Turkish Journal of Mathematics

Volume 48 | Number 4 Article 5

7-3-2024

Stacks in Einstein gravity

KADRİ İLKER BERKTAV

Recommended Citation

BERKTAV, KADRİ İLKER (2024) "Stacks in Einstein gravity," *Turkish Journal of Mathematics*: Vol. 48: No. 4, Article 5. https://doi.org/10.55730/1300-0098.3534
Available at: https://journals.tubitak.gov.tr/math/vol48/iss4/5

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 673 – 700
© TÜBİTAK
doi:10.55730/1300-0098.3534

Stacks in Einstein gravity

Kadri İlker BERKTAV*
Institute of Mathematics, University of Zurich, Zurich, Switzerland

Received: 24.07.2023 • **Accepted/Published Online:** 15.03.2024 • **Final Version:** 03.07.2024

Abstract: In this paper, we examine stacky structures in certain Einstein gravity theories. In brief, using the classical formulation of (vacuum) gravity, with vanishing cosmological constant, we first construct the stack of solutions to Einstein field equations on any given fixed manifold. Using a similar approach and setup, we also study Einstein's gravity on families of manifolds and define another stack encoding this situation. Later on, we focus on the gauge theoretical interpretation of 3D gravity and provide a natural stack associated with that interpretation. Finally, in a particular setup, we give a natural morphism between the two stacks arising from different descriptions of 3D gravity.

Key words: Stacks, classical 3D Einstein gravity, Cartan's formalism, 3D gravity as a gauge theory

1. Introduction

Stacks are interesting higher spaces that appear in geometry and physics. Regarding physics-related problems, for example, Costello and Gwilliam [8] study gauge theories and factorization algebras in the context of derived algebraic geometry (DAG), which is a handy framework that combines algebraic geometry with homotopy theory using a higher categorical dictionary. In that respect, it offers new ways of organizing information for various purposes [1, 15]. Let us go back to the examples of interest: Benini et al. [3] describe a stacky formulation of Yang–Mills fields on Lorentzian manifolds; Benini and Schenkel [2] examine higher structures in algebraic quantum field theory; and Ludewig and Stoffel [12] study geometric functorial field theories. This is, of course, not a complete list. There are many other interesting examples on stacks and neighboring subjects in the literature.

The current work is centered around the fact that the phase spaces of our interest have the structure of a *groupoid*, rather than a *set*. To be more specific, for ordinary field theories, the collection of fields have the structure of a *set*, and hence two fields f, f' are said to be the *same* if and only if the equation $f = f'$ holds set theoretically. However, for gauge theories, two gauge fields A, A' are the "same" if there exists a gauge transformation $g : A \to A'$ relating them.

Due to the extra data mentioned above, points in the corresponding phase space naturally form a groupoid: i.e. the data should include the points (the *fields* of our theory), along with invertible (gauge) transformations between them. Consequently, the phase space of a gauge theory turns out to be a "higher space" (called a *stack*) rather than an "ordinary space". More details can be found in [2, 3].

*Correspondence: kadriilker.berktav@math.uzh.ch
2020 *AMS Mathematics Subject Classification:* 14A20, 14A30, 14D23, 18F20, 70S15, 81T35, 83C99

Of course, one can naturally ask for similar kinds of relations between gauge transformations themselves. For instance, if there are gauge transformations between gauge transformations, then the underlying structure of the collection of points will be encoded by "2-groupoids". One can play the same game for these "2nd level transformations" and ends up with 3-groupoids, etc. Using a higher categorical dictionary, this essentially leads to the notion of an *infinite tower of equivalences.* Therefore, if we allow higher symmetries in gauge theory, the natural framework will be encoded by ∞-groupoids; hence, the corresponding phase space becomes a *higher stack.* For details, we again refer to [2, 3].

It should be clear by now why it is natural to investigate similar structures in Einstein's theory of general relativity: *Once symmetries are involved as a part of the data, one should interpret phase spaces as higher spaces, rather than just ordinary spaces.* This slogan can eventually lead to a new way of formalizing the data and make certain higher algebraic tools available. In this paper, we only consider "first" level symmetries of the theory. Therefore, stacks naturally enter the picture, and they are good enough to encode the underlying structure of the phase space. In short, *stacks are good enough for our purposes*, and thus, we concentrate on stacky constructions for Einstein's theory of gravity.

In this paper, we give some nontrivial "stacky" constructions in the case of certain gravity theories and investigate their possible consequences. In short, we define the stacks of Ricci-flat metrics on a fixed manifold and on families of manifolds; a gauge theoretical stack of 3D gravity; and a natural transformation between the two stacks arising from the different models of 3D gravity.

Let us report our results in detail. Using the homotopy theory of stacks (cf. §2.1), we first give an elementary construction of the so-called *moduli stack of vacuum Einstein gravity on a Lorentzian spacetime with vanishing cosmological constant.* More precisely, we prove the following theorem.

Theorem 1.1 *Given a Lorentzian n-manifold M, let $\mathcal{C}$ be the category of open subsets of M that are diffeomorphic to $\mathbb{R}^n$, with morphisms being canonical inclusions between open subsets whenever $U \subset V$. Then the presheaf $\mathcal{E} \in PSh(\mathcal{C}, Grpds)$*

$$\mathcal{C}^{op} \longrightarrow Grpds, \quad U \mapsto \mathcal{E}(U)$$

is a stack of Ricci-flat Lorentzian metrics on $\mathcal{C}$, where for an object U of $\mathcal{C}$, $\mathcal{E}(U)$ is a groupoid such that the objects *of $\mathcal{E}(U)$ form the set $Ob(\mathcal{E}(U)) := \big\{ g \in \Gamma(Sym^2(T^*U)) : Ric(g) = 0 \big\}$, and a* morphism *in $\mathcal{E}(U)$ is determined by an automorphism of $Sym^2(T^*U)$.*

Here, *Grpds* denotes the 2-category of groupoids. Roughly speaking, $\mathcal{E}$ is a prestack *(a presheaf of groupoids)* that preserves certain structures and possesses the descent property. The precise description of $\mathcal{E}$, as a prestack, is given in Lemma 3.1, while the descent property and the site structure are discussed in §3.1.

Theorem 1.1 provides a suitable stack that in fact captures the *contravariance and locality* behaviors of the Ricci-flat geometric structure on the underlying manifold M. On the other hand, in the context of moduli theory, it is natural to study *smoothly varying families of manifolds* as well. Therefore, we also investigate Ricci-flat Lorentzian metrics on families of manifolds and define a new stack encoding this situation.

To be more specific, we require geometric structures to vary in families, parametrized over cartesian spaces. In brief, this can be achieved by replacing the category $\mathcal{C}$ in Theorem 1.1 by the site Fam_n of families of manifolds, where its objects are submersions $\pi : M \to S$, with n-dimensional fibers, and morphisms are fiberwise open embeddings. With this modification, we prove the following result (cf. §3.2).

Theorem 1.2 *Let Fam_n be the site of families of manifolds (with n-dimensional fibers). Denote an object of Fam_n by M/S. Then the presheaf $\mathcal{E}^{fam}$ on Fam_n*

$$Fam_n^{op} \longrightarrow Grpds, \;\; M/S \mapsto \mathcal{E}^{fam}(M/S)$$

is a stack, where for each object M/S in Fam_n, $\mathcal{E}^{fam}(M/S)$ is a groupoid such that its objects *form a set $\{g \in \Gamma(Sym^2(T^*(M/S))) : Ric(g) = 0\}$, with* morphisms *determined by certain automorphisms of $Sym^2(T^*(M/S))$. Note that $T^*(M/S)$ denotes the relative cotangent bundle $T^*(M/S) = Coker(T^*S \to T^*M)$.*

Additionally, we examine the so-called "equivalence" of 3D gravity with gauge theory. Our setup consists of vacuum 3D Einstein gravity (with vanishing cosmological constant Λ) on Lorentzian spacetimes of the form $M := \Sigma \times \mathbb{R}$, where Σ is a closed Riemann surface of genus $g > 1$.

