt{B}-\mathtt{A}}{100}\mathbf{p} + 10^{-6}\int_0^{10} \mho(\mathbf{p},\mathbf{q}) \frac{(0.1+|\sin(\mathbf{w})|)}{\left[\int_0^1 (0.1+|\sin(\mathbf{w}(\tau))|)\Delta\tau\right]^2}\Delta\mathbf{q},$$

where

$$\mho(\mathbf{p},\mathbf{q}) = \frac{1}{100}\begin{cases} \mathbf{p}\,(100-\sigma(\mathbf{q}))\,, & \mathbf{p}\le\mathbf{q},\\ \sigma(\mathbf{q})\,(100-\mathbf{p})\,, & \sigma(\mathbf{q})\le\mathbf{p}. \end{cases}$$

Since

$$\begin{aligned}\mho^{\star} &= \max_{\mathbf{p}\in[0,100]_{\mathbb{S}}}\int_0^{10} |\mho(\mathbf{p},\mathbf{q})|\Delta\mathbf{q}\\ &= \max_k\left[\lim_{n\to\infty}\int_{\frac{1}{10^n}}^{\frac{1}{10^k}} 10\mathbf{q}\left(10^2-\frac{1}{10^k}\right)\Delta\mathbf{q} + \int_{\frac{1}{10^k}}^{10}\frac{1}{10^k}\left(10^2-10\mathbf{q}\right)\Delta\mathbf{q}\right]\\ &= \max_k\left[\frac{1}{11\times10^{2k-2}} + \frac{9}{10^k}\left(110+\frac{10^k-1}{9\times10^{k-1}}-\frac{10^{2k+4}-1}{99\times10^{2k+1}}\right)\right]\\ &\approx \frac{991}{11}\times10^2,\end{aligned}$$

it follows that $\lambda\mathtt{M}\mho^{\star}<1$. *Furthermore, if* $\mathbf{z}\in\mathrm{C}^2_{\mathrm{rd}}([0,\sigma(1)]_{\mathbb{S}},\mathbb{R})$ *satisfies*

$$\left|\mathbf{z}^{\Delta^2}(\mathbf{p}) + \frac{10^{-6}(0.1+|\sin(\mathbf{z})|)}{\left[\int_0^1 (0.1+|\sin(\mathbf{z}(\tau))|)\,\Delta\tau\right]^2}\right| \le \varepsilon,$$

then by Corollary 3, there exists a solution $\mathbf{w}$ *of DTBVP* (27)-(28) *satisfying*

$$|\mathbf{z}(\mathbf{p})-\mathbf{w}(\mathbf{p})| \le \frac{\mho^{\star}}{1-\lambda\mathtt{M}\mho^{\star}}\varepsilon, \quad \mathbf{p}\in[0,\sigma^2(1)]_{\mathbb{S}}.$$

Now, from the above data, we find that $\dfrac{\mho^{\star}}{1-\lambda\mathtt{M}\mho^{\star}} = 10101.11306$. *Hence, DTBVP* (27)-(28) *has the Hyers–Ulam stability with HUS constant* $\mathfrak{N} = 10101.11306$.

5. Conclusion

We considered a boundary value problem of thermistor type with two-point boundary conditions in the time scale domain. We established sufficient conditions for the existence of at least one solution employing the Schaefer fixed point theorem. The existence of unique nontrivial solution is obtained by employing the Rus fixed point theorem involving two metrics. Furthermore, we discussed the continuous dependence of solutions on functions. The main novelty of the present work is considering a new dynamic boundary value problem (1)-(2) of thermistor type and then establishing qualitative results its of solutions. The results obtained in the present work are new in the literature and we believe that they would be helpful for further analysis. We believe that the other qualitative properties like stability, oscillations and nonoscillations of solutions would be an interesting subject for future work.

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