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Bernstein-Nikol'skii-Markov-type inequalities for algebraic polynomials in a weighted Lebesgue space in regions with cusps

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Abstract: In this paper, we study Bernstein-Nikol'skii-Markov type inequalities for arbitrary algebraic polynomials with respect to a weighted Lebesgue space, where the weight functions have some singularities on a given contour. We consider curves which can contain a finite number of exterior and interior corners with power law tangency of the boundary arcs at those points where the weight functions have both zeros and poles of finite order. The estimates are given for the growth of the module of derivatives for algebraic polynomials on the closure of a region bounded by a given curve, depending on the behavior of weight functions, on the property of curve, and on the degree of contact of the boundary arcs, which form zero angles on the boundary.

Key words: Algebraic polynomials, conformal mapping, quasicircle

1. Introduction and definitions

Let $\mathbb{C}$ be a complex plane, $\overline{\mathbb{C}} =: \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region (without loss of generality, let $0 \in G$) and $L := \partial G$, $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$. Let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z) := \sum\limits_{j=0}^{n} a_j z^j$ of degree at most $n \in \mathbb{N}$.

Let $0 < p \leq \infty$ and $h(z)$ be a some weight function. For a rectifiable Jordan curve L, we denote:

$$\|P_n\|_p := \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int\limits_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p}, \ 0 < p < \infty,$$

$$\|P_n\|_\infty := \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \ p = \infty.$$

Clearly, $\|\cdot\|_p$ is a quasinorm (i.e. a norm for $1 \leq p \leq \infty$ and a $p-$norm for $0 < p < 1$).

The classical Markov inequality [30] says

$$\max_{x \in [-1,1]} |P_n'(x)| \leq n^2 \max_{x \in [-1,1]} |P_n(x)|. \tag{1.1}$$

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Bernstein [21] indicated an analogue of this result for the unit disk instead of the real interval $[-1,1]$ as follows:

$$\max_{|z|\leq 1}|P_n'(z)| \leq n\max_{|z|\leq 1}|P_n(z)|. \tag{1.2}$$

In 1933, Jackson [26] for $L=\{z:|z|=1\}$ and $0<p<\infty$ established the following estimate for $P_n(z)$:

$$\|P_n\|_\infty \leq 2n^{\frac{1}{p}}\left(\int_0^{2\pi}\left|P_n(e^{it})\right|^p dt\right)^{\frac{1}{p}}. \tag{1.3}$$

During almost one hundred years, the estimates of (1.1–1.3) type and their generalizations for higher order derivatives, as well as similar estimates in various weighted spaces, have been studied by many mathematicians. See, e.g., Szegö and Zygmund [43], Suetin [41], Suetim [42], Mamedhanov [28], Mamedhanov and Dadashova[29], Nikol'skii [33, pp. 122–123], Dzyadyk [25], Andrashko [16], Nevai and Totik [32], Milovanovic et al. [31], Pritsker [38], Ditzian and Prymak [23], Ditzian and Tikhonov [24], Andrievskii [18], [19] (and the references therein).

For the last few years, the estimates of (1.3) type and their analogues for the weighted Bergman class under some $m\geq 0$, $h(z)$, L and $0<p\leq\infty$ have been obtained in [3]-[14], [20], [35], [36], [40] and others.

In this work, we continue to study the estimates of (1.3)-type for $m-th$ derivatives, $m=0,1,2,...,$ for polynomials $P_n(z)$ in the weighted Lebesgue spaces $\mathcal{L}_p(h,L)$, $p>1$, in various regions of the complex plane.

For $t\in\mathbb{C}$ and $\delta>0$, let $\Delta(t,\delta):=\{w\in\mathbb{C}:|w-t|>\delta\};\Delta:=\Delta(0,1)$ and $B(t,\delta):=\{w\in\mathbb{C}: |w-t|<\delta\}$; $B:=B(0,1)$. Let $\Phi:\Omega\to\Delta$ be a univalent conformal mapping normalized by $\Phi(\infty)=\infty$ and $\lim_{z\to\infty}\frac{\Phi(z)}{z}>0$; $\Psi:=\Phi^{-1}$. For $t\geq 1$, let us set:

$$L_t:=\{z:\ |\Phi(z)|=t\},\ L_1\equiv L,\ G_t:=int\ L_t,\ \Omega_t:=ext\ L_t.$$

Let $\{z_j\}_{j=1}^{l}$ be a fixed system of distinct points on curve L which is located in the positive direction. For some fixed R_0, $1<R_0<\infty$, and $z\in G_{R_0}$, consider a generalized Jacobi weight function $h(z)$ which is defined as follows:

$$h(z):=h_0(z)\prod_{j=1}^{l}|z-z_j|^{\gamma_j}, \tag{1.4}$$

where $\gamma_j>-1$ for all $j=1,2,...,l$, and h_0 is uniformly separated from zero in G_{R_0}, i.e. there exists a constant $c_0:=c_0(G_{R_0})>0$ such that $h_0(z)\geq c_0>0$ for all $z\in G_{R_0}$.

Let $\varphi:G\to B$ be a conformal and univalent map which is normalized by $\varphi(0)=0$, $\varphi'(0)>0$; $\psi:=\varphi^{-1}$. Following [37] and [27, p. 100], a bounded Jordan region G is called a κ *-quasidisk,* $0\leq\kappa<1$, if any conformal mapping ψ can be extended to a K -quasiconformal homeomorphism, $K=\frac{1+\kappa}{1-\kappa}$, of the plane $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$. In that case, the curve $L:=\partial G$ is called a κ *-quasicircle.* The region G (curve L) is called a *quasidisk (quasicircle)*, if it is a κ*-quasidisk* (κ *-quasicircle)* for some $0\leq\kappa<1$. We denote this class by $Q(\kappa)$, $0\leq\kappa<1$, and say that $L=\partial G\in Q(\kappa)$, if $G\in Q(\kappa)$, $0\leq\kappa<1$. Note that quasicircles can be nonrectifiable (see, e.g., [22], [27, p. 104]). Therefore, we will say that $G\in\widetilde{Q}(\kappa)$, $0\leq\kappa<1$, if $G\in Q(\kappa)$ and ∂G is rectifiable. Furthermore, we mean that $G(L)\in Q$ (or $\widetilde{Q}$), if $G(L)\in Q(\kappa)$ (or $\widetilde{Q}(\kappa)$) for some $0\leq\kappa<1$. Recall that there is a geometric definition [27, p. 102] of quasicircle (quasiconformal curve). A curve L is said to be quasiconformal if for arbitrary points $z_1 \in L$ and $z_2 \in L$, the diameter of the shorter arc $l(z_1, z_2)$ of the curve L joining points z_1, z_2 satisfies the inequality:

$$\frac{\operatorname{diam} l(z_1,z_2)}{|z_1-z_2|} \le c < +\infty. \tag{1.5}$$

In [35, Th. 2.5], it is proved that:

Theorem A. *Let $0 < p \le \infty$; $L \in \widetilde{Q}(\kappa)$ for some $0 \le \kappa < 1$ and $h(z)$ be defined by (1.4). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 0, 1, 2, \ldots$*

$$\|P_n^{(m)}\|_\infty \le c_1 n^{\left(\frac{1+\gamma^*}{p}+m\right)(1+k)} \|P_n\|_p,$$

where

$$\gamma^* := \max\left\{0; \gamma_j,\ j=\overline{1,l}\right\}.$$

In particularly, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, \ldots$ we have the following sharp estimate for all $0 \le \kappa < 1$ (see the well-known sharp Markov inequality, [15, $m = 1$] and [35, Cor. 2.6]):

$$\|P_n^{(m)}\|_\infty \le c_1 n^{m(1+k)} \|P_n\|_\infty. \tag{1.6}$$

A simple example of curve

$$L^* := [0,1] \cup [1, 1+i] \cup \{z = x + ix^\alpha,\ x \in [0.1],\ \alpha > 1\}$$

shows that the curve L^* does not satisfy (1.5) and, consequently, is not a quasicircle.

In this work, we study this problem for regions bounded by piecewise rectifiable quasicircles having a finite number interior and exterior zero angles on the boundary.

We start with the corresponding definitions.

Definition 1.1 *A Jordan arc ℓ is called $\kappa-$ quasiarc for some $0 \le \kappa < 1$, if ℓ is a part of some κ-quasicircle for the same $0 \le \kappa < 1$.*

Now, we define a new class of regions bounded by piecewise quasicircle having interior and exterior cusps at the connecting points of boundary arcs.

Throughout this paper, $c, c_{0,} c_1, c_2, \ldots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depend on G (in general) and on parameters inessential for the argument; otherwise, such a dependence will be explicitly stated.

We say that a bounded Jordan curve (arc) L is a *locally $\kappa-$ quasicircle* ($\kappa-$quasiarc) at the point $z \in L$, if there exists a closed subarc $\ell \subset L$ containing z such that every open subarc of ℓ containing z is the *$\kappa-$ quasicircle* ($\kappa-$quasiarc).