Let us denote the aforementioned gravity theory by $GR^{3D}_{\Lambda=0}(M)$ and the corresponding gauge theory by $CS^{3D}_{ISO(2,1)}(M)$. By *equivalence*, we essentially mean the existence of an isomorphism between the phase spaces of these theories (i.e. moduli spaces of solutions to the corresponding field equations)

$$Mod(GR^{3D}_{\Lambda=0}(M)) \xrightarrow{\sim} Mod(CS^{3D}_{ISO(2,1)}(M)),$$

which sends a flat pseudo-Riemannian metric $[g]$ to the corresponding flat gauge field $[A^g]$. More details will be discussed in subsection 2.2.3, but the upshot is that once there exists such an equivalence on the classical level, one can construct a natural stack morphism between the stacks of these theories. Here, by a *stack of a theory*, we mean the stack of solutions to the corresponding field equations of the theory under consideration. This approach essentially encodes nontrivial stacky structures on top of the naïve moduli spaces of solutions and then provides a stacky extension for the map between these moduli spaces. In this regard, we prove the following result.

Theorem 1.3 *Suppose that $M = \Sigma \times (0, \infty)$ is a Lorentzian 3-manifold, where Σ is a closed Riemann surface of genus $g > 1$. Let $\mathcal{E}$ and $\mathcal{M}$ denote the moduli stacks of $GR^{3D}_{\Lambda=0}(M)$ and $CS^{3D}_{ISO(2,1)}(M)$, respectively. Then there exists an induced natural transformation $\Phi : \mathcal{M} \Rightarrow \mathcal{E}$ (cf. Construction (3.13)).*

Now, let us outline the remainder of this paper. Section 2 includes preliminaries. It begins by reviewing Hollander's study [9] on the homotopy theory of stacks. In subsection 2.2, we discuss different formulations of 3D gravity and their consequences. In subsection 3.1, we first present an elementary construction of the moduli prestack of Einstein gravity (cf. Lemma 3.1). Then we give the proof of Theorem 1.1 using the homotopy theory of stacks. In subsection 3.2, we explain the content of Theorem 1.2 in more detail and give a sketch of the proof. Finally, subsection §3.3 provides the proof of Theorem 1.3. We also have Appendices A and B to support some ideas in the text.

2. Recollection

2.1. Background from the homotopy theory of stacks

It is very well-known that by Yoneda's embedding, one can consider spaces as *functors* in addition to the standard ringed-space formulation [15]. In this paper, we follow the same approach to define stacks. More precisely, we work within the context of Hollander's theory of stacks [9]. In what follows, we present some key notions and constructions. We mostly follow [3, 9].

Groupoids. Recall that a *groupoid* is a category in which all morphisms are isomorphisms. Since each groupoid is a category itself (i.e. it has own objects and morphisms between any pair of objects with a list of axioms), the collection of groupoids has the structure of a *2-category*.

By a *2-category* $\mathcal{C}$, we mean a *category enriched over the cartesian monoidal category* Cat, where Cat is the category with small categories as objects and with functors as morphisms. Thus, $\mathcal{C}$ has a collection of objects, and for each pair of objects A, B, the mapping space $Hom(A,B)$ has the structure of a category, rather than a set. In that case, we call the objects of $Hom(A,B)$ *1-morphisms of* $\mathcal{C}$, and call the morphisms of $Hom(A,B)$ *2-morphisms of* $\mathcal{C}$; and all relations are up to 2-isomorphisms*. Then we have the following example.

Example 2.1 *(*2-category of groupoids $Grpds$*) Objects of* $Grpds$ *are just groupoids; 1-morphisms in* $Grpds$ *are functors* $\mathcal{F}: G \to H$ *between two groupoids; and 2-morphisms are natural transformations* $\eta: \mathcal{F} \Rightarrow \mathcal{G}$ *of functors, where* $\mathcal{F}, \mathcal{G} \in Fun(G,H)$. *In this example, there are no nontrivial higher* n*-morphisms for* $n > 2$. *Once we allow such types of morphisms, we land in the territory of higher categories.*

Groupoids form a model category. The other important feature of $Gpds$ is that one can do homotopy theory with groupoids. This is possible because $Gpds$ has a suitable structure, the *model structure*, which makes it a *model category*. In brief, a *model structure*† consists of three distinguished classes of morphisms, namely *weak equivalences*, *fibrations*, and *cofibrations* with a big list of axioms, see [9, 10]. Then we have:

Theorem 2.2 *The 2-category* $Gpds$ *admits a* **model structure**, *where*

1. *A morphism* $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ *in* $Gpds$ *is a* **weak equivalence** *if it is fully faithful and essentially surjective.*
2. *A morphism* $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ *in* $Gpds$ *is a* **fibration** *if for each object* A *in* $\mathcal{C}$ *and each morphism* $\varphi: \mathcal{F}(A) \xrightarrow{\sim} D$ *in* $\mathcal{D}$, *there exist an object* B *and a morphism* $f: A \xrightarrow{\sim} B$ *in* $\mathcal{C}$ *such that* $\mathcal{F}(f) = \varphi$. *A morphism* $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ *is a* **cofibration** *if it is injective on objects.*

Homotopy limits in groupoids. Hollander [9] provides simple and tractable models for the *homotopy limits* of a cosimplicial diagram in $Grpds$. The key observation of [9] is that the homotopy approach encodes the classical descent conditions for stacks in a compact way. Let us start with some terminology.

Definition 2.3 *Denote by* Δ *the category of finite ordered sets, where objects are finite ordered sets* $[n] := \{0 < 1 < 2 < \cdots < n\}$ *and morphisms are* $f: [n] \to [m]$ *nondecreasing functions. Given a category* $\mathcal{C}$, *a* **cosimplicial object in a category** $\mathcal{C}$ *is a functor* $X_\bullet : \Delta \longrightarrow \mathcal{C}$, *Denote the image by* $X_\bullet([n]) =: X_n$.

Definition 2.4 *Given a cosimplicial object* $X_\bullet$ *in* $\mathcal{C}$, *one obtains a sequence of objects* $\{X_n\}$ *in* $\mathcal{C}$, *together with the morphisms* $X_\bullet(d_i^n): X_{n-1} \longrightarrow X_n$ *and* $X_\bullet(s_i^n): X_{n+1} \longrightarrow X_n$, *where* d_i^n *and* s_i^n *are the usual coface and codegeneracy maps, respectively. Then by abusing the notation and omitting the codegeneracy maps, we define the* **cosimplicial diagram in** $\mathcal{C}$ *by*

$$X_\bullet = \Big(X_0 \rightrightarrows X_1 \overset{\textstyle\rightarrow}{\rightrightarrows} X_2 \overset{\textstyle\rightrightarrows}{\rightrightarrows} \cdots \Big). \tag{2.1}$$

*That is, a 2-category is a higher category, where on top of the objects and morphisms, there are also 2-morphisms.

†With a model structure, which was originally defined by Quillen, one can localize the given category $\mathcal{C}$ by formally inverting a special class of morphisms, the *weak equivalences*, and define the *homotopy category* of $\mathcal{C}$.

