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The flow-geodesic curvature and the flow-evolute of spherical curves

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Abstract: We introduce and study a deformation of the geodesic curvature for a given spherical curve $\gamma$. Also, we define a new type of evolute and two Fermi-Walker type derivatives for $\gamma$. Some concrete examples are detailed with a special attention towards space curves with a constant torsion.

Key words: Spherical curve, flow-geodesic curvature, flow-evolute

1. Introduction

The subject of curves on a given Euclidean surface $S \subset \mathbb{R}^3$ is a classical one but still preserves the flavor of a charming framework. Even more so if the given surface is a remarkable one, e.g. the unit sphere; recently, the curve shortening flow was studied on $S^2$ in [6]. The purpose of this work is to contribute to this setting with a deformation of the well-known geodesic curvature, somehow in the spirit of [9].

Recall that the geodesic curvature $k_g$ of a curve $\gamma \subset S \subset \mathbb{R}^3$ is provided by an orthonormal frame $\mathcal{F}(\gamma, S)$ adapted to both $\gamma$ and $S$; for $S^2$ we denote by $\mathcal{F}_s$ with $s$ from "spherical". Our idea is to rotate this frame in the normal-radial bundle $\gamma^+ := \{(x,v) \in (\text{Im } \gamma) \times S^2; v \perp x = \gamma(t) \in \mathbb{R}^3, t \in I\}$ (the notations are explained below) by an angle equal exactly to the parameter $t$ of $\gamma$. Then we call this new one flow-frame and since we use an orthogonal transformation, i.e. a matrix from $SO(3)$, this frame yields a new curvature called flow-geodesic curvature; for the case of plane curves, this notion is already studied in [4]. In turn, this new function gives a new evolute for the given curve. As new tools in studying spherical curves we introduce a spherical, as well as a flow-spherical Fermi-Walker derivative, and both these derivatives are computed for our main vector fields along $\gamma$.

The contents of the paper is as follows. The first section is a short survey on spherical curves and we point out the relationship between $k_g$ and the pair (curvature, torsion) of the given spherical curve $\gamma$ considered a space curve. The second section gives the new curvature and the new evolute; a main result establishes the computational expression of the new curvature. The third section is concerned with several examples and some related remarks; a special attention is devoted to find flow-flat curves, i.e. spherical curves having a vanishing flow-geodesic curvature. Our study is connected through two examples with the subject of space curves having a constant torsion, a theme of great interest in contemporary differential geometry of curves.

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2. Preliminaries: spherical curves

The setting of this section is provided by the space $\mathbb{R}^3$ which is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v \rangle = u^1v^1 + u^2v^2 + u^3v^3, u = (u^1, u^2, u^3), v = (v^1, v^2, v^3) \in \mathbb{R}^3, 0 \leq \|u\|^2 = \langle u, u \rangle. \quad (2.1)$$

Let $S^2 = SO(3)/SO(2) = SO(4)/U(2)$ be the unit sphere of $\mathbb{E}^3 := (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ and fix a smooth regular space curve which is a spherical one, $\gamma : I \subseteq \mathbb{R} \rightarrow S^2 \subset \mathbb{R}^3$. Its spherical Frenet frame is:

$$\mathcal{F}_s := \begin{pmatrix} \gamma \\ t \\ n \end{pmatrix}, \quad t(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}, \quad n(t) := \gamma(t) \times t(t), \quad (2.2)$$

and the corresponding spherical Frenet equation is provided by [11, p. 338]:

$$\frac{d}{dt} \mathcal{F}_s(t) = \|\gamma'(t)\| \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & k_g(t) \\ 0 & -k_g(t) & 0 \end{pmatrix} \mathcal{F}_s(t). \quad (2.3)$$

The smooth real function $k_g : I \rightarrow \mathbb{R}$ is called the geodesic curvature of $\gamma$ and its computational formula is:

$$k_g(t) := \frac{\langle \mathcal{F}_s'(t), \mathcal{F}_s(t) \rangle}{\|\gamma'(t)\|^3} = \frac{\det(\gamma(t), \gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3}. \quad (2.4)$$