For any $k \ge 0$ and $m > k$, the notation $i = \overline{k,m}$ means $i = k, k+1, \ldots, m$. For any $i = 1, 2, \ldots,$ $k = 0, 1, 2$ and $\varepsilon_1 > 0$, we denote by $f_i : [0, \varepsilon_1] \to \mathbb{R}$ and $g_i : [0, \varepsilon_1] \to \mathbb{R}$ twice differentiable functions such that

$$f_i(0) = g_i(0) = 0,\ f_i^{(k)}(x) > 0,\ \ g_i^{(k)}(x) > 0,\ 0 < x \le \varepsilon_1. \tag{1.7}$$

Definition 1.2 *We say that a Jordan region* $G \in \widetilde{PQ}(\kappa; f_i,\ g_i)$, *for some* $0 \le \kappa < 1$, $f_i = f_i(x), i = \overline{1, l_1}$ *and* $g_i = g_i(x), i = \overline{l_1+1, l}$, *defined as in (1.7), if* $L = \partial G = \bigcup_{i=0}^{l} L_i$ *is a union of the finite number of rectifiable* κ_i*-quasiarcs,* $0 \le \kappa_i < 1$, $(\kappa = \max\{\kappa_i,\ 0 \le i \le l\})$ L_i, *connecting at the points* $\{z_i\}_{i=0}^{l} \in L$ *and such that* L *is a locally* κ*-quasiarc at* $z_0 \in L \backslash\ \{z_i\}_{i=1}^{l}$ *and, in the* (x, y) *local coordinate system with its origin at the* z_i, $1 \le i \le l$, *the following conditions are satisfied:*

a) for every $z_i \in L,\ i = \overline{1, l_1},\ l_1 \le l$,

$$\left\{z = x + iy: \ |z| \le \varepsilon_1,\ \overset{i}{c}_{11} f_i(x) \le y \le \overset{i}{c}_{12} f_i(x),\ 0 \le x \le \varepsilon_1\right\} \subset \overline{G},$$

$$\{z = x + iy : |z| \le \varepsilon_1,\ |y| \ge \varepsilon_2 x,\ 0 \le x \le \varepsilon_1\} \subset \overline{\Omega};$$

b) for every $z_i \in L,\ i = \overline{l_1 + 1, l}$,

$$\left\{z = x + iy : |z| < \varepsilon_3, \quad c_{21}^i g_i(x) \le y \le c_{22}^i g_i(x),\ 0 \le x \le \varepsilon_3\right\} \subset \overline{\Omega},$$

$$\{z = x + iy : |z| < \varepsilon_3, \quad |y| \ge \varepsilon_4 x,\ 0 \le x \le \varepsilon_3\} \subset \overline{G},$$

for some constants $-\infty < c_{11}^i < c_{12}^i < \infty$, $-\infty < c_{21}^i < c_{22}^i < \infty$ *and* $\varepsilon_s > 0$, $s = \overline{1, 4}$.

It is clear from Definition 1.2, that each region $G \in \widetilde{PQ}(\kappa; f_i, g_i)$ may have l_1 interior and $l - l_1$ exterior zero angles (with respect to $\overline{G}$) at the points $\{z_i\}_{i=1}^{l} \in L$. If a region G does not have interior zero angles ($l_1 = 0$) (exterior zero angles ($l_1 = l$)), then it is written as $G \in \widetilde{PQ}(\kappa; 0, g_i)$ ($G \in \widetilde{PQ}(\kappa; f_i, 0)$). If a region G does not have such angles ($l = 0$), then G is bounded by a rectifiable $\kappa-$quasicircle, and in this case we set $\widetilde{PQ}(\kappa, 0, 0) \equiv \widetilde{Q}(\kappa)$.

Throughout this work, we will assume that the points $\{\xi_i\}_{i=1}^{l} \in L$ defined in (1.4) and the points $\{z_i\}_{i=1}^{l} \in L$ defined in Definition 1.2 coincide. Without loss of generality, we also will assume that the points $\{z_i\}_{i=0}^{l}$ are ordered in the positive direction on the curve L such that G has interior zero angles at the points $\{z_i\}_{i=1}^{l_1}$, if $l_1 \ge 1$ and exterior zero angles at the points $\{z_i\}_{i=l_1+1}^{l}$, if $l \ge l_1 + 1$.

2. Main results

For $L = \partial G$ and $0 < \delta_j < \delta_0 := \frac{1}{4} \min\left\{|z_i - z_j| : i, j = \overline{1, l},\ i \ne j\right\}$. For $t \ge 1$, we set: $\Omega_t(z_j,\ \delta_j) := \Omega_t \cap \{z : |z - z_j| \le \delta_j\}$; $\delta := \min_{1 \le j \le l} \delta_j$; $\Omega_t(\delta) := \bigcup_{j=1}^{l} \Omega_t(z_j,\ \delta)$, $\widehat{\Omega}_t(\delta) := \Omega_t \backslash\ \Omega_t(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega_t(z_j,\ \delta))$, $\Delta_t(\delta) := \bigcup_{j=1}^{l} \Phi(\Omega_t(z_j,\ \delta))$, $\widehat{\Delta}_t(\delta) := \Delta_t \backslash \Delta_t(\delta)$. $\Omega_{t,j} := \Psi(\Delta'_{t,j})$, $L_t^j := L_t \cap \overline{\Omega}_{t,j}$, $i = 1, 2, ..., l$. Clearly, $\Omega_t = \bigcup_{j=1}^{l} \Omega_{t,j}$. $F_t^i := \Phi(L_t^i) = \overline{\Delta}'_{t,i} \cap \{\tau : |\tau| = t\}$, $i = \overline{1, l}$.

Throughout this paper, for any $i=1,2,...;\ \ \gamma_i>-1,\ \alpha_i\geq 0,\ \beta_i>0,\ 0<\kappa<1,\ \widetilde{\kappa}=\left\{\begin{array}{ll} 1, & \alpha_i>0, \\ \kappa, & \alpha_i=0, \end{array}\right.$;

$$\begin{aligned} & p_6^*(m;i):=\frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-m(1+\kappa)};\ \ p_7:=\frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{\beta_2-\kappa}; \\ & p_8:=\frac{(\gamma_2+1)(1+\kappa)-(1+\widetilde{\kappa})(1+\beta_2)}{(1+\widetilde{\kappa})(1+\beta_2)-(1+\kappa)};\ \ \gamma_i^*:=\max\{0;\gamma_i\};\ \ \widetilde{\gamma}_2:=\frac{\gamma_2\,(1+\kappa)}{(1+\kappa)+(1+\beta_2)}\cdot\frac{2+\widetilde{\kappa}}{1+\widetilde{\kappa}}; \\ & \widetilde{\gamma}_4:=2\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{(1+\kappa)}-1\right];\ \ \widetilde{\gamma}_1(2):=\frac{(\gamma_2+1+p)(1+\kappa)}{(1+\widetilde{\kappa})(1+\beta_2)}-p-1. \end{aligned} \tag{2.1}$$

Now, we start with formulating some new results. We note that all parameters p and γ with different labels are taken from (2.1).

Therefore, let us begin with the evaluations for $\left|P_n^{(m)}(z)\right|,\ m\geq 0$.

Theorem 2.1 *Let $p>1;\ G\in\widetilde{PQ}(\kappa;f_i,\ g_i)$ for some $0\leq\kappa<1,\ \ f_i(x)=c_ix^{1+\alpha_i},\ \alpha_i\geq 0,\ i=\overline{1,l_1},$ and $g_i(x)=c_ix^{1+\beta_i},\beta_i>0,\ i=\overline{l_1+1,l}$. Suppose that $h(z)$ is defined by (1.4). Then, for any $\gamma_i>-1,\ i=\overline{1,l}$, and $P_n\in\wp_n,\ n\in\mathbb{N}$, the following inequality holds:*

$$\left\|P_n^{(m)}\right\|_\infty\leq c_1\left(\sum_{i=1}^{l_1}M_{n,1}^i(m)+\sum_{i=l_1+1}^{l}M_{n,2}^i(m)\right)\|P_n\|_p\,, \tag{2.2}$$

where $c_1=c_1(L,\gamma_i,\beta,p)>0$ is a constant independent of n and z;

$$M_{n,1}^i(m):=\left\{\begin{array}{lll} n^{\left(\frac{\gamma_i^*+1}{p}+m\right)(1+\widetilde{\kappa})}, & p>1, & m\geq 1, \\ n^{\frac{\gamma_i^*+1}{p}(1+\widetilde{\kappa})}, & p<1+(\gamma_i^*+1)(1+\widetilde{\kappa}), & m=0, \\ (n\ln n)^{1-\frac{1}{p}}, & p=1+(\gamma_i^*+1)(1+\widetilde{\kappa}), & m=0, \\ n^{1-\frac{1}{p}}, & p>1+(\gamma_i^*+1)(1+\widetilde{\kappa}), & m=0, \end{array}\right.; \tag{2.3}$$

$$M_{n,2}^i(m):=\left\{\begin{array}{llll} n^{\left(\frac{\gamma_i^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_i}}, & p>1, & \beta_i<m(1+\kappa)-1, & m\geq 1, \\ n^{\left(\frac{\gamma_i^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_i}}, & p<\frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-m(1+\kappa)}, & \beta_i\geq m(1+\kappa)-1, & m\geq 1, \\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-m(1+\kappa)}, & \beta_i\geq m(1+\kappa)-1, & m\geq 1, \\ n^{1-\frac{1}{p}}, & p>\frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-m(1+\kappa)}, & \beta_i\geq m(1+\kappa)-1, & m\geq 1, \\ n^{\left(\frac{\gamma_i^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_i}}, & p<1+(\gamma_i^*+1)\frac{(1+\kappa)}{(1+\beta_i)}, & \beta_i>0, & m=0, \\ (n\ln n)^{1-\frac{1}{p}}, & p=1+(\gamma_i^*+1)\frac{(1+\kappa)}{(1+\beta_i)}, & \beta_i>0, & m=0, \\ n^{1-\frac{1}{p}}, & p>1+(\gamma_i^*+1)\frac{(1+\kappa)}{(1+\beta_i)}, & \beta_i>0, & m=0. \end{array}\right.$$

For $i=1,2;\ l_1=1,\ l=2,$ we obtain the following.