In the case of *Grpds*, from [9, Corollary 2.11], we have the following result regarding the homotopy limits of a cosimplicial diagram in *Grpds*, which will be useful to formulate the classical definition of stack in the language of homotopy theory.

Lemma 2.5 *(Homotopy limits in groupoids) Given a cosimplicial diagram $X_\bullet$ in Grpds*

$$X_\bullet = \Big(X_0 \rightrightarrows X_1 \substack{\rightarrow\\\rightarrow\\\rightarrow} X_2 \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \Big),$$

where each X_i is a groupoid, then the homotopy limit $\mathbf{holim}_{\mathbf{Grpds}}(\mathbf{X}_\bullet)$ of a cosimplicial diagram $X_\bullet$ is a groupoid for which

(i) **objects** *are the pairs (x,h), where x is an object in X_0, $h : d_1^1(x) \to d_0^1(x)$ is a morphism in X_1 such that*

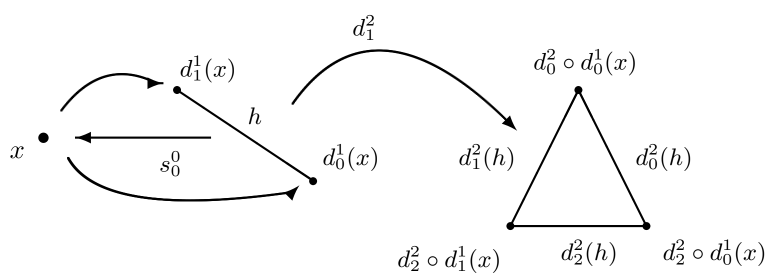
$$(a) \quad s_0^0(h) = id_x, \tag{2.2}$$

$$(b) \quad d_0^2 \circ d_2^2(h) = d_1^2(h). \tag{2.3}$$

Note that x and h can be realized as 0- and 1-simplicies in $X_\bullet$, respectively, such that, by using the properties of d_i^n and s_j^n, those conditions correspond to the commutativity of the diagram

$$\begin{tikzcd}
d_2^2 \circ d_1^1(x) \arrow[r, "d_2^2(h)"] \arrow[d, "“=”"] & d_2^2 \circ d_0^1(x) = d_0^2 \circ d_1^1(x) \arrow[r, "d_0^2(h)"] & d_0^2 \circ d_0^1(x) \arrow[d, "“=”"] \\
d_1^2 \circ d_1^1(x) \arrow[rr, "d_1^2(h)"] & & d_1^2 \circ d_0^1(x);
\end{tikzcd}$$

hence, we geometrically have



(ii) **morphisms** *are the arrows of pairs $(x,h) \to (x',h')$ that consist of a morphism $f : x \to x'$ in X_0 such that the following diagram commutes.*

$$\begin{tikzcd}
d_1^1(x) \arrow[r, "d_1^1(f)"] \arrow[d, "h"'] & d_1^1(x') \arrow[d, "h'"] \\
d_0^1(x) \arrow[r, "d_0^1(f)"'] & d_0^1(x')
\end{tikzcd}$$

Here, d_i^n's are in fact covariant functors between groupoids.

Stacks as homotopy sheaves. Classically, stacks were defined either as categories fibered in groupoids or lax presheaves satisfying the descent conditions.

Denote by $PSh(\mathcal{C}, Grpds)$ the *category of presheaves of groupoids on* $\mathcal{C}$. Instead of the classical definitions above, Hollander [9] first proved that it is enough to work with actual objects in $PSh(\mathcal{C}, Grpds)$. Then it was proven in [9] that the homotopy sheaf condition is equivalent to the descent conditions; hence, stacks can be described as *homotopy sheaves*.

The desired sheaf condition is in fact based on the model structure on $PSh(\mathcal{C}, Grpds)$. It has been shown in [9] that there exists a suitable model structure on $PSh(\mathcal{C}, Grpds)$ such that stacks arise as the *fibrant* objects in the model structure on $PSh(\mathcal{C}, Grpds)$. More precisely, we have the following definitions/theorems [9]:

Definition 2.6 *Let $\mathcal{C}$ be a category with a Grothendieck topology, a* site. *An object $\mathcal{X} \in PSh(\mathcal{C}, Grpds)$ in the model structure is* **fibrant** *if for each covering family $\{U_i \to U\}$ of U in $\mathcal{C}$, the canonical morphism*

$$\mathcal{X}(U) \longrightarrow \mathrm{holim}_{Grpds}\big(\mathcal{X}(U_\bullet)\big) \tag{2.4}$$

is a weak equivalence in Grpds, where $\mathcal{X}(U_\bullet)$ is the cosimplicial diagram in Grpds

$$\mathcal{X}(U_\bullet) := \Big(\prod_i \mathcal{X}(U_i) \rightrightarrows \prod_{ij} \mathcal{X}(U_{ij}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{ijk} \mathcal{X}(U_{ijk}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots\Big),$$

such that holim_{Grpds} *is the homotopy limit in Lemma 2.5; and $U_{i_1 i_2 \ldots i_m}$ is the fibered product of U_{i_n}'s in U.*

Definition 2.7 *Let $\mathcal{C}$ be a site. A presheaf of groupoids $\mathcal{X}$ on $\mathcal{C}$ is called a* **stack** *if it is fibrant.*

Example 2.8 (Manifolds as stacks) *Denote by $Cart$ the* category of cartesian spaces*, where an object is an open subset of $\mathbb{R}^n$ that is diffeomorphic to $\mathbb{R}^n$, and morphisms are smooth maps. To turn $Cart$ into a site, we use open covers for which every intersection of those open subsets U_i's in U is either empty or diffeomorphic to $\mathbb{R}^n$. Let $\mathcal{C} = Man_n$, the category of n-manifolds, then any manifold M can be considered as a functor*

$$\mathcal{F}_M : Cart^{op} \to Sets \subset Grpds, \; U \mapsto \mathcal{F}_M(U) := C^\infty(U, M),$$

where the set $C^\infty(U, M)$ is a groupoid with objects being the elements of $C^\infty(U, M)$ and morphisms being just identities. Since $C^\infty(-, M)$ is a sheaf on $Cart$, the functor $\mathcal{F}_{(-)} : Man_n \to PSh(Cart, Grpds)$ is fully faithful and takes values in stacks. Thus, manifolds can be seen as particular stacks. For details, see [3, § 2.3].

Example 2.9 (Classifying stack of G-bundles) *Given a Lie group G, define an object BG in $PSh(Cart, Grpds)$ as the functor $BG : Cart^{op} \to Grpds,\; U \mapsto BG(U)$, such that $BG(U)$ is a groupoid with one object $\{*\}$ and morphisms are from $C^\infty(U, G)$. The composition in $BG(U)$ is given as the pointwise product. Using similar arguments as before, BG becomes a stack. For details, see [3, § 2.3].*

Example 2.10 (Quotient stacks) *Let X be a stack and G a group object in stacks acting on X. Then we can define the* quotient stack *of X by G as the following (homotopy) colimit of the simplicial diagram:*

$$[X/G] = \mathrm{colim}\Big(X \leftleftarrows X \times G \substack{\leftarrow\\\leftarrow\\\leftarrow} X \times G \times G \substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow} \cdots\Big) \tag{2.5}$$

where the maps are given by the action and projection. When $X = \{*\}$, the classifying stack in Example 2.9 can be recovered as $BG = [\{\star\}/G]$ with the simplicial diagram coming from the nerve of the groupoid $BG(\{*\})$.