Moreover, the usual curvature $k$ of $\gamma$ as a space curve is $k = \sqrt{k_g^2 + 1} \geq 1$ and the torsion of $\gamma$ is $\tau = \frac{k_g'}{k_g^2 + 1}$. Recall also that $\gamma$ is convex if $k_g > 0$ and if $k_g = 0$, we say that $\gamma$ is a spherical-flat curve. Sometimes, another adapted frame is used, namely the Darboux-Ribaucour frame, which is connected to $\mathcal{F}_s$ through a cubic root of the unit matrix $I_3$:

$$\begin{pmatrix} t \\ n \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mathcal{F}_s(t), \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in SO(3), \quad R^3 = I_3. \quad (2.6)$$

The rotation matrix $R$ is denoted by $\hat{q}$ at page 276 of [1]. The evolute of $\gamma$ is a new spherical curve:

$$Ev(\gamma)(t) := \frac{k_g(t)}{k(t)} \gamma(t) + \frac{1}{k(t)}n(t) \in S^2. \quad (2.5)$$

An important tool in dynamics along curves is the Fermi-Walker derivative. Let $\mathcal{X}_\gamma$ be the set of vector fields along the curve $\gamma$. Then the Fermi-Walker derivative is the map ([5]) $\nabla^{FW}_\gamma : \mathcal{X}_\gamma \rightarrow \mathcal{X}_\gamma$:

$$\nabla^{FW}_{\gamma}(X) := \frac{d}{dt} X + \|\gamma'(\cdot)\|k_g(\langle X, N \rangle t - \langle X, t \rangle N) \quad (2.6)$$

for $(t, N, B)$ the usual Frenet frame of $\gamma$. Inspired by this expression, we introduce a spherical Fermi-Walker derivative:

$$\nabla^{FW}_{\gamma}(X) := \frac{d}{dt} X + \|\gamma'(\cdot)\|k_g(\langle X, n \rangle t - \langle X, t \rangle n). \quad (2.7)$$
The Fermi-Walker derivative for our main vector fields is:
\[
\left\{ \begin{array}{l}
\nabla_{\gamma}^{FW}(\gamma)(t) = \gamma'(t), \quad \nabla_{\gamma}^{FW}(t) = 0, \quad \nabla_{\gamma}^{FW}(N)(t) = \|\gamma'(t)\|\tau(t)B(t), \\
\nabla_{\gamma}^{FW}(Ev(\gamma))(t) = \frac{d}{dt} \left( \frac{k_{\gamma}(t)}{k_{\gamma}(t)} \right) \gamma(t) + \|\gamma'(t)\|k_{\gamma}(t)n(t), N(t)t(t) + \frac{d}{dt} \left( \frac{1}{k_{\gamma}(t)} \right) n(t).
\end{array} \right.
\]
(2.8)

Also, the spherical Fermi-Walker derivative of our main vector fields is:
\[
\left\{ \begin{array}{l}
\nabla_{\gamma}^{s}(\gamma)(t) = \gamma'(t), \quad \nabla_{\gamma}^{s}(t)(t) = -\|\gamma'(t)\|\gamma(t), \quad \nabla_{\gamma}^{s}(n) = 0, \\
\nabla_{\gamma}^{s}(Ev(\gamma))(t) = \frac{d}{dt} \left( \frac{k_{\gamma}(t)}{k_{\gamma}(t)} \right) \gamma(t) + \|\gamma'(t)\|k_{\gamma}(t)t(t) + \frac{d}{dt} \left( \frac{1}{k_{\gamma}(t)} \right) n(t).
\end{array} \right.
\]
(2.9)

3. The flow-geodesic curvature and the flow-evolute of a spherical curve

The aim of this short note is to introduce a new curvature in order to find possible new features of spherical curves; our model is the case of plane curves studied in [4]. More precisely, we first introduce a new frame along \( \gamma \), denoted by \( \mathcal{F}^f \) and called the flow-spherical frame through:
\[
\mathcal{F}^f_s(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}, \quad \mathcal{F}^f_s(t) = \begin{pmatrix} \gamma & E^f_1 \\ E^f_2 \end{pmatrix} \quad (t)
\]
(3.1)

and the \( 3 \times 3 \) matrix above being an element of the subgroup \{1\} \( \times \) \( SO(2) \) of the special orthonormal group \( SO(3) \), we have that \( \mathcal{F}^f_s \) is also a positive oriented frame for \( \gamma \). It follows that its moving equation:
\[
\frac{d}{dt} \mathcal{F}^f_s(t) = \|\gamma'(t)\| \begin{pmatrix} 0 & \cos t & \sin t \\ -\cos t & 0 & k_{\gamma}^{f}(t) \\ -\sin t & -k_{\gamma}^{f}(t) & 0 \end{pmatrix} \mathcal{F}^f_s(t)
\]
(3.2)

defines a new smooth function \( k^f_{\gamma} : I \rightarrow \mathbb{R} \) which we call the flow-geodesic curvature of \( \gamma \) and then if \( k^f_{\gamma} = 0 \), we say that \( \gamma \) is a flow-flat spherical curve. We introduce the flow-evolute of \( \gamma \) as another spherical curve:
\[
Ev^f(\gamma)(t) := \frac{k^f_{\gamma}(t)}{\sqrt{|k^2_{\gamma}(t)| + 1}} \gamma(t) + \frac{1}{\sqrt{|k^2_{\gamma}(t)| + 1}} E^f_2(t) \in S^2.
\]
(3.3)

We point out that \( t, n, E^f_1, \) and \( E^f_2 \) are also spherical curves.

A straightforward computation yields:

**Theorem 3.1** The expression of the flow-geodesic curvature is:
\[
k^f_{\gamma}(t) = k_{\gamma}(t) - \frac{1}{\|\gamma'(t)\|} = \frac{\det(\gamma(t), \gamma'(t), \gamma''(t)) - \|\gamma'(t)\|^2}{\|\gamma'(t)\|^3} < k_{\gamma}(t).
\]
(3.4)

Therefore, \( \gamma \) is a flow-flat spherical curve if and only if:
\[
\det(\gamma(t), t(t), \gamma''(t)) = \|\gamma'(t)\|.
\]
(3.5)

In particular, if \( \gamma \) is parametrized by the arc-length \( s \) and is flow-flat, then we have the conservation law:
\[
\det(\gamma(s), t(s) = \gamma'(s), \gamma''(s)) = \text{constant} = 1.
\]
A setting where flow-flat curves may appear interesting is as follows: fix a remarkable map \( \varphi : M^n \to S^2 \) from a smooth \( n \)-dimensional manifold \( M^n \), and a smooth curve \( \Gamma : I \to M \). Then we call \( \Gamma \) as being \( \varphi \)-flow-flat if its image through \( \varphi \) is a flow-flat spherical curve. For example, any harmonic map from a simply connected Riemann surface \( \Sigma \) to \( S^2 \) gives rise to a spherical surface with singularities, called spherical frontals; here spherical surface means a surface in \( \mathbb{R}^3 \) with constant and positive Gaussian curvature, see the excellent survey in [3].

**Example 3.2** Another remarkable example of a map with the 2-sphere as target is the Hopf map, \( H : \mathbb{C}^2 \setminus \{0\} \to S^2 \subset \mathbb{C} \times \mathbb{R} : \)

\[
H(u, v) = \left( \frac{2uv}{|u|^2 + |v|^2}, \frac{|u|^2 - |v|^2}{|u|^2 + |v|^2} \right)
\]

and then a curve in \( \mathbb{C}^2 \setminus \{0\} \) will be Hopf-flow-flat if its image through \( H \) is a flow-flat spherical curve. □

Following the approach of the first section, we define now a flow-spherical Fermi-Walker derivative:

\[
\nabla^{f_s}_\gamma(X) := \frac{d}{dt} X + \|\gamma'\| k_9[(X, n) t - (X, t)n] = 0.
\]

The flow-spherical Fermi-Walker derivative of our main vector fields is:

\[
\begin{align*}
\nabla^{f_s}_\gamma(\gamma)(t) &= \gamma'(t), \\
\nabla^{f_s}_\gamma(t)(t) &= -\|\gamma'(t)\|\gamma(t) + n(t), \\
\nabla^{f_s}_\gamma(n)(t) &= -t(t), \\
\nabla^{f_s}_\gamma(Ev(\gamma))(t) &= \frac{d}{dt} \left( \kappa_3(t) \right) \gamma(t) + \|\gamma'(t)\| \kappa_3(t) t(t) \frac{d}{dt} \left( \frac{1}{\kappa_3(t)} \right) n(t).
\end{align*}
\]

and then the flow-spherical Fermi-Walker derivative for the elements of the flow-spherical frame is:

\[
\nabla^{f_s}_\gamma(E_i^f)(t) = -(\|\gamma'(t)\| \cos t) \gamma(t), \quad \nabla^{f_s}_\gamma(E_i^f)(t) = -(\|\gamma'(t)\| \sin t) \gamma(t).
\]

4. Examples and remarks

In what follows we are interested in computing this new function for some remarkable spherical curves.

**Example 4.1** Recall the spherical coordinates \((u, v) \in [0, 2\pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}]\) giving the well-known parametrization of \( S^2 \):

\[
S^2 : \tilde{r}(u, v) = (\cos u \cos v, \sin u \cos v, \sin v).
\]

Fix \( m \in \mathbb{R} \) and the corresponding Clelia curve [7, p. 60]:

\[
\gamma_m(t) = (\cos t \cos(mt), \sin t \cos(mt), \sin(mt)) = \tilde{r}(u = t, v = mt), \quad t \in \mathbb{R}.
\]

Then:

\[
\begin{align*}
\gamma'_m(t) &= (-\sin t \cos(mt) - m \cos t \sin(mt), \cos t \cos(mt) - m \sin t \sin(mt), m \cos(mt)), \\
\|\gamma'_m(t)\| &= \sqrt{m^2 + \cos^2(mt)} \geq \max\{|m|, 1\} > 0
\end{align*}
\]

which says that \( \gamma_m \) is a regular curve. It follows:

\[
\begin{align*}
\kappa_9(t) &= \frac{1}{\sqrt{m^2 + \cos^2(mt)}} \left( m \sin t - \frac{1}{2} \cos t \sin(2mt), -m \cos t - \frac{1}{2} \sin t \sin(2mt), \cos^2(mt) \right), \\
k_3(t) &= \frac{\sin(mt) [2m^2 + \cos^2(mt)]}{(m^2 + \cos^2(mt))^2}
\end{align*}
\]

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and hence the arc $t \in (0, \frac{\pi}{m})$ is convex. The flow-geodesic curvature is:

$$k_f^g(t) = \sin(mt)[2m^2 + \cos^2(mt)] - \frac{1}{(m^2 + \cos^2(mt))^{\frac{3}{2}}}.$$ \hspace{1cm} (4.5)

\[\square\]

**Example 4.2** A spherical curve with prescribed constant geodesic curvature $k_g = K$ and parametrized by the arc-length $s$ is:

$$\gamma_K(s) = \frac{1}{\sqrt{1 + K^2}} \left( \cos(\sqrt{1 + K^2}s), \sin(\sqrt{1 + K^2}s), K \right) = \frac{1}{K} \left( \cos(ks), \sin(ks), K \right), s \in \mathbb{R} \hspace{1cm} (4.6)$$

with the spherical coordinates ($u = u(s) = ks, v = constant = \arcsin \left( \frac{K}{K} \right)$); its evolute is the constant unit vector $Ev(\gamma_K) = (0, 0, 1) = \tilde{k}$ and its binormal is also constant $B = k(0, 0, 1)$. Then the flow-geodesic curvature of $\gamma_K$ is the constant $k_f^g = K - 1$. It follows that $\gamma_1$ is a flow-flat convex spherical curve, $\gamma_1(s) = \frac{1}{\sqrt{2}}(\cos(\sqrt{2}s), \sin(\sqrt{2}s), 1)$, having the flow-evolute:

$$Ev^f(\gamma_1)(s) = E_2^f(s) = (\sin s) \cdot (\sin(\sqrt{2}s), \cos(\sqrt{2}s), 0) - \frac{\cos s}{\sqrt{2}} \cdot (\cos(\sqrt{2}s), \sin(\sqrt{2}s), -1). \hspace{1cm} (4.7)$$

The stereographic projection from the North Pole $N(0, 0, 1)$ (respectively from the South Pole $S(0, 0, -1)$) of the parallel $\gamma_1 \in S^2$ is the plane circle centered in the origin $(0, 0)$ and having the radius $2 + \sqrt{2}$ (respectively the radius $2 - \sqrt{2}$). Concerning the example 2.2 the hypercone $H^{-1}(\gamma_1)$ of $\mathbb{C}^2 \setminus \{0\}$ is given by $|u| = (\sqrt{2} + 1)|v|$ and then any curve in this hypersurface will be a Hopf-flow-flat curve. \[\square\]

**Figure 1.** The flow-flat curve $\gamma_1$ of the example 4.2

**Example 4.3** The tangent indicatrix of the given $\gamma$ is exactly the map $t \in I \rightarrow \mathbf{t}(t) \in S^2$. Its spherical Frenet frame is:

$$\mathcal{F}^t_s := \begin{pmatrix} \mathbf{t} \\ \mathbf{t}^t \\ \mathbf{n}^t \end{pmatrix}, \mathbf{t}^t(t) := \frac{1}{k(t)}[-\gamma(t) + k_g(t)\mathbf{n}(t)], \hspace{0.5cm} \mathbf{n}^t(t) := \frac{1}{k(t)}[k_g(t)\gamma(t) + \mathbf{n}(t)]. \hspace{1cm} (4.8)$$

Since $||\mathbf{t}^t(t)|| = k(t)||\gamma'(t)||$, we get the geodesic curvature of this new spherical curve:

$$k_g^f(t) = \frac{k_g^f(t)}{||\gamma'(t)||k^3(t)} \hspace{1cm} (4.9)$$

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and then the flow-geodesic curvature of the tangent indicatrix is:

\[ k^f_{g}(t) = \frac{k'_g(t) - (1 + k^3_g(t))}{\|\gamma'(t)\|k^3(t)}. \]  

(4.10)

Suppose now that \( \gamma \) is parametrized by arc-length. Then the tangent indicatrix is a flow-flat curve if and only if \( \gamma \) has the geodesic curvature \( k_g(s) = \tan s \), equivalently the curvature \( k(s) = \frac{-1}{\cos s} \) for \( s \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \); it follows the evolute \( Ev(\gamma)(t) = -[\sin t \gamma(t) + \cos t n(t)] \). However, this curvature corresponds exactly to the expression (5) of [8, p. 363] for the constant torsion \( \tau = 1 \) and an explicit formula for \( \gamma \) involving hypergeometric functions is provided by the cited paper. The functions total curvature and total flow-geodesic curvature (on \( \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \)) of \( \gamma \) are:

\[ \int k(t)dt = -\ln \frac{\cos \frac{t}{2} + \sin \frac{t}{2}}{\cos \frac{t}{2} - \sin \frac{t}{2}}, \quad \int k_g(t)dt = -\ln(-\cos t). \]  

(4.11)

\( \square \)

**Remark 4.4** In the paper [10], the Delaunay variational problem defined by the arc-length functional acting on the space of curves with constant torsion \( \tau = 1 \) is studied. A main characterization is that a biregular curve is a critical point of the Delaunay functional if and only if the associated binormal curve \( \gamma \) is an elastic spherical curve, i.e. there exists \( \lambda \in \mathbb{R} \) such that:

\[ (k_g)_s + \frac{3}{2}k^3_g + (1 - \lambda)k_g = 0. \]  

(4.12)

Then we define the \( \lambda \)-elastic curvature \( k^e_{\lambda} \) of the spherical curve \( \gamma \) through the left-hand-side of the equation above. For our example 4.3 with \( k_g(s) = \frac{\cos s}{\sin s} \), we have:

\[ k^f_{g}(s) = \frac{\cos s}{\sin s} - 1, \quad k^3_{e}(s) = \frac{\cos s(4 - 3\cos^2 s)}{2\sin^3 s} + (1 - \lambda)\frac{\cos s}{\sin s} \]  

(4.13)

and then a zero of \( k^f_{g} \) is provided by the angle \( \frac{3\pi}{4} \). \( \square \)

**Remark 4.5** Since we arrive at the subject of curves with constant torsion, we connect our study with the proposition 1.1 from [2, p. 216]. Fix \( \gamma \) a spherical curve parametrized by arc-length and a constant \( \tau \neq 0 \). Using the pair \( (\gamma, \tau) \), a new space curve is considered:

\[ \Gamma(\gamma, \tau) := \frac{1}{\tau} \int \gamma \times \gamma' ds \]  

(4.14)

and the cited theorem gives that the curvature \( k_\Gamma \) and torsion \( \tau_\Gamma = \tau \) are related to the geodesic curvature of \( \gamma \) through: \( k_g = k_\Gamma \cdot \tau \); then the flow-geodesic curvature of \( \gamma \) is \( k^f_{g} = k_\Gamma \cdot \tau - 1 \). Hence, we return to the curve \( \gamma_K \) of the previous example and the corresponding \( \Gamma \) is:

\[ \Gamma_K(s) = \frac{1}{k_\Gamma} \left( -\frac{K}{k} \sin(k s), \frac{K}{k} \cos(k s), s \right) \]  

(4.15)

satisfying then \( k_{\Gamma K} = \frac{K}{k} \) and \( ||\Gamma_K'(s)|| = \frac{1}{|\tau|} = \text{constant} \). Having both curvature and torsion as constants \( \Gamma_K \) is a helix lying on the cylinder \( C : x^2 + y^2 = \frac{K^2}{k^2|\tau|} \). Its arc-length parametrization is:

\[ \Gamma_K(u) = \frac{1}{k_\Gamma} \left( -\frac{K}{k} \sin(k \tau u), \frac{K}{k} \cos(k \tau u), \tau u \right). \]  

(4.16)
Example 4.6 The spherical nephroid is presented in [11, p. 353] as:

\[ \gamma(t) = \left( \frac{3}{4} \cos t - \frac{1}{4} \cos 3t, \frac{3}{4} \sin t - \frac{1}{4} \sin 3t, \frac{\sqrt{3}}{2} \cos t \right). \] (4.17)

Its geodesic curvature is:

\[ k_g(t) = \frac{\cos t}{|\sin t|} \] (4.18)

and then we restrict the parameter to \( t \in (0, \pi) \); it results: \( k(t) = \frac{1}{\sin t}, \tau = -1 \). The flow-curvature of \( \gamma \) is:

\[ k_f^g(t) = \frac{\cos t}{\sin t} - \frac{1}{\sqrt{3} \sin t} \] (4.19)

and hence a zero \( t_0 \) of \( k_f^g \) is exactly the magic angle \( t_0 = \arccos \left( \frac{1}{\sqrt{3}} \right) \simeq 0.955 \). The total flow-geodesic curvature function is:

\[ \int k_f^g(t) dt = \ln(\sin t) + \frac{1}{\sqrt{3}} \ln \cot \frac{t}{2}. \] (4.20)

Conflict of interest
The author declares that there are no conflicts of interest.

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