Corollary 2.2 *Let $p>1;\ G\in\widetilde{PQ}(\kappa;f_1,\ g_2)$, for some $0\leq\kappa<1,\ f_1(x)=C_1x^{1+\alpha_1},\ \alpha_1\geq 0,$ and $g_2(x)=C_2x^{1+\beta_2},\beta_2>0$. Suppose that $h(z)$ is defined by (1.4) for $l=2$. Then, for any $\gamma_i>-1,\ i=1,2,$ and*

$P_n \in \wp_n,\ n \in \mathbb{N},$ the following inequality holds:

$$\left\|P_n^{(m)}\right\|_\infty \leq c_2 M_{n,3}(m) \left\|P_n\right\|_p, \tag{2.4}$$

where $c_2 = c_2(L, \gamma_i, \beta, p) > 0$ is a constant independent of n and z; $M_{n,3}(m) := M^1_{n,1}(m) + M^2_{n,2}(m)$ and $M^1_{n,1}(m)$, $M^2_{n,2}(m)$ are defined as follows:

$$M^1_{n,1}(m) := \begin{cases} n^{\left(\frac{\gamma_1^*+1}{p}+m\right)(1+\widetilde{\kappa})}, & p > 1, & m \geq 1, \\ n^{\frac{\gamma_1^*+1}{p}(1+\widetilde{\kappa})}, & p < 1 + (\gamma_1^*+1)(1+\widetilde{\kappa}), & m = 0, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + (\gamma_1^*+1)(1+\widetilde{\kappa}), & m = 0, \\ n^{1-\frac{1}{p}}, & p > 1 + (\gamma_1^*+1)(1+\widetilde{\kappa}), & m = 0, \end{cases}$$

$$M^2_{n,2}(m) := \begin{cases} n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_2}}, & p > 1, & \beta_2 < m(1+\kappa) - 1, & m \geq 1, \\ n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_2}}, & p < \frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \beta_2 \geq m(1+\kappa) - 1, & m \geq 1, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \beta_2 \geq m(1+\kappa) - 1, & m \geq 1, \\ n^{1-\frac{1}{p}}, & p > \frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \beta_2 \geq m(1+\kappa) - 1, & m \geq 1, \\ n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_2}}, & p < 1 + (\gamma_2^*+1)\frac{(1+\kappa)}{(1+\beta_2)}, & \beta_2 > 0, & m = 0, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + (\gamma_2^*+1)\frac{(1+\kappa)}{(1+\beta_2)}, & \beta_2 > 0, & m = 0, \\ n^{1-\frac{1}{p}}, & p > 1 + (\gamma_2^*+1)\frac{(1+\kappa)}{(1+\beta_2)}, & \beta_2 > 0, & m = 0. \end{cases}$$

In particular,

$$\left\|P_n'\right\|_\infty \leq c_2 M_{n,3}(1) \left\|P_n\right\|_p, \tag{2.5}$$

where $M_{n,3}(1) := M^1_{n,1}(1) + M^2_{n,2}(1)$, and it is defined as follows:

$$M_{n,3}(1) :=$$

$$\begin{cases} n^{\left(\frac{\gamma_1+1}{p}+1\right)(1+\widetilde{\kappa})}, & p < \frac{(\gamma_2+1)(1+\kappa)-(1+\widetilde{\kappa})(1+\beta_2)}{(1+\widetilde{\kappa})(1+\beta_2)-(1+\kappa)}, & \gamma_1 \geq \frac{(\gamma_2+1+p)(1+\kappa)}{(1+\widetilde{\kappa})(1+\beta_2)} - p - 1, & \gamma_2 \geq 2\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{(1+\kappa)} - 1\right], \\ n^{\left(\frac{\gamma_2+1}{p}+1\right)\frac{1+\kappa}{1+\beta_2}}, & p < \frac{(\gamma_2+1)(1+\kappa)-(1+\widetilde{\kappa})(1+\beta_2)}{(1+\widetilde{\kappa})(1+\beta_2)-(1+\kappa)}, & 0 < \gamma_1 < \frac{(\gamma_2+1+p)(1+\kappa)}{(1+\widetilde{\kappa})(1+\beta_2)} - p - 1, & \gamma_2 \geq 2\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{(1+\kappa)} - 1\right], \\ n^{\left(\frac{\gamma_1+1}{p}+1\right)(1+\widetilde{\kappa})}, & p \geq \frac{(\gamma_2+1)(1+\kappa)-(1+\widetilde{\kappa})(1+\beta_2)}{(1+\widetilde{\kappa})(1+\beta_2)-(1+\kappa)}, & \gamma_1 > 0, & \gamma_2 \geq 2\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{(1+\kappa)} - 1\right], \\ n^{\left(\frac{\gamma_1^*+1}{p}+1\right)(1+\widetilde{\kappa})}, & p > 1, & \gamma_1 > -1, & -1 < \gamma_2 < 2\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{(1+\kappa)} - 1\right]. \end{cases}$$

Corollary 2.3 *Let $p > 1$; $G \in \widetilde{PQ}(\kappa; f_,, 0)$, for some $0 < \kappa < 1$, $f_i(x) = C_1 x^{1+\alpha_i}$, $\alpha_i \geq 0$. Suppose that $h(z)$ is defined by (1.4) for $l = l_1$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $m \geq 1$, there exists $c_3 = c_3(G, p, \kappa, \gamma_1) > 0$ such that:*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_3 \left(\sum_{i=1}^{l_1} n^{\left(\frac{\gamma_i^*+1}{p}+m\right)(1+\widetilde{\kappa})}\right) \left\|P_n\right\|_p,$$

and, consequently, we obtain a global estimate

$$\left\|P_n^{(m)}\right\|_\infty \leq c_3 n^{\left(\frac{\gamma^* \max + 1}{p}+m\right)(1+\widetilde{\kappa})} \left\|P_n\right\|_p, \tag{2.6}$$

where $\gamma^{*\max} := \max\left\{0; \gamma_i, i = \overline{1, l_1}\right\}$.

Similarly to the above, for $G \in \widetilde{PQ}\left(\kappa; 0, g_i\right)$, we get the following.

Corollary 2.4 *Let* $p > 1$; $G \in \widetilde{PQ}\left(\kappa; 0, g_i\right)$, *for some* $0 < \kappa < 1$, $g_i(x) = C_i x^{1+\beta_i}, \beta_i > 0$, $i = \overline{l_1+1, l}$. *Suppose that* $h(z)$ *is defined by (1.4). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $\gamma_i > -1$, $i = \overline{l_1+1, l}$, $m \geq 1$, *there exists* $c_4 := c_4(G, p, \kappa, \gamma_i, \beta_i) > 0$ *such that:*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_4 \left(\sum_{i=l_1+1}^{l} M_{n,2}^i(m)\right) \|P_n\|_p, \tag{2.7}$$

where $M_{n,2}^i(m)$ *are taken from (2.3) for* $m \geq 1$, *and, therefore, we get*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_4 \widetilde{M}_{n,2}^i(m) \|P_n\|_p, \tag{2.8}$$

where

$$\widetilde{M}_{n,2}^i(m) := \begin{cases} n^{\left(\frac{\gamma^*_{\max}+1}{p}+m\right)\frac{1+\kappa}{1+\beta_{\min}}}, & p > 1, & \beta_i < m(1+\kappa)-1, & \textit{for all } i = \overline{l_1+1, l}, \\ n^{\left(\frac{\gamma^*_{\max}+1}{p}+m\right)\frac{1+\kappa}{1+\beta_{\min}}}, & p < \frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-m(1+\kappa)}, & \beta_i \geq m(1+\kappa)-1, & \textit{for all } i = \overline{l_1+1, l}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-m(1+\kappa)}, & \beta_i \geq m(1+\kappa)-1, & \textit{for all } i = \overline{l_1+1, l}, \\ n^{1-\frac{1}{p}}, & p > \frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-m(1+\kappa)}, & \beta_i \geq m(1+\kappa)-1, & \textit{for all } i = \overline{l_1+1, l}, \end{cases}$$

$\gamma^*_{\max} := \max\left\{0; \gamma_i, i = \overline{l_1+1, l}\right\}$, $\beta_{\min} := \min\left\{\beta_i, i = \overline{l_1+1, l}\right\}$.

Remark 1. a) Obviously, $G = B \subset \widetilde{PQ}\left(0; 0,\ 0\right) \equiv \widetilde{Q}(0)$. Then, for $h \equiv 1$, $m = 0$, the estimate (1.3) follows from (2.2).

b) Theorem 2.1 for $\alpha_i = \beta_i = 0$ coincides with [35, Th. 2.5], and, therefore, it generalizes [35, Th. 2.5] to the case of regions with interior and exterior zero angles;

c) The second sum in Theorem 2.1 gives a better estimate for $m = 0$ than the corresponding estimate in [5, Th. 2.1].

Remark 2. Theorem 2.1 and their corollaries, for some $p > 1$, $\kappa = 0$, and $\alpha_i = 0, \beta_i = 0$ are sharp.

3. Some auxiliary results

For $a > 0$ and $b > 0$ we use notations "$a \preceq b$" and "$a \asymp b$" if $a \leq cb$ and $c_1 a \leq b \leq c_2 a$ for some constants $c, c_1, and c_2$, respectively.

Lemma 3.1 *[1] Let* G *be a quasidisk,* $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. *Then,* *a) The statements* $|z_1 - z_2| \preceq |z_1 - z_3|$ *and* $|w_1 - w_2| \preceq |w_1 - w_3|$ *are equivalent. Therefore,* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$ *also are equivalent.*

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_1} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_2},$$

where $0 < r_0 < 1$ *a constant, depending on* G.

Corollary 3.2 *Under the conditions of Lemma 3.1, we have*

$$|w_1 - w_2|^{c_1} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{\varepsilon},$$

where $\varepsilon = \varepsilon(G) < 1$.

Lemma 3.3 *Let* $G \in Q(\kappa)$ *for some* $0 \leq \kappa < 1$. *Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\kappa},$$

for all $w_1, w_2 \in \overline{\Delta}$.

This fact follows from an appropriate result for the mapping $f \in \sum(\kappa)$ [37, p. 287] and estimation for Ψ' [17, Th.2.8]:

$$\frac{d(\Psi(\tau), L)}{|\tau| - 1} \asymp |\Psi'(\tau)|. \tag{3.1}$$

Let $\{z_j\}_{j=1}^{l}$ be a fixed system of the points on L and the weight function $h(z)$ defined by (1.4).

Lemma 3.4 *[2] Let* $L = \partial G$ *be a rectifiable Jordan curve and* $P_n(z)$, $\deg P_n \leq n$, $n = 1, 2, ...$, *be an arbitrary polynomial, and weight function* $h(z)$ *satisfies the condition (1.4). Then for any* $R > 1$, $p > 0$ *and* $n = 1, 2, ...$

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \leq R^{n + \frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \ \gamma^* = \max\{0; \gamma_j : 1 \leq j \leq l\}.$$

4. Proofs of theorems

Proof of Theorem 2.1. Suppose that $G \in \widetilde{PQ}(\kappa; f_i, g_i)$, for some $0 \leq \kappa < 1$, $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1, l_1}$, and $g_i(x) = c_i x^{1+\beta_i}, \beta_i > 0$, $i = \overline{l_1 + 1, l}$. First of all, we introduce some notations. Let $w_j := \Phi(z_j), \varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. For simplicity in our next calculations, we assume that

$$l_1 = 1, \ l = 2, \ i = 1, 2; \ z_1 = -1, \ z_2 = 1; \ (-1, 1) \subset G; \ R = 1 + \frac{\varepsilon_0}{n}, \tag{4.1}$$

and let the local coordinate axis in Definition 1.2 be parallel to OX and OY in the OXY coordinate system; $L = L^+ \cup L^-$, where $L^+ := \{z \in L : \mathrm{Im} z \geq 0\}$, $L^- := \{z \in L : \mathrm{Im} z < 0\}$. Let $w^{\pm} := \left\{w = e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$,

$z^{\pm} \in \Psi(w^{\pm})$ and L^i be arcs, connecting the points $z^+,\ z_i,\ z^- \in\ L;\ L^{i,\pm} := L^i \cap L^{\pm},\ i=1,2$. Let z_0 be taken as an arbitrary point on L^+ (or on L^- subject to the chosen direction). For simplicity, without loss of generality, we assume that $z_0 = z^+$ $(z_0 = z^-)$. Analogously to the previous notations, we introduce the following: $L_R = L_R^+ \cup L_R^-$, where $L_R^+ := \{z \in L_R : \mathrm{Im} z \geq 0\}$, $L_R^- := \{z \in L_R : \mathrm{Im} z < 0\}$. Let $w_R^{\pm} := \left\{w = Re^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, $z_R^{\pm} \in \Psi(w_R^{\pm})$. We set: $z_{i,R} \in L_R$, such that $d_{i,R} = |z_i - z_{i,R}|$ and $\zeta^{\pm} \in L^{\pm}$, such that $d(z_{2,R}, L^2 \cap L^{\pm}) := d(z_{2,R}, L^{\pm})$; $z_i^{\pm} := \left\{\zeta \in L^i :\ |\zeta - z_i| = c_i d(z_i, L_R)\right\}$, $z_{i,R}^{\pm} := \left\{\zeta \in L_R^i :\ |\zeta - z_{i,R}| = c_i d(z_{i,R}, L_R)\right\}$, $w_{i,R}^{\pm} = \Phi(z_{i,R}^{\pm})$. Let $L_R^i,\ i=1,2$, denote the arcs connecting the points $z_R^+,\ z_{i,R},\ z_R^- \in\ L_R,\ L_R^{i,\pm} := L_R^i \cap L_R^{\pm}$ and $l_{i,R}^{\pm}(z_{i,R}^{\pm}, z_R^{\pm})$ denote the arcs, connecting the points $z_{i,R}^{\pm}$ with $z_R^{\pm}$, respectively, and $\left| l_{i,R}^{\pm} \right| := mes\ l_{i,R}^{\pm}(z_{i,R}^{\pm}, z_R^{\pm}), i=1,2$. We denote

$$
\begin{aligned}
E_{1,R}^{i,\pm} &:= \left\{\zeta \in L_R^{i,\pm} :\ |\zeta - z_i| < c_i d_{i,R}\right\}, \qquad (4.2)\\
E_{2,R}^{i,\pm} &:= \left\{\zeta \in L_R^{i,\pm} : c_i d_{i,R} \leq |\zeta - z_i| \leq \left| l_{i,R}^{\pm}\right|\right\},\ F_{j,R}^{i,\pm} := \Phi(E_{j,R}^{i,\pm});\\
E_1^{i,\pm} &:= \left\{\zeta \in L^{i,\pm} :\ |\zeta - z_i| < c_i d_{i,R}\right\},\\
E_2^{i,\pm} &:= \left\{\zeta \in L^{i,\pm} : c_i d_{i,R} \leq |\zeta - z_i| \leq \left| l_{i,R}^{\pm}\right|\right\},\ F_j^{i,\pm} := \Phi(E_j^{i,\pm}),\ i,j=1,2.
\end{aligned}
$$

Now let us start the proof. Let $p>1$. The Cauchy integral formulas for $m-th$ derivatives for the region $G_R,\ R>1$, give

$$P_n^{(m)}(z) = \frac{1}{2\pi i}\int\limits_{L_R} P_n(\zeta)\frac{d\zeta}{(\zeta - z)^{m+1}},\ z \in G_R.$$

Let $z \in L$ be arbitrary fixed. Multiplying the numerator and denominator of the integrand by $h^{1/p}(\zeta)$, and applying the Hölder inequality, we obtain

$$\left|P_n^{(m)}(z)\right| \preceq \frac{1}{2\pi}\left(\int\limits_{L_R} h(\zeta)\left|P_n(\zeta)\right|^p |d\zeta|\right)^{\frac{1}{p}} \times \left(\int\limits_{L_R} \frac{|d\zeta|}{\prod\limits_{j=1}^{l} |\zeta - z_j|^{(q-1)\gamma_j}\,|\zeta - z|^{q(m+1)}}\right)^{\frac{1}{q}} = \frac{1}{2\pi}Y_{n,1} \times Y_{n,2}. \qquad (4.3)$$

According to Lemma 3.4, we have:

$$\left|P_n^{(m)}(z)\right| \preceq Y_{n,1}\cdot Y_{n,2} \preceq \left\|P_n\right\|_p \cdot Y_{n,2},\ z \in L. \qquad (4.4)$$

In order to evaluate the integral $Y_{n,2}$, we get

$$Y_{n,2} =: \left(\sum_{i=1}^{l} Y_{n,2}^i\right)^{\frac{1}{q}} \leq \sum_{i=1}^{l}\left(Y_{n,2}^i\right)^{\frac{1}{q}}, \qquad (4.5)$$

where

$$Y_{n,2}^i := \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{(q-1)\gamma_i}\,|\zeta - z|^{q(m+1)}},\ i = \overline{1,l}, \qquad (4.6)$$

since the points $\{z_i\}_{i=1}^{l} \in L$ are distinct. Let us estimate the integrals $Y_{n,2}^{i}$ for each $i = \overline{1, l}$.

According to the above notation, under changing the variable $\tau = \Phi(\zeta)$, from (3.1) and (4.6), we have

$$Y_{n,2}^{i} \asymp \sum_{i,j=1}^{2} \int\limits_{F_{j,R}^{i,\pm}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{(q-1)\gamma_i}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)}} \tag{4.7}$$

$$\asymp \sum_{i,j=1}^{2} \int\limits_{F_{j,R}^{i,\pm}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{(q-1)\gamma_i}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)}\,(|\tau|-1)} =: \sum_{i,j=1}^{2} Y(F_{j,R}^{i,\pm}).$$

Thus, we need to evaluate the integrals $Y(F_{j,R}^{i,\pm})$ for each $i,j = 1,2$. Let

$$\left\| P_n^{(m)} \right\|_\infty =: |P_n(z')|, \; z' \in L = L^1 \cup L^2, \tag{4.8}$$

and let $w' = \Phi(z')$.

1) Suppose that $z' \in L^1$.

1.1) If $z' \in E_1^{1,\pm}$, then

$$Y(F_{1,R}^{1,+}) + Y(F_{1,R}^{1,-}) \preceq n \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{(q-1)\gamma_1}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \tag{4.9}$$

$$\preceq n \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|d\tau|}{\left[\min\left\{|\Psi(\tau)-\Psi(w_1)|\,;|\Psi(\tau)-\Psi(w')|\right\}\right]^{(q-1)\gamma_1+q(m+1)-1}}$$

$$\preceq n \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|d\tau|}{\left[\min\left\{|\tau-w_1|\,;|\tau-w'|\right\}\right]^{[(q-1)\gamma_1+q(m+1)-1](1+\widetilde{\kappa})}}$$

$$\preceq \begin{cases} n^{[(q-1)\gamma_1+q(m+1)-1](1+\widetilde{\kappa})}, & p>1 & m\geq 1, \\ n^{[(q-1)\gamma_1+q-1](1+\widetilde{\kappa})}, & [(q-1)\gamma_1+q-1]\,(1+\widetilde{\kappa})>1, & m=0, \\ n\ln n, & [(q-1)\gamma_1+q-1]\,(1+\widetilde{\kappa})=1, & m=0, \\ n, & [(q-1)\gamma_1+q-1]\,(1+\widetilde{\kappa})<1, & m=0, \end{cases}$$

for $\gamma_1 > 0$, and

$$Y(F_{1,R}^{1,+}) + Y(F_{1,R}^{1,-}) \preceq n \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-(q-1)\gamma_1}\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \tag{4.10}$$

$$\preceq n \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \preceq n \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|d\tau|}{|\tau-w'|^{[q(m+1)-1](1+\widetilde{\kappa})}}$$

$$\preceq \begin{cases} n^{[q(m+1)-1](1+\widetilde{\kappa})}, & p>1, & m\geq 1, \\ n^{[q-1](1+\widetilde{\kappa})}, & [q-1]\,(1+\widetilde{\kappa})>1, & m=0, \\ n\ln n, & [q-1]\,(1+\widetilde{\kappa})=1, & m=0, \\ n, & [q-1]\,(1+\widetilde{\kappa})<1, & m=0, \end{cases}$$

for $-1<\gamma_1\leq 0$.

1.2) If $z'\in E_2^{1,\pm}$, then

$$Y(F_{1,R}^{1,+})+Y(F_{1,R}^{1,-})\preceq n\int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{(q-1)\gamma_1}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \tag{4.11}$$

$$\preceq n\int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}}\frac{|d\tau|}{[\min\{|\Psi(\tau)-\Psi(w_1)|\,;|\Psi(\tau)-\Psi(w')|\}]^{(q-1)\gamma_1+q(m+1)-1}}$$

$$\preceq n\int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}}\frac{|d\tau|}{[\min\{|\tau-w_1|\,;|\tau-w'|\}]^{((q-1)\gamma_1+q(m+1)-1)(1+\widetilde{\kappa})}}$$

$$\preceq\begin{cases} n^{[(q-1)\gamma_1+q(m+1)-1](1+\widetilde{\kappa})}, & p>1, \qquad m\geq 1,\\ n^{[(q-1)\gamma_1+q(m+1)-1](1+\widetilde{\kappa})}, & (q-1)\gamma_1+q-1>\frac{1}{1+\widetilde{\kappa}}, \qquad m=0,\\ n\ln n, & (q-1)\gamma_1+q-1=\frac{1}{1+\widetilde{\kappa}}, \qquad m=0,\\ n, & [(q-1)\gamma_1+q-1]<\frac{1}{1+\widetilde{\kappa}}, \qquad m=0,\end{cases}$$

for all $\gamma_1>0$ and

$$Y(F_{1,R}^{1,+})+Y(F_{1,R}^{1,-})\preceq n\int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}}\frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma_1}\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}}\preceq n\int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}}$$

$$\preceq n\int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}}\frac{|d\tau|}{|\tau-w'|^{[q(m+1)-1](1+\widetilde{\kappa})}}\preceq\begin{cases} n^{[q(m+1)-1](1+\widetilde{\kappa})}, & p>1, \qquad m\geq 1,\\ n^{[q-1](1+\widetilde{\kappa})}, & [q-1]\,(1+\widetilde{\kappa})>1, \qquad m=0,\\ n\ln n, & [q-1]\,(1+\widetilde{\kappa})=1, \qquad m=0,\\ n, & [q-1]\,(1+\widetilde{\kappa})<1, \qquad m=0,\end{cases}$$

for $-1<\gamma_1\leq 0$.

1.3) If $z'\in E_1^{1,\pm}$, then

$$Y(F_{2,R}^{1,+})+Y(F_{2,R}^{1,-})\preceq n\int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{(q-1)\gamma_1}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \tag{4.12}$$

$$\preceq n\int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}}\frac{|d\tau|}{\min\{|\Psi(\tau)-\Psi(w_1)|\,;|\Psi(\tau)-\Psi(w')|\}^{(q-1)\gamma_1+q(m+1)-1}}$$

$$\preceq n\int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}}\frac{|d\tau|}{\min\{|\tau-w_1|\,;|\tau-w'|\}^{[(q-1)\gamma_1+q(m+1)-1](1+\widetilde{\kappa})}}$$

$$\preceq\begin{cases} n^{[(q-1)\gamma_1+q(m+1)-1](1+\widetilde{\kappa})}, & p>1, \qquad m\geq 1,\\ n^{[(q-1)\gamma_1+q-1](1+\widetilde{\kappa})}, & [(q-1)\gamma_1+q-1]\,(1+\widetilde{\kappa})>1, \qquad m=0,\\ n\ln n, & [(q-1)\gamma_1+q-1]\,(1+\widetilde{\kappa})=1, \qquad m=0,\\ n, & [(q-1)\gamma_1+q-1]\,(1+\widetilde{\kappa})<1, \qquad m=0,\end{cases}$$

for $\gamma_1 \geq 0$ and

$$Y(F_{2,R}^{1,+}) + Y(F_{2,R}^{1,-}) \preceq n \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-(q-1)\gamma_1}\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \preceq n \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \tag{4.13}$$

$$\preceq n \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{|\tau-w'|^{[q(m+1)-1](1+\widetilde{\kappa})}} \preceq \begin{cases} n^{[q(m+1)-1](1+\widetilde{\kappa})}, & p>1, & m\geq 1, \\ n^{[q-1](1+\widetilde{\kappa})}, & [q-1]\,(1+\widetilde{\kappa})>1, & m=0, \\ n\ln n, & [q-1]\,(1+\widetilde{\kappa})=1, & m=0, \\ n, & [q-1]\,(1+\widetilde{\kappa})<1, & m=0, \end{cases}$$

for $-1<\gamma_1<0$.

1.4) If $z' \in E_2^{1,\pm}$, then

$$Y(F_{2,R}^{1,+}) + Y(F_{2,R}^{1,-}) \preceq n \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{(q-1)\gamma_1}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \tag{4.14}$$

$$\preceq n \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{\left[\min\left\{|\Psi(\tau)-\Psi(w_1)|\,;|\Psi(\tau)-\Psi(w')|\right\}\right]^{(q-1)\gamma_1+q(m+1)-1}}$$

$$\preceq n \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{\left[\min\left\{|\tau-w_1|\,;|\tau-w'|\right\}\right]^{[(q-1)\gamma_1+q(m+1)-1](1+\widetilde{\kappa})}}$$

$$\preceq \begin{cases} n^{[(q-1)\gamma_1+q(m+1)-1](1+\widetilde{\kappa})}, & p>1, & m\geq 1, \\ n^{[(q-1)\gamma_1+q-1](1+\widetilde{\kappa})}, & (q-1)\gamma_1+q-1>\frac{1}{1+\widetilde{\kappa}}, & m=0, \\ n\ln n, & (q-1)\gamma_1+q-1=\frac{1}{1+\widetilde{\kappa}}, & m=0, \\ n, & (q-1)\gamma_1+q-1<\frac{1}{1+\widetilde{\kappa}}, & m=0, \end{cases}$$

for $\gamma_1>0$, and

$$Y(F_{2,R}^{1,+}) + Y(F_{2,R}^{1,-}) \preceq n \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \tag{4.15}$$

$$\preceq n \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{|\tau-w'|^{[q(m+1)-1](1+\widetilde{\kappa})}} \preceq \begin{cases} n^{[q(m+1)-1](1+\widetilde{\kappa})}, & p>1, & m\geq 1, \\ n^{[q-1](1+\widetilde{\kappa})}, & [q-1]\,(1+\widetilde{\kappa})>1, & m=0, \\ n\ln n, & [q-1]\,(1+\widetilde{\kappa})=1, & m=0, \\ n, & [q-1]\,(1+\widetilde{\kappa})<1, & m=0, \end{cases}$$

for $-1<\gamma_1\leq 0$. Combining the relations (4.9)–(4.15), in the case of $z' \in L^1$ for each $\gamma_1>-1$ we get

$$\left(Y_{n,2}^1\right)^{\frac{1}{q}} \preceq \begin{cases} n^{\left(\frac{\gamma_1^*+1}{p}+m\right)(1+\widetilde{\kappa})}, & p>1, & \gamma_1>-1, & m\geq 1, \\ n^{\frac{\gamma_1^*+1}{p}(1+\widetilde{\kappa})}, & p<1+(\gamma^*+1)(1+\widetilde{\kappa}), & \gamma_1>-1, & m=0, \\ (n\ln n)^{1-\frac{1}{p}}, & p=1+(\gamma^*+1)(1+\widetilde{\kappa}), & \gamma_1>-1, & m=0, \\ n^{1-\frac{1}{p}}, & p>1+(\gamma^*+1)(1+\widetilde{\kappa}), & \gamma_1>-1, & m=0. \end{cases} \tag{4.16}$$

2) Now suppose that $z' \in L^2$. Replacing the variable $\tau=\Phi(\zeta)$ and according to (3.1), we have

$$Y_{n,2}^{i} \asymp \sum_{i,j=1}^{2} \int\limits_{F_{j,R}^{i,\pm}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)}\,(|\tau|-1)} =: \sum_{i,j=1}^{2} Y(F_{j,R}^{i,\pm}). \tag{4.17}$$

2.1) If $z' \in E_1^{2,\pm}$, then

$$Y(F_{1,R}^{2,+})+Y(F_{1,R}^{2,-}) \preceq n \int\limits_{F_{1,R}^{2,+}\cup F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \tag{4.18}$$

$$\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} + n \int\limits_{F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}},$$

for all $\gamma_2 > -1$. The last two integrals are evaluated identically; therefore, we give an estimate for the first on. When $\tau \in F_{1,R}^{2,+}$, for the $|\Psi(\tau)-\Psi(w')|$, we obtain

$$|\Psi(\tau)-\Psi(w')| \succeq \max\left\{|\Psi(\tau)-\Psi(w_2)|;\ \left|\Psi(\tau)-z_2^{+}\right|\right\} = |\Psi(\tau)-\Psi(w_2)| \succeq \left|\Psi(\tau)-z_2^{+}\right|^{\frac{1}{1+\beta_2}}.$$

Then,

$$Y(F_{1,R}^{2,+}) \preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}}} \preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^{+}\right|^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}}$$

$$\preceq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)>1, \\ n\ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)=1, \\ n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)<1, \end{cases}$$

if $\gamma_2 > 0$, and

$$Y(F_{1,R}^{2,+}) \preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|\Psi(\tau)-\Psi(w_2)|^{-(q-1)\gamma_2}\,|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{\frac{q(m+1)-1}{1+\beta_2}}} \preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^{+}\right|^{\frac{q(m+1)-1}{1+\beta_2}(1+\kappa)}}$$

$$\preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)>1, \\ n\ln n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)=1, \\ n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)<1, \end{cases},$$

if $-1 < \gamma_2 \le 0$, and so, we get

$$Y(F_{1,R}^{2,+})+Y(F_{1,R}^{2,-}) \preceq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)>1, \\ n\ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)=1, \\ n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)<1, \end{cases} \tag{4.19}$$

if $\gamma_2 > 0$, and

$$Y(F_{1,R}^{2,+}) + Y(F_{1,R}^{2,-}) \preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) > 1, \\ n \ln n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) = 1, \\ n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) < 1, \end{cases}$$

if $-1 < \gamma_2 \leq 0$.

2.2) If $z' \in E_2^{2,\pm}$, then

$$Y(F_{1,R}^{2,+}) + Y(F_{1,R}^{2,-}) \preceq n \int\limits_{F_{1,R}^{2,+} \cup F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2} |\Psi(\tau) - \Psi(w')|^{q(m+1)-1}}$$

for all $\gamma_2 > -1$. When $\tau \in F_{1,R}^{2,+}$ for $|\Psi(\tau) - \Psi(w')|$, we obtain: $|\Psi(\tau) - \Psi(w')| \succeq \left|\Psi(\tau) - z_2^+\right|$ and similarly to previous case, we get

$$Y(F_{1,R}^{2,+}) \preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2} \left|\Psi(\tau) - z_2^+\right|^{q(m+1)-1}} \tag{4.20}$$

$$\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}}} \preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}(1+\kappa)}}$$

$$\preceq \begin{cases} n^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}(1+\kappa) > 1, \\ n \ln n, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}(1+\kappa) = 1, \\ n, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}(1+\kappa) < 1, \end{cases}$$

if $\gamma_2 > 0$, and

$$Y(F_{1,R}^{2,+}) \preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|\Psi(\tau) - \Psi(w_2)|^{-(q-1)\gamma_2} |d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{q(m+1)-1}} \preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{q(m+1)-1}}$$

$$\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{[q(m+1)-1](1+\kappa)}} \preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) > 1, \\ n \ln n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) = 1, \\ n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) < 1, \end{cases}$$

if $-1 < \gamma_2 < 0$.

Therefore, from (4.19) and (4.20), for this case, we have

$$Y(F_{1,R}^{2,+}) + Y(F_{1,R}^{2,-}) \preceq \begin{cases} n^{\frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}(1+\kappa) > 1, \quad \gamma_2 > -1, \\ n \ln n, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}(1+\kappa) = 1, \quad \gamma_2 > -1, \\ n, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}(1+\kappa) < 1, \quad \gamma_2 > -1. \end{cases} \tag{4.21}$$

2.3) If $z' \in E_1^{2,\pm}$, then

$$Y(F_{2,R}^{2,+}) + Y(F_{2,R}^{2,-}) \preceq n \int\limits_{F_{2,R}^{2,+} \cup F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2} |\Psi(\tau) - \Psi(w')|^{q(m+1)-1}} \tag{4.22}$$

$$\preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} + n \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}},$$

for $\gamma_2 > 0$. The last two integrals are again estimated identically. Thus, we evaluate the first one. For $\tau \in F_{2,R}^{2,+}$ and $z' \in E_1^{2,\pm}$, we have:

$$|\Psi(\tau)-\Psi(w')| \succeq \left|\Psi(\tau)-z_2^+\right|; \;\; |\Psi(\tau)-\Psi(w_2)| \succeq d_{2,R} \succeq \left|z_{2,R}-z_2^+\right|^{\frac{1}{1+\beta_2}} \succeq \left(\frac{1}{n}\right)^{\frac{1}{1+\beta_2}(1+\kappa)}.$$

Then

$$Y(F_{2,R}^{2,+}) \preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^+\right|^{(q-1)\gamma_2}\;\left|\Psi(\tau)-z_2^+\right|^{q(m+1)-1}} \preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^+\right|^{\left[(q-1)\gamma_2+q(m+1)-1\right]\frac{(1+\kappa)}{1+\beta_2}}}$$

$$\preceq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa) > 1, \quad \gamma_2 > 0, \\ n\ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa) = 1, \quad \gamma_2 > 0, \\ n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa) < 1, \quad \gamma_2 > 0, \end{cases}$$

and for $\gamma_2 > 0$ we obtain

$$Y(F_{2,R}^{2,+}) + Y(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa) > 1, \quad \gamma_2 > 0, \\ n\ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa) = 1, \quad \gamma_2 > 0, \\ n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa) < 1, \quad \gamma_2 > 0. \end{cases}$$

For $-1 < \gamma_2 \leq 0$, we get

$$Y(F_{2,R}^{2,+}) + Y(F_{2,R}^{2,-}) \preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^+\right|^{q(m+1)-1}} \tag{4.23}$$

$$\preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^+\right|^{\frac{[q(m+1)-1]}{1+\beta_2}(1+\kappa)}} \preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) > 1, \quad \gamma_2 \leq 0, \\ n\ln n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) = 1, \quad \gamma_2 \leq 0, \\ n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa) < 1, \quad \gamma_2 \leq 0. \end{cases}$$

Then, in this case we have

$$Y(F_{2,R}^{2,+}) + Y(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}(1+\kappa) > 1, \quad \gamma_2 > -1, \\ n\ln n, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}(1+\kappa) = 1, \quad \gamma_2 > -1, \\ n, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}(1+\kappa) < 1, \quad \gamma_2 > -1. \end{cases} \tag{4.24}$$

2.4) If $z' \in E_2^{2,+}$, then for $\gamma_2 > 0$,

$$Y(F_{2,R}^{2,+}) \preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{(q-1)\gamma_2+q(m+1)}} \tag{4.25}$$

$$\preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\tau-w'|^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}} \preceq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)>1, \quad \gamma_2>0, \\ n\ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)=1, \quad \gamma_2>0, \\ n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)<1, \quad \gamma_2>0, \end{cases}$$

and

$$Y(F_{2,R}^{2,-}) \preceq n \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\tau-w'|^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}} \tag{4.26}$$

$$\preceq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)>1, \quad \gamma_2>0, \\ n\ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)=1, \quad \gamma_2>0, \\ n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}(1+\kappa)<1, \quad \gamma_2>0. \end{cases}$$

The case when $z' \in E_2^{2,-}$ is absolutely identical to the case if $z' \in E_2^{2,+}$.

If $-1<\gamma_2\leq 0$, then

$$Y(F_{2,R}^{2,+}) \preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q(m+1)-1}} \preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)>1, \quad \gamma_2\leq 0, \\ n\ln n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)=1, \quad \gamma_2\leq 0, \\ n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)<1, \quad \gamma_2\leq 0, \end{cases}$$

and

$$Y(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}(1+\kappa)}, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)>1, \quad \gamma_2\leq 0 \\ n\ln n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)=1, \quad \gamma_2\leq 0, \\ n, & \frac{q(m+1)-1}{1+\beta_2}(1+\kappa)<1, \quad \gamma_2\leq 0. \end{cases} \tag{4.27}$$

Combining the estimations (4.17), (4.19)–(4.27), we obtain

$$Y_{n,2}^1 \preceq \begin{cases} n^{\left(\frac{\gamma_2+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p>1, & \gamma_2>0, & \beta_2<m(1+\kappa)-1, & m\geq 1, \\ n^{\left(\frac{\gamma_2+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p<\frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \gamma_2>0, & \beta_2\geq m(1+\kappa)-1, & m\geq 1, \\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \gamma_2>0, & \beta_2\geq m(1+\kappa)-1, & m\geq 1, \\ n^{1-\frac{1}{p}}, & p>\frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \gamma_2>0, & \beta_2\geq m(1+\kappa)-1, & m\geq 1, \\ n^{\left(\frac{\gamma_2+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p<\frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, & \gamma_2>0, & \beta_2>0, & m=0, \\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, & \gamma_2>0, & \beta_2>0, & m=0, \\ n^{1-\frac{1}{p}}, & p>\frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, & \gamma_2>0, & \beta_2>0, & m=0, \end{cases}$$

$$+\begin{cases} n^{\left(\frac{1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p>1, & \gamma_2\leq 0, & \beta_2<m(1+\kappa)-1, & m\geq 1 \\ n^{\left(\frac{1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p<\frac{(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \gamma_2\leq 0, & \beta_2\geq m(1+\kappa)-1, & m\geq 1 \\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \gamma_2\leq 0, & \beta_2\geq m(1+\kappa)-1, & m\geq 1 \\ n^{1-\frac{1}{p}}, & p>\frac{(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-m(1+\kappa)}, & \gamma_2\leq 0, & \beta_2\geq m(1+\kappa)-1, & m\geq 1 \\ n^{\left(\frac{1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p<\frac{(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, & \gamma_2\leq 0, & \beta_2>0, & m=0, \\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, & \gamma_2\leq 0, & \beta_2>0, & m=0, \\ n^{1-\frac{1}{p}}, & p>\frac{(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, & \gamma_2\leq 0, & \beta_2>0, & m=0. \end{cases}$$

$$
= \begin{cases}
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p>1, \quad \gamma_2>-1, \quad \beta_2<m(1+\kappa)-1, \quad m\geq 1,\\
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p<p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
(n\ln n)^{1-\frac{1}{p}}, & p=p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
n^{1-\frac{1}{p}}, & p>p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p<\frac{(\gamma_2^*+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0,\\
(n\ln n)^{1-\frac{1}{p}}, & p=\frac{(\gamma_2^*+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0,\\
n^{1-\frac{1}{p}}, & p>\frac{(\gamma_2^*+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0.
\end{cases}
$$

Therefore, for $l_1=1,\ l=2,$ and any $p>1,$ we get

$$
Y_{n,2}^1+Y_{n,2}^2 \preceq \begin{cases}
n^{\left(\frac{\gamma_1^*+1}{p}+m\right)(1+\widetilde{\kappa})}, & p>1, \quad \gamma_1>-1, \quad m\geq 1,\\
n^{\frac{\gamma_1^*+1}{p}(1+\widetilde{\kappa})}, & p<1+(\gamma^*+1)(1+\widetilde{\kappa}), \quad \gamma_1>-1, \quad m=0,\\
(n\ln n)^{1-\frac{1}{p}}, & p=1+(\gamma^*+1)(1+\widetilde{\kappa}), \quad \gamma_1>-1, \quad m=0,\\
n^{1-\frac{1}{p}}, & p>1+(\gamma^*+1)(1+\widetilde{\kappa}), \quad \gamma_1>-1, \quad m=0,
\end{cases} \tag{4.28}
$$

$$
+ \begin{cases}
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p>1, \quad \gamma_2>-1, \quad \beta_2<m(1+\kappa)-1, \quad m\geq 1,\\
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p<p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
(n\ln n)^{1-\frac{1}{p}}, & p=p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
n^{1-\frac{1}{p}}, & p>p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{(1+\kappa)}{1+\beta_2}}, & p<\frac{(\gamma_2^*+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0,\\
(n\ln n)^{1-\frac{1}{p}}, & p=\frac{(\gamma_2^*+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0,\\
n^{1-\frac{1}{p}}, & p>\frac{(\gamma_2^*+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0.
\end{cases}
$$

Then, from (4.3)–(4.8), (4.16), and (4.28), we obtain

$$
|P_n(z)| \preceq \|P_n\|_p \begin{cases}
n^{\left(\frac{\gamma_1^*+1}{p}+m\right)(1+\widetilde{\kappa})}, & p>1, \quad \gamma_1>-1, \quad m\geq 1,\\
n^{\frac{\gamma_1^*+1}{p}(1+\widetilde{\kappa})}, & p<1+(\gamma_1^*+1)(1+\widetilde{\kappa}), \quad \gamma_1>-1, \quad m=0,\\
(n\ln n)^{1-\frac{1}{p}}, & p=1+(\gamma_1^*+1)(1+\widetilde{\kappa}), \quad \gamma_1>-1, \quad m=0,\\
n^{1-\frac{1}{p}}, & p>1+(\gamma_1^*+1)(1+\widetilde{\kappa}), \quad \gamma_1>-1, \quad m=0,
\end{cases}
$$

$$
+ \begin{cases}
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_2}}, & p>1, \quad \gamma_2>-1, \quad \beta_2<m(1+\kappa)-1, \quad m\geq 1,\\
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_2}}, & p<p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
(n\ln n)^{1-\frac{1}{p}}, & p=p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
n^{1-\frac{1}{p}}, & p>p_6^*(m;2), \quad \gamma_2>-1, \quad \beta_2\geq m(1+\kappa)-1, \quad m\geq 1,\\
n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{1+\kappa}{1+\beta_2}}, & p<1+(\gamma_2^*+1)\frac{(1+\kappa)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0,\\
(n\ln n)^{1-\frac{1}{p}}, & p=1+(\gamma_2^*+1)\frac{(1+\kappa)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0,\\
n^{1-\frac{1}{p}}, & p>1+(\gamma_2^*+1)\frac{(1+\kappa)}{(1+\beta_2)}, \quad \gamma_2>-1, \quad \beta_2>0, \quad m=0.
\end{cases}
$$

Since $z\in L$ is arbitrary, we complete the proof of Theorem 2.1. □

Proof of Remark 2. The sharpness of this estimate can be argued as follows. These inequalities can be interpreted as a successive application of the well-known sharp Markov inequalities $\left\|P_n^{(m)}\right\|_\infty \preceq n^m \left\|P_n\right\|_\infty$, $m \geq 1$, with inequality [14]

$$\|P_n\|_\infty \leq c_1 n^{\frac{(1+\gamma^*)(1+k)}{p}} \|P_n\|_p, \tag{4.29}$$

and the sharpness of the last inequality can be verified by the following examples. For the polynomial $T_n(z) = 1 + z + ... + z^n$, $h^*(z) = h_0(z)$, $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$, $L := \{z : |z| = 1\}$ and any $n \in \mathbb{N}$ there exist $c_3 = c_3(h^*, p) > 0$, $c_3' = c_3'(h^{**}, p) > 0$ such that

$$a)\ \|T\|_\infty \geq c_3 n^{\frac{1}{p}} \|T\|_{\mathcal{L}_p(h^*,\ L)}, \quad p > 1;$$

$$b)\ \|T\|_\infty \geq c_3' n^{\frac{\gamma+1}{p}} \|T\|_{\mathcal{L}_p(h^{**},\ L)}, \quad p > \gamma + 1.$$

Indeed, if $L := \{z : |z| = 1\}$,then, $L \in \widetilde{Q}(0)$. Pick for

a) $h^*(z) \equiv 1$; b) $h^{**}(z) = |z-1|^\gamma$, $\gamma > 0$.

Obviously,

$$|T(z)| \leq \sum_{j=0}^{n-1} \left|z^j\right| = n, \ \ |z| = 1;\ |T(1)| = n.$$

Thus,

$$\|T\|_\infty = n.$$

On the other hand, according to [43, p. 236], we have

$$\|T\|_{\mathcal{L}_p(h^*,L)} \asymp n^{1-\frac{1}{p}},\ p > 1,$$

and

$$\|T\|_{\mathcal{L}_p(h^{**},L)} \asymp n^{1-\frac{\gamma+1}{p}}, \quad p > \gamma + 1.$$

Therefore,

$$a)\ \|T\|_\infty = n \asymp n^{\frac{1}{p}} \|T\|_{\mathcal{L}_p(h^*,L)},\ p > 1;$$

$$b)\ \|T\|_\infty = n = n \cdot n^{1-\frac{\gamma+1}{p}} \cdot n^{\frac{\gamma+1}{p}-1} \asymp n^{\frac{\gamma+1}{p}} \|T\|_{\mathcal{L}_p(h^{**},L)},\ p > \gamma + 1.$$

□

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References

[1] Abdullayev FG, Andrievskii VV. On the orthogonal polynomials in the domains with K -quasiconformal boundary. Izvestiya Akademii Nauk Azerb. SSR Serıya Fiziko-Tekhnicheskikh i Matematicheskikh Nauk 1983; 4 (1): 7-11 (in Russian).

[2] Abdullayev FG. On some properties of the orthogonal polynomials over the region of the complex plane (Part III). Ukrainian Mathematical Journal 2001; 53 (12): 1934-1948. https://doi.org/10.1023/A:1015419521005

[3] Abdullayev FG, Değer U. On the orthogonal polynomials with weight having singularities on the boundary of regions in the complex plane. Bulletin of the Belgian Mathematical Society-Simon Stevin 2009; 16 (2): 235-250. https://doi.org/10.36045/bbms/1244038136

[4] Abdullayev FG, Aral ND. On Bernstein–Walsh-type lemmas in regions of the complex plane. Ukrainian Mathematical Journal 2011; 63 (3): 337-350. https://doi.org/10.1007/s11253-011-0507-y

[5] Abdullayev FG, Özkartepe P. Uniform and pointwise polynomial inequalities in regions with cusps in the weighted Lebesgue space. Jaen Journal on Approximation 2015; 7 (2): 231-261.

[6] Abdullayev FG, Özkartepe P. On the growth of algebraic polynomials in the whole complex plane. Journal of the Korean Mathematical Society 2015; 52 (4): 699-725. https://doi.org/10.4134/JKMS.2015.52.4.699

[7] Abdullayev FG, Özkartepe P. Uniform and pointwise Bernstein-Walsh-type inequalities on a quasidisk in the complex plane. Bulletin of the Belgian Mathematical Society-Simon Stevin 2016; 23 (2): 285-310. https://doi.org/10.36045/bbms/1464710119

[8] Abdullayev FG, Özkartepe P. Polynomial inequalities in Lavrentiev regions with interior and exterior zero angles in the weighted Lebesgue space. Publications de l’Institut Mathématique 2016; 100 (114): 209-227. https://doi.org/10.2298/PIM1614209A

[9] Abdullayev FG. Polynomial inequalities in regions with corners in the weighted Lebesgue spaces. Filomat 2017; 31 (18): 5647-5670. https://doi.org/10.2298/FIL1718647A

[10] Abdullayev FG, Tunç T, Abdullayev GA. Polynomial inequalities in quasidisks on weighted Bergman space. Ukrainian Mathematical Journal 2017; 69 (5): 675-695. https://doi.org/10.1007/s11253-017-1388-5

[11] Abdullayev FG, Gün CD. Bernstein-Nikolskii-type inequalities for algebraic polynomials in Bergman space in regions of complex plane. Ukrainian Mathematical Journal 2021; 73 (4): 513-531. https://doi.org/10.1007/s11253-021-01940-z

[12] Abdullayev FG, Gün CD. Bernstein-Walsh type inequalities for derivatives of algebraic polynomials. Bulletin of the Korean Mathematical Society 2022; 59 (1): 45-72. https://doi.org/10.4134/BKMS.b210023

[13] Abdullayev FG. Bernstein-Walsh-type inequalities for derivatives of algebraic polynomials in quasidisks. Open Mathematics 2021; 19 (1): 1847-1876. https://doi.org/10.1515/math-2021-0138

[14] Abdullayev FG, Imashkyzy M. On the growth of mth derivatives of algebraic polynomials in the weighted Lebesgue space. Applied Mathematics in Science and Engineering 2022; 30 (1): 249-282. https://doi.org/10.1080/27690911.2022.2056170

[15] Anderson JM, Gehring FW, Hinkkanen A. Polynomial Approximation in Quasidisks. In: Chavel I, Farkas HM (editors). Differential Geometry and Complex Analysis. Berlin, Heidelberg: Springer, 1985, pp. 75-86. $https://doi.org/10.1007/978-3-642-69828-6\_6$

[16] Andrashko MI. Inequalities for a derivative of algebraic polynomial in the metric $L_p, (p \geq 1)$ in domains with corner. Ukrainian Mathematical Journal 1964; 16: 439-444 (in Russian).

[17] Andrievskii VV, Belyi VI, Dzjadyk VK. Conformal Invariants in Constructive Theory of Functions of Complex Variable. Atlanta, GA, USA: World Federation Publisher, 1995.

[18] Andrievskii VV. Weighted L_p Bernstein-type inequalities on a quasismooth curve in the complex plane. Acta Mathematica Hungarica 2012; 135: 8-23. https://doi.org/10.1007/s10474-011-0155-6

[19] Andrievskii VV. Weighted polynomial inequalities in the complex plane. Journal of Approximation Theory 2012; 164 (9): 1165-1183. https://doi.org/10.1016/j.jat.2012.05.012

[20] Balc S, Imash-kyzy M, Abdullayev FG. Polynomial inequalities in regions with interior zero angles in the Bergman space. Ukrainian Mathematical Journal 2018; 70 (3): 362-384. https://doi.org/10.1007/s11253-018-1505-0

[21] Bernstein SN. Sur L'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné. Hayez, Bruxelles: Imprimeur des Académies Royales, 1912 (in French).

[22] Belinskii PP. General Properties of Quasiconformal Maps. Novosibirsk, Russia: Nauka, 1974 (in Russian).

[23] Ditzian Z, Prymak A. Nikol'skii inequalities for Lorentz spaces. Rocky Mountain Journal of Mathematics 2010; 40 (1): 209-223. https://doi.org/10.1216/RMJ-2010-40-1-209

[24] Ditzian Z, Tikhonov S. Ul'yanov and Nikol'skii-type inequalities. Journal of Approximation Theory 2005; 133 (1): 100-133. https://doi.org/10.1016/j.jat.2004.12.008

[25] Dzjadyk VK. Introduction to the Theory of Uniform Approximation of Functiıon by Polynomials. Moskow, Russia: Nauka, 1977 (in Russian).

[26] Jackson D. Certain problems on closest approximations. Bulletin of the American Mathematical Society 1933; 39: 889-906. https://doi.org/10.1090/S0002-9904-1933-05759-2

[27] Lehto O, Virtanen KI. Quasiconformal Mapping in the Plane. Berlin, Germany: Springer Verlag, 1973.

[28] Mamedhanov DI. Inequalities of S.M. Nikol'skii type for polynomials in the complex variable on curves. Soviet Mathematics-Doklady 1974; 15: 34-37.

[29] Mamedkhanov JI, Dadashova IB. S. N. Bernstein type estimations in the mean on the curves in a complex plane. Abstract and Applied Analysis 2009; 2009: 1-19. https://doi.org/10.1155/2009/165194

[30] Markov AA. On a problem of D.I. Mendeleev. Zapishi Imperatorskoy Akademii Nauk 1889; 62: 1-24 (in Russian).

[31] Milovanovic GN, Mitrinovic DS, Rassias ThM. Topics in Polynomials: Extremal Problems, Inequalities, Zeros. Singapore: World Scientific, 1994.

[32] Nevai P, Totik V. Sharp Nikolskii inequalities with exponential weights. Analysis Mathematica 1987; 13 (4): 261-267. https://doi.org/10.1007/BF01909432

[33] Nikol'skii SM. Approximation of Function of Several Variable and Imbeding Theorems. New York, NY, USA: Springer-Verlag, 1975.

[34] Özkartepe NP, Tunç T, Abdullayev FG. Uniform and pointwise estimates for algebraic polynomials in regions with interior and exterior zero angles. Filomat 2019; 33 (2): 403-413. https://doi.org/10.2298/FIL1902403O

[35] Özkartepe P, Imashkyzy M, Abdullayev FG. Bernstein-Nikolskii-Markov-type inequalities for algebraic polynomials in a weighted Lebesgue space. Filomat 2023; 37 (17): 5701-5715. https://doi.org/10.2298/FIL2317701O

[36] Özkartepe P, Abdullayev FG. On the growth of m-th derivatives of algebraic polynomials in regions with corners in a weighted Bergman space. Proceedings of the Institute of Mathematics and Mechanics National Academy of Sciences of Azerbaijan 2023; 49 (1): 109-153. https://doi.org/10.30546/2409-4994.2023.49.1.109

[37] Pommerenke Ch. Univalent Functions. Göttingen, Germany: Vandenhoeck & Ruprecht, 1975.

[38] Pritsker IE. Comparing norms of polynomials in one and several variables. Journal of Mathematical Analysis and Applications 1997; 216 (2): 685-695. https://doi.org/10.1006/jmaa.1997.5699

[39] Özkartepe P, Abdullayev FG. The uniform and pointwise estimates for polynomials on the weighted Lebesgue spaces in the general regions of complex plane. Hacettepe Journal of Mathematics and Statistics 2019; 48 (1): 87-101. https://doi.org/10.15672/HJMS.2018.639

[40] Özkartepe NP, Gün CD, Abdullayev FG. Bernstein-Walsh-type inequalities for derivatives of algebraic polynomials on the regions of complex plane. Turkish Journal of Mathematics 2022; 46: 2572-2609. https://doi.org/10.55730/1300-0098.3289

[41] Suetin PK. The ordinally comparison of various norms of polynomials in the complex domain. Matematicheskie Zapiski Ural'skogo Gosudarstvennogo Universiteta 1966; 5 (4): 91-100 (in Russian).

[42] Suetim PK. Bounds for polynomials orthogonal on a contour when the contour and weight function can have singularities. Siberian Mathematical Journal 1967; 8: 817-823. https://doi.org/10.1007/BF01040657

[43] Szegö G, Zygmund A. On certain mean values of polynomials. Journal d'Analyse Mathématique 1953; 3 (1): 225-244. https://doi.org/10.1007/BF02803592