2.2. Overview of different formulations of 3D gravity

In this section, we briefly outline different formulations 3D Einstein gravity: the metric formalism; a geometric description via model spacetimes; and the Cartan formalism. We also study infinitesimal symmetries (see Appendix A) and a relation between 3D gravity and gauge theory. We essentially follow [6, 7, 14].

2.2.1. The metric formalism of 3D gravity and a geometrical description

Definition 2.11 *Let M be a manifold. A* **classical field theory** *on M, in the sense of Lagrangian formalism, consists of a piece of data $(F_M, \mathcal{S}, G; crit(\mathcal{S}))$, where F_M denotes the* space of fields *on the base manifold M; $\mathcal{S}$ is a* smooth action functional *on F_M; G is a certain* group encoding the symmetries *of the system; and $crit(\mathcal{S})$ is the* critical locus $crit(\mathcal{S})$ *of $\mathcal{S}$. We call the defining equations for $crit(\mathcal{S})$ the* field equations.

In the sense of Definition 2.11, *3D gravity with vanishing cosmological constant* consists of the following data. The metric tensor is the fundamental *field* of study. Given a 3-manifold M, we have $G := ISO(M)$, with the usual pullback action; and the *Einstein-Hilbert action* for the metric is given as

$$\mathcal{I}_{EH}[g] := \kappa \int_M \mathrm{dx}^3 R\sqrt{-det(g)}. \tag{2.6}$$

Here, κ is some constant, R is the Ricci scalar, g is the metric tensor field, and $det(g)$ denotes the determinant of the metric tensor matrix. Then the *vacuum Einstein field equations*, with cosmological constant $\Lambda = 0$, are given as[‡]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{2.7}$$

Observe that after contracting with $g^{\mu\nu}$, one has $R = 0$. Therefore, from substituting this back into Equation (2.7), we get $R_{\mu\nu} = 0$. Then we introduce the following definition.

Definition 2.12 *Denote by $\mathcal{EH}(M)$ the moduli space of solutions to the field equations above, then $\mathcal{EH}(M)$ is the* moduli space of Ricci-flat Lorentzian metrics on M *(i.e. $R_{\mu\nu} = 0$).*

Remark 2.13 *It should also be noted that, Weyl tensor in 3D is identically zero. Thus, the curvature tensor of a 3-manifold is determined completely by its Ricci tensor. Therefore,* any solution of the vacuum Einstein field equations (2.7) in 3D, with vanishing cosmological constant, is locally flat. *In physics, we then say 3D gravity is a theory without local gravitational degrees of freedom. It means it has curvature only where there is matter, and there are no gravitational waves [11].*

By Remark 2.13, the critical locus $\mathcal{EH}(M)$ can be seen as the moduli space of *flat geometric structures on M*. We will not formally discuss the notion of a geometric structure in detail, but with this interpretation in hand, one can equivalently say that for each vacuum solution g to 3D Einstein field equations, (M, g) is locally modeled on $(ISO(2,1), \mathbb{R}^{2,1})$, where $\mathbb{R}^{2,1}$ denotes the usual Minkowski spacetime. In fact, we have a one-to-one correspondence (cf. [6])

$$\zeta : \mathcal{EH}(M) \longrightarrow \Big\{ (ISO(2,1), \mathbb{R}^{2,1}) \text{ structures on } M \Big\}. \tag{2.8}$$

[‡]More generally, the Einstein-Hilbert action for gravity coupled to matter, with nonvanishing cosmological constant Λ is of the form $\mathcal{I}_{EH}[g] := \kappa \int_M \mathrm{dx}^3 (R - 2\Lambda)\sqrt{-det(g)} + \int_{\text{matter}}$ with a constant κ. Then the field equations are $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - 2\Lambda) = -\ell T_{\mu\nu}$ for some constant ℓ and $T_{\mu\nu}$ the energy-momentum tensor.

More precisely, the map (2.8) sends (equivalence classes) of flat Lorentzian metrics to the induced flat geometric structure $(ISO(2,1), \mathbb{R}^{2,1})$ on M. The converse map is clear since such structure carries a locally flat metric. In this paper, we are mostly interested in *Lorentzian vacuum 3D gravity with* $\Lambda = 0$.

In general, the geometric description of (vacuum) 3D gravity can be given by (quotients of) certain model spacetimes [6]. When $\Lambda \neq 0$, the vacuum field equations are

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu}. \tag{2.9}$$

In that case, Remark 2.13 also implies that any vacuum solution to the Einstein equations with general cosmological constant (2.9) will give a geometric structure of *constant curvature*. In this regard, the symmetry groups of gravitational interest and the corresponding model spacetimes are as follows:

$$(G, X_\Lambda) := \begin{cases} (SO(3,1), dS_\Lambda), & \Lambda > 0 \text{ (de Sitter)} \\ (ISO(2,1), \mathbb{R}^{2,1}), & \Lambda = 0 \text{ (Minkowski)} \\ (SO(2,2), AdS_\Lambda), & \Lambda < 0 \text{ (Anti de Sitter)}, \end{cases} \tag{2.10}$$

where the stabilizer group in each case is the Lorentz group $H = SO(2,1)$. We then introduce the following definition.

Definition 2.14 *(Geometric description for 3D gravity) Let* $\Lambda \in \mathbb{R}$ *and* M *be a base manifold. A* **Lorentzian (vacuum) 3D gravity for** Λ *is a classical field theory on* M *in the sense of Definition 2.11 such that the space* $\mathcal{EH}(M, \Lambda)$ *of vacuum solutions to 3D Einstein field equations is equivalent to the space of geometric structures on* M *w.r.t the models in (2.10). Note that when* $\Lambda = 0$*, we have the map* ζ *in (2.8) and* $\mathcal{EH}(M, 0) =: \mathcal{EH}(M)$*.*

Note that the geometric description of 3D gravity in Definition 2.14 is in fact an example of Thurston's geometric structures in the case of a $(ISO(2,1), \mathbb{R}^{2,1})$ structure [6]. In general, a (G, X)-manifold is an n-dimensional manifold M locally modeled on X, the *model space* equipped with a G-action, just as an ordinary manifold is modeled on $\mathbb{R}^n$ equipped with a GL_n-action. For details, we refer to [6, 7].

2.2.2. The first-order formalism of 3D gravity

The action functional of gravity was originally given as a functional on the space of Lorentzian metrics as we introduced in subsection 2.2.1. Later, it was shown that the action may alternatively be given in a more general form as a functional on the space of connections with values in the Poincaré Lie algebra $iso(2,1)$. This version is called the *Eintein-Cartan gravity*, which also has the advantage of being interpreted as a gauge theory. In what follows, we discuss the basics of Eintein-Cartan gravity in the context of Cartan geometry.

Some background material on Cartan geometry can be found in [16]. For the gravitational interpretation, let us assume w.l.o.g that the underlying 3-manifold M has a topology of the form $\mathbb{R} \times \Sigma$, with Σ a closed oriented surface, and $G = ISO(2,1) = SO(2,1) \ltimes \mathbb{R}^{2,1}$. Then we have the following definition.

Definition 2.15 *Let* M, G *be as above. The* **Eintein-Cartan theory of gravity** *for the pair* (M, G) *consists of a* G*-frame bundle* $LM \xrightarrow{\pi} M$ *over* M*, a Cartan connection* $A \in \Omega^1(LM, \mathfrak{iso}(2,1))$*, and the action in (2.11), such that* A *is expressed uniquely as a decomposition* $A = \omega + e$*, where* $\omega \in \Omega^1(LM, \mathfrak{so}(2,1))$ *is the* $\mathfrak{so}(2,1)$*-valued Ehresmann 1-form on* LM *(the* spin connection*) and* $e \in \Omega^1(LM, \mathbb{R}^{2,1})$ *is the* coframe field *(or* triad*).*

The *Einstein-Hilbert action in triad-spin-connection language* can be defined as

$$\mathcal{I}'_{EH}[e,\omega] = \int\limits_M e^a \wedge \left(\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \right). \tag{2.11}$$

Here $e^a = e^a_\mu \mathrm{d}x^\mu$ and $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}\mathrm{d}x^\mu$, where $\mu, \nu, \dots$ label the *space indices* with respect to a local chart; and the others $a, b, \dots$ are the *Lorentz indices*. Let P^a and J^a, for $a = 0,1,2$, be the generators of $iso(2,1)$ corresponding to *translations* and *Lorentz generators*, respectively, with the structure relations

$$[J^a, J^b] = \epsilon^{abc}J_c, \quad [J^a, J^b] = \epsilon^{abc}P_c, \quad [P^a, P^b] = 0.$$

Then we can write $\omega = \omega^a J_a$ and $e = e^a P_a$ such that $A = \omega + e$. Notice that the action (2.11) is invariant under both local $SO(2,1)$ transformations

$$\delta e^a = \epsilon^{abc}e_b\tau_c \;\text{ and }\; \delta\omega^a = d\tau^a + \epsilon^{abc}\omega_b\tau_c, \tag{2.12}$$

and local translations

$$\delta e^a = d\rho^a + \epsilon^{abc}\omega_b\rho_c \;\text{ and }\; \delta\omega^a = 0. \tag{2.13}$$

Now, *Einstein field equations in triad-spin-connection language* are obtained as

$$T^a = \mathrm{d}e_b + \epsilon^{abc}\omega_b \wedge e_c = 0, \tag{2.14}$$

$$\Omega[\omega]^a = \mathrm{d}\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c = 0. \tag{2.15}$$

It means any vacuum solutions to Einstein field equations must have vanishing torsion and curvature. In fact, these equations have the following consequences [6, 7]:

1. The triad can be used to define a Lorentzian metric $g(e)$ via $g(e)_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and $g(e)^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab}$, where η denotes the usual Minkowski metric.

2. We can obtain ω as a function $\omega[e]$ of e by solving Equation (2.14). If we substitute $\omega[e]$ into Equation (2.15), the resulting equations will be equal to the ordinary vacuum Einstein field equations $R_{\mu\nu}[g(e)] = 0$ for the Lorentzian metric $g(e)$ defined by the triad. As discussed before, such metrics in 3D are flat. Therefore, the space of solutions to the field equations for $\mathcal{I}'_{EH}[e,\omega]$ can thus be identified with the set of flat Lorentzian metrics on M.

3. (**3D gravity as a Chern-Simons gauge theory**) It has been shown in [17] that the pair (e,ω) can be combined into an actual gauge field A, a *Lie algebra-valued connection 1-form*, with the gauge group $ISO(2,1)$. Note that using the generators P^a, J^a of $iso(2,1)$, for $a = 0,1,2$, given as above, we can define an invariant nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ on $iso(2,1)$ by

$$\langle J_a, P_b\rangle = \eta_{ab} \quad \langle J_a, J_b\rangle = \langle P_a, P_b\rangle = 0,$$

and introduce the gauge field A as $A := P_a e^a + J_a\omega^a$. [17] shows that the action in (2.11) is equal to the Chern-Simons action $\mathcal{I}_{CS}$ for A, with the gauge group $G = ISO(2,1)$ and the bilinear form $\langle\cdot,\cdot\rangle$ above, where

$$\mathcal{I}_{CS}[A] = \int\limits_M \langle A, \mathrm{d}A + \frac{2}{3}A \wedge A\rangle. \tag{2.16}$$

The gauge group acts naturally on the space of $ISO(2,1)$-connections: For $\rho \in G$ and a connection A, we have $A \bullet \rho := \rho^{-1} \cdot A \cdot \rho + \rho^{-1} \cdot \mathrm{d}\rho$. The corresponding field equation in this case turns out to be

$$F_A = \mathrm{d}A + A \wedge A = 0.$$

In brief, the field equations now reduce to the requirement for A to be flat, and the gauge transformations (2.12)–(2.13) can be identified with standard $ISO(2,1)$ transformations. Note that spacetime diffeomorphisms do not correspond to independent gauge symmetries. Let us explain the situation.

Remark 2.16

1. *$\mathcal{I}'_{EH}[e,\omega]$ is also invariant under the action $Diff(M)$. But, it has been shown by Witten in [17] that diffeomorphisms in the connected component of the identity are equivalent to transformations combining local Lorentz transformations and local translations mentioned above. In other words, when we identify the phase space of 3D gravity with that of the associated 3D Chern-Simons theory,* infinitesimal Chern-Simons gauge transformations are equivalent to infinitesimal diffeomorphisms.

2. *The aforementioned equivalence does not hold for "large" diffeomorphisms, i.e. those are not infinitesimally generated. Large diffeomorphisms in fact require different treatment, and they are important for the quantum theory [11]. Therefore, when we discuss an equivalence between some transformations, we always consider them "infinitesimally generated". (See also Appendix A.)*

2.2.3. Equivalence of 3D gravity with gauge theory

In this section, we will establish a relation between a Lorentzian 3D gravity on M for $\Lambda = 0$ and the space of gauge-equivalence classes of flat $ISO(2,1)$-connections on M. For simplicity, assume also that M has a topology of the form $\Sigma \times \mathbb{R}$, with Σ a closed oriented surface.

Observe that given a Lorentzian 3D gravity on M with $\Lambda = 0$, there is an induced map

$$\varphi : \mathcal{EH}(M) \longrightarrow \mathcal{M}_{flat,\Sigma,ISO(2,1)}, \tag{2.17}$$

sending a (equivalence class of) flat Lorentzian metric $[g]$ to the corresponding (equivalence class of) flat gauge connection, denoted by $[A^g]$. For details, see Appendix B and the composition (B.4).

The moduli space $\mathcal{M}_{flat,\Sigma,ISO(2,1)}$ in (2.17) can be seen as the phase space of the Chern-Simons gauge theory, with gauge group $ISO(2,1)$. As we discussed in subsection 2.2.2, this theory can be equivalently obtained by using the first order formulation of 3D gravity in the language of triad-spin-connection. In that case, a flat connection on the frame bundle of M is determined by its holonomies; and holonomies are determined by $(ISO(2,1), \mathbb{R}^{2,1})$-structures, etc. (see Appendix B). Thus, our previous constructions fit into the current discussion.

Remark 2.17 *There is a more general version of the map φ in (2.17), denoted by*

$$\varphi_\Lambda : \mathcal{EH}(M,\Lambda) \longrightarrow \mathcal{M}_{flat,\Sigma,G_\Lambda}, \tag{2.18}$$

where G_Λ is one of the symmetry groups in (2.10). That is, any (equivalence class of) vacuum solution $[g]$ (either flat or of constant curvature) of the Einstein equations determines a (class of) flat G_Λ-connection $[A^g]$. This is because the composition (B.4) will still be valid for any symmetry group in (2.10) and any manifold topologically of the form $\Sigma \times \mathbb{R}$, with Σ a closed oriented surface.

Note also that the map φ_Λ in (2.18) relates the vacuum 3D gravity for the pair (M, G_Λ) to the Chern-Simons theory with the gauge group G_Λ. However, we end up with the following question: *Are the resulting theories equivalent (in some sense)?* This leads to the following definition.

Definition 2.18 *We say that 3D gravity is* **equivalent** *to gauge theory in the sense of the canonical formalism if the map φ_Λ in (2.18) is an isomorphism*[§].

Remark 2.19 *It should be noted that φ_Λ* need not to be invertible *in the first place. We may have two possible approaches to that problem.*

1. *If we adopt the first order formalism in subsection 2.2.2, the map φ_Λ in (2.18) happens to be invertible if every such flat connection can be transformed into a form in which the triad is invertible (uniquely up to a diffeomorphism/local Lorentz transformation) [18, §6.1].*

2. *If we adopt the geometric description in subsection 2.2.1, we then ask whether the holonomy group of a (G, X) structure is sufficient to determine a geometry. The answer is no, in general. But there are some positive answers. In fact, we address this issue in Remark B.1.*

In this regard, combining the terminology in Definition 2.18 with [13, Prop. 2] (cf. Thm. B.2), we get the following important result, which will be central for us in §3.3.

Theorem 2.20 *For vacuum Einstein gravity on $M = \Sigma \times (0, \infty)$, with $\Lambda = 0$, and Σ a closed Riemann surface of genus $g > 1$, there exists an equivalence of gravity with gauge theory in the sense of Definition 2.18. In that case, the map φ_Λ in (2.18) reduces to the map φ in (2.17), and hence we have the identification $\mathcal{EH}(M) \simeq \mathcal{M}_{flat,\Sigma,ISO(2,1)}$.*

3. Proofs of the main results

In what follows, we give more explanations about the contents of Theorems 1.1, 1.2, and 1.3 and the proofs of these results.

3.1. Proof of Theorem 1.1

In this section, we will present the proof of Theorem 1.1. Inspired by [3], we first prove the following result encoding the prestacky part of the construction of interest.

Lemma 3.1 *Given a Lorentzian n-manifold M, let $\mathcal{C}$ be the category of open subsets of M that are diffeomorphic to $\mathbb{R}^n$, with morphisms being canonical inclusions between open subsets whenever $U \subset V$. Then the functor $\mathcal{E} : \mathcal{C}^{op} \to Grpds$ described below is a prestack.*

1. **The action of $\mathcal{E}$ on the objects of $\mathcal{C}$.** For each object U of $\mathcal{C}$, we have a groupoid $\mathcal{E}(U)$ of Ricci-flat pseudo-Riemannian metrics on U, where *objects of $\mathcal{E}(U)$* form the *set*

$$FMet(U) := \{g \in \Gamma(Sym^2(T^*U)) : Ric(g) = 0\}.$$

[§] We could have asked for the invertibility of the map (B.5) with a general 3-manifold M, instead. But in 3D gravity, we mostly consider $M = \Sigma \times \mathbb{R}$ with Σ a closed oriented surface; hence, the map (B.5) reduces to φ_Λ.

Morphisms in $\mathcal{E}(U)$. Let $Aut(Sym^2(T^*U))$ be the group of automorphisms of the bundle $Sym^2(T^*U)$ over U, and $\cdot\varphi$ denotes the action of φ on the sections. We may sometimes use φ^* for the action as well because of the natural motivation coming from the pulling-back operation.

By the *action of* φ, we mean that φ is a bundle isomorphism making the diagram

$$\begin{tikzcd} Sym^2(T^*U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"] & & Sym^2(T^*U) \arrow[dl, "\pi"'] \\ & U \arrow[ul, bend left, "g"] \arrow[ur, bend right, "g'"'] & \end{tikzcd}$$

commute such that it acts on each fiber isomorphically; that is, for each $p \in U$ there is an isomorphism $\varphi_p : Sym^2(T_p^*U) \xrightarrow{\sim} Sym^2(T_p^*U)$ such that

$$g'_p = \varphi_p(g_p). \tag{3.1}$$

In the context of GR, we consider particular automorphisms that are induced from *infinitesimal diffeomorphisms* of the underlying spacetime. Following Remarks A.1 and 2.16, we consider the infinitesimal diffeomorphisms acting on the metric g as

$$g_{\mu\nu}(p) \longrightarrow g_{\mu\nu}(p) + \mathcal{L}_X g_{\mu\nu}(p),$$

where $X \in \Gamma(TU)$ is a vector field over U, $p \in U$, and $\mathcal{L}_X$ is the Lie derivative operator along X. Here, $\mathcal{L}_X g$ serves as a variation δg of g as in Remark A.1.

Since any combinations of infinitesimal diffeomorphisms are also meaningful for our construction, considering the C^∞-module generated by these infinitesimal generators over U, we formally define

$$L(U) = \big\langle \mathcal{L}_X : [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]},\ X, Y \in \Gamma(TU) \big\rangle \tag{3.2}$$

as an algebra over $C^\infty(U)$. Then we also have the following definition.

Definition 3.2 *Let* $g \in \mathcal{E}(U)$. *By an* infinitesimal diffeomorphism φ, *we mean a transformation determined by an element* $\hat{\varphi} \in L(U)$ *such that for each* $p \in U$, g *transforms under this infinitesimal diffeomorphism as*

$$g_{\mu\nu}(p) \xrightarrow{\varphi} g_{\mu\nu}(p) + \hat{\varphi}(g_{\mu\nu})(p). \tag{3.3}$$

In this case, we also use $\cdot\varphi$ *to denote the action of this infinitesimal transformation on the space of metrics. As mentioned before, if* $g_{\mu\nu}$ *satisfies the corresponding Einstein field equations, so does its variation* $g_{\mu\nu} \cdot \varphi$.

Definition 3.3 *We define a* morphism $g \to g'$ *in* $\mathcal{E}(U)$ *if there exists an infinitesimal diffeomorphism* φ *such that* $g' = g \cdot \varphi$. *Then the set of morphisms is given by*

$$Hom_{\mathcal{E}(U)}(g, g') = \big\{\varphi \in Aut(Sym^2(T^*U)) : g' = g \cdot \varphi \text{ in } \mathcal{E}(U)\big\}.$$

We denote a morphism $g \to g'$ in $Hom_{\mathcal{E}(U)}(g, g')$ by (g, φ) or just by φ if the meaning is clear from the context. It is also clear from the construction that all morphisms in $Hom_{\mathcal{E}(U)}(g, g')$ are invertible.

Compositions in $\mathcal{E}(U)$. Given two morphisms $g \xrightarrow{\psi} g'$ and $g' \xrightarrow{\varphi} g''$ in $\mathcal{E}(U)$, using Equation (3.3), the composition of two morphisms is given as the standard composition

$$(g \cdot \psi) \cdot \varphi : g \to g'',$$

where $\hat{\varphi}, \hat{\psi} \in L(U)$ representing the corresponding operators. More precisely, w.l.o.g, we assume $\hat{\varphi} \equiv \mathcal{L}_X$ and $\hat{\psi} \equiv \mathcal{L}_Y$ for some vector fields X, Y on U. Then one obtains

$$\begin{aligned} g''_{\mu\nu}(p) &= g'_{\mu\nu}(p) \cdot \varphi \\ &= g'_{\mu\nu}(p) + \mathcal{L}_X g'_{\mu\nu}(p) \\ &= g_{\mu\nu}(p) + \mathcal{L}_Y g_{\mu\nu}(p) + \mathcal{L}_X \big(g_{\mu\nu}(p) + \mathcal{L}_Y g_{\mu\nu}(p)\big) \\ &= g_{\mu\nu}(p) + \big(\mathcal{L}_Y + \mathcal{L}_X + \mathcal{L}_X \mathcal{L}_Y\big) g_{\mu\nu}(p), \end{aligned}$$

where $\big(\mathcal{L}_Y + \mathcal{L}_X + \mathcal{L}_X \mathcal{L}_Y\big) \in L(U)$, and we get a morphism $g \to g''$ represented by the element $\psi + \varphi + (\varphi \circ \psi)$. Following our notation, we use "$\varphi \circ \psi$" to represent the composition, by which we mean $g \cdot (\varphi \circ \psi) = (g \cdot \psi) \cdot \varphi$.

2. **The action of $\mathcal{E}$ on the morphisms in $\mathcal{C}$.** To each morphism $U \xrightarrow{f} V$ in $\mathcal{C}$, it assigns a functor of categories $\mathcal{E}(f) : \mathcal{E}(V) \to \mathcal{E}(U)$, whose action on both objects and morphisms of $\mathcal{E}(V)$ is given as follows.

   (a) For any object $g \in Ob(\mathcal{E}(V)) = FMet(V)$, we set $g \xrightarrow{\mathcal{E}(f)} f^* g$, where

   $$f^* g = g \circ f = g|_U \in FMet(U).$$

   Notice that the pullback of a Ricci-flat metric, in general, may no longer be Ricci-flat. But, in the case of particular canonical inclusions $f : U \hookrightarrow V$, with U, V open subsets, if a metric g is Ricci-flat on V, so is $f^* g$ on U. This is because $f^* g$ is just the restriction $g|_U$ of g to the open subset U.

   (b) For any morphism $(g, \varphi) \in Hom_{\mathcal{E}(V)}(g, g')$, by the definition of φ, there exists an isomorphism $g_{\mu\nu}(p) \to g'_{\mu\nu}(p) = g_{\mu\nu}(p) + \hat{\varphi}(g_{\mu\nu})(p)$ for all $p \in U \subset V$ as well. Therefore, due to the fiberwise action given in Equation (3.3), φ induces an isomorphism $\varphi_p : Sym^2(T_p^* U) \xrightarrow{\sim} Sym^2(T_p^* U)$, and hence a subbundle isomorphism. Thus, we get the desired transformation over the smaller open subset U in V. We denote this induced isomorphism by $\varphi|_U$ (or $f^* \varphi$), and write

   $$\Big(g \xrightarrow[(g,\varphi)]{\sim} g'\Big) \xrightarrow{\mathcal{E}(f)} \Big(g|_U \xrightarrow[(g|_U, \varphi|_U)]{\sim} g'|_U\Big).$$

3. Given a composition of morphisms $U \xrightarrow{f} V \xrightarrow{h} W$ in $\mathcal{C}$, there exists an invertible natural transformation (arising naturally from properties of the action)

$$\varphi_{h \circ f} : \mathcal{E}(h \circ f) \Rightarrow \mathcal{E}(f) \circ \mathcal{E}(h),$$

together with the compatibility condition.

Proof of Lemma 3.1. It is enough to prove the following two statements:

(i) Given a composition of morphisms in $\mathcal{C}$

$$\begin{tikzcd} U \arrow[r, "f"'] \arrow[rr, bend left, "h\circ f"] & V \arrow[r, "h"'] & W, \end{tikzcd}$$

there is an invertible natural transformation

$$\begin{tikzcd} \mathcal{E}(W) \arrow[rr, bend left, "\mathcal{E}(h\circ f)"] \arrow[rr, bend right, "\mathcal{E}(f)\circ\mathcal{E}(h)"'] \arrow[rr, phantom, "\Downarrow \psi_{h,f}"] & & \mathcal{E}(U). \end{tikzcd}$$

(ii) Given a composition of morphisms $U \xrightarrow{f} V \xrightarrow{h} W \xrightarrow{p} Z$ in $\mathcal{C}$, the associativity condition holds in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathcal{E}(p\circ h\circ f) \arrow[r, Rightarrow, "\psi_{p,h\circ f}"'] \arrow[d, Rightarrow, "\psi_{p\circ h,f}"'] & \mathcal{E}(h\circ f)\circ\mathcal{E}(p) \arrow[d, Rightarrow, "\psi_{h,f}\star id_{\mathcal{E}(p)}"] \\ \mathcal{E}(f)\circ\mathcal{E}(p\circ h) \arrow[r, Rightarrow, "id_{\mathcal{E}(f)}\star\psi_{p,h}"'] & \mathcal{E}(f)\circ\mathcal{E}(h)\circ\mathcal{E}(p) \end{tikzcd}$$

Proof of (i). First, we need to analyze *objectwise*: For any object $g \in FMet(W)$, we have the following *strong condition* by which the rest of the proof will become rather straightforward.

$$\mathcal{E}(h\circ f)(g) = (h\circ f)^* g = f^* h^* g = \big(\mathcal{E}(f)\circ\mathcal{E}(h)\big)(g) \in FMet(U). \tag{3.4}$$

As we have identical metrics $\mathcal{E}(h\circ f)(g) = \mathcal{E}(f)\circ\mathcal{E}(h)(g)$ for any $g \in FMet(W)$, there is, by construction, a unique identity morphism

$$\big(\mathcal{E}(h\circ f)(g), id\big) \in Hom_{FMet(U)}\Big(\mathcal{E}(h\circ f)(g),\ \mathcal{E}(f)\circ\mathcal{E}(h)(g)\Big)$$

such that

$$\mathcal{E}(h\circ f)(g) \xrightarrow[\left(\mathcal{E}(h\circ f)(g), id\right)]{\sim} \mathcal{E}(f)\circ\mathcal{E}(h)(g) = \big(\mathcal{E}(h\circ f)(g)\big)\cdot id = \mathcal{E}(h\circ f)(g).$$

Thus, one has the natural choice of a collection of morphisms

$$\big\{m_g : \mathcal{E}(h\circ f)(g) \longrightarrow \mathcal{E}(f)\circ\mathcal{E}(h)(g)\big\},$$

where $m_g = \big(\mathcal{E}(h\circ f)(g), id\big)$ for all $g \in FMet(W)$.

Just for the sake of notational simplicity, we let

$$\mathcal{F} := \mathcal{E}(h \circ f) \;\; and \;\; \mathcal{G} := \mathcal{E}(f) \circ \mathcal{E}(h).$$

Then for each morphism $g \xrightarrow[(g,\varphi)]{\sim} g'$ in $\mathcal{E}(W)$, we get

$$\begin{aligned}
\mathcal{F}((g,\varphi)) &= \mathcal{E}(h \circ f)((g,\varphi)) \\
&= \big((h \circ f)^* g, (h \circ f)^* \varphi\big) \\
&= \big(f^* \circ h^*(g), f^* \circ h^*(\varphi)\big) \\
&= \big(\mathcal{E}(f) \circ \mathcal{E}(h)(g), f^* \circ h^*(\varphi)\big) \\
&= \mathcal{E}(f) \circ \mathcal{E}(h)((g,\varphi)) \\
&= \mathcal{G}((g,\varphi)).
\end{aligned}$$

The computation above implies the commutativity of the diagram.

$$\begin{tikzcd}
\mathcal{F}(g) \arrow[r, "\mathcal{F}((g,\varphi))"] \arrow[d, "m_g"'] & \mathcal{F}(g') \arrow[d, "m_{g'}"] \\
\mathcal{G}(g) \arrow[r, "\mathcal{G}((g,\varphi))"'] & \mathcal{G}(g')
\end{tikzcd}$$

Furthermore, it is clear from Equation (3.4) and from the construction that $\psi_{h,f} : \mathcal{E}(h \circ f) \Rightarrow \mathcal{E}(f) \circ \mathcal{E}(h)$ is in fact invertible. In other words, we have $\mathcal{E}(h \circ f) \cong \mathcal{E}(f) \circ \mathcal{E}(h)$ up to invertible natural transformation.

This completes the proof of (i).

Proof of (ii). If $U \xrightarrow{f} V \xrightarrow{h} W$ in $\mathcal{C}$ is a composition, then we have

$$(1) \quad \mathcal{F}(g) = \mathcal{G}(g) \;\text{ for any }\; g \in Ob(\mathcal{E}(W)), \tag{3.5}$$

$$(2) \quad \mathcal{F}((g,\varphi)) = \mathcal{G}((g,\varphi)) \text{ for any } g \xrightarrow[(g,\varphi)]{\sim} g' \;\; in \; \mathcal{E}(W), \tag{3.6}$$

where $\mathcal{F} := \mathcal{E}(h \circ f)$ and $\mathcal{G} := \mathcal{E}(f) \circ \mathcal{E}(h)$.

Now, let $U \xrightarrow{f} V \xrightarrow{h} W \xrightarrow{p} Z$ be a composition of morphisms in $\mathcal{C}$, then it suffices to show that the associativity condition holds both *objectwise* and *morphismwise*.

- Let $g \in Ob(\mathcal{E}(Z))$, then we have

$$\begin{aligned}
\mathcal{E}(p \circ (h \circ f))(g) &= \mathcal{E}(h \circ f) \circ \mathcal{E}(p)(g) \quad from\ Eq.(3.5)\ with\ \psi_{p,h\circ f} \\
&= \mathcal{E}(f) \circ \mathcal{E}(h) \circ \mathcal{E}(p) \quad from\ Eq.(3.5)\ with\ \psi_{h,f} \star id_{\mathcal{E}(p)} \\
&= \mathcal{E}(f) \circ \mathcal{E}(p \circ h)(g) \quad from\ Eq.(3.5)\ with\ id_{\mathcal{E}(f)} \star \psi_{p,h} \\
&= \mathcal{E}((p \circ h) \circ f)(g) \quad from\ Eq.(3.5)\ with\ \psi_{p\circ h,f}
\end{aligned}$$

This gives the commutativity of the diagram objectwise.

- Let $g \xrightarrow[(g,\varphi)]{\sim} g'$ in $\mathcal{E}(Z)$, then we have

$$\begin{aligned}
\mathcal{E}(p \circ (h \circ f))((g,\varphi)) &= \mathcal{E}(h \circ f) \circ \mathcal{E}(p)((g,\varphi)) && \textit{from Eqn.}(3.6) \textit{ with } \psi_{p,h\circ f} \\
&= \mathcal{E}(f) \circ \mathcal{E}(h) \circ \mathcal{E}(p)((g,\varphi)) && \textit{from Eqn.}(3.6) \textit{ with } \psi_{h,f} \star id_{\mathcal{E}(p)} \\
&= \mathcal{E}(f) \circ \mathcal{E}(p \circ h)((g,\varphi)) && \textit{from Eqn.}(3.6) \textit{ with } id_{\mathcal{E}(f)} \star \psi_{p,h} \\
&= \mathcal{E}((p \circ h) \circ f)((g,\varphi)) && \textit{from Eqn.}(3.6) \textit{ with } \psi_{p\circ h,f}.
\end{aligned}$$

This completes the proof of (ii), and hence that of Lemma 3.1.

□

Let $\mathcal{E} : \mathcal{C}^{op} \to Grpds$ be the prestack defined in Lemma 3.1. Now, introducing a suitable site structure on $\mathcal{C}$, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. As in the case of [3], we first endow $\mathcal{C}$ with an appropriate Grothendieck topology τ by defining the covering families $\{U_i \to U\}$ of U in $\mathcal{C}$ to be "good" open covers $\{U_i \subseteq U\}$ meaning that the fibered products $U_{i_1 i_2 \dots i_m} := U_{i_1} \times_U U_{i_2} \times_U \cdots \times_U U_{i_m}$ corresponding to the intersection of those open subsets U_i's in U are either empty or open subsets diffeomorphic to $\mathbb{R}^n$. Here each morphism $U_i \hookrightarrow U$ is the canonical inclusion (and hence a morphism in $\mathcal{C}$).

Let U be an object in $\mathcal{C}$. Given $\{U_i \subseteq U\}$ a covering family for U, one has the following cosimplicial diagram in $Grpds$

$$\mathcal{E}(U_\bullet) := \Big(\prod_i \mathcal{E}(U_i) \rightrightarrows \prod_{ij} \mathcal{E}(U_{ij}) \substack{\to\\\to\\\to} \prod_{ijk} \mathcal{E}(U_{ijk}) \substack{\to\\\to\\\to\\\to} \cdots \Big),$$

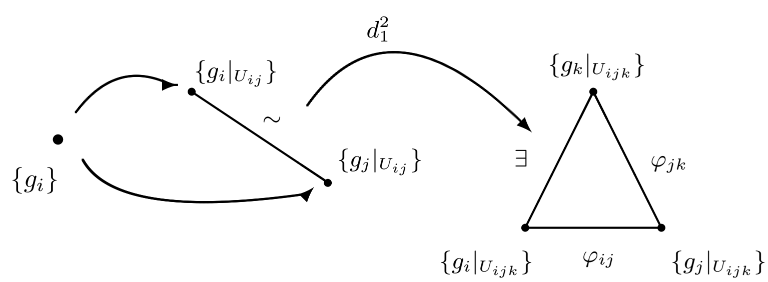
where $U_{i_1 i_2 \dots i_m}$ denotes the fibered product of U_{i_n}'s in U as above. Note that for a family

$$\{g_i\} \ \text{in} \ \prod_i \mathcal{E}(U_i),$$

where $\mathcal{E}(U_i) = FMet(U_i)$, the coface maps d^1_0 and d^1_1 correspond to the suitable restrictions of each component $g_i|_{U_{ij}}$ and $g_j|_{U_{ij}}$, respectively.

Now, it follows from the Lemma 2.5 that $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$ is indeed a particular groupoid and can be defined as follows.

1. *Objects* are the pairs (x,h), where $x := \{g_i\} \in \prod_i \mathcal{E}(U_i)$. That is, it is a family of Ricci-flat pseudo-Riemannian metrics on U_i's, along with the diagram



where $g_j|_{U_{ij}} = g_i|_{U_{ij}} \cdot \varphi_{ij}$ for some $\varphi_{ij} \in Aut(Sym^2(T^*U_{ij}))$. The "triangle" on the RHS of the diagram above implies that for all i, j, k, we have

$$\begin{aligned} g_k|_{U_{ijk}} &= g_j|_{U_{ijk}} \cdot \varphi_{jk} \\ &= (g_i|_{U_{ijk}} \cdot \varphi_{ij}) \cdot \varphi_{jk} \\ &= g_i|_{U_{ijk}} \cdot (\varphi_{jk} \circ \varphi_{ij}). \end{aligned} \tag{3.7}$$

It means that there exists a morphism $\varphi_{ik} : g_i|_{U_{ijk}} \xrightarrow{\sim} g_k|_{U_{ijk}}$. Therefore, we define the morphism h in $\prod \mathcal{E}(U_{ij})$ as a family

$$\Big\{ g_i|_{U_{ij}} \xrightarrow[(g_i|_{U_{ij}}, \varphi_{ij})]{\sim} g_j|_{U_{ij}} : g_j|_{U_{ij}} = g_i|_{U_{ij}} \cdot \varphi_{ij} \ \& \ \varphi_{ij} \in Aut(Sym^2(T^*U_{ij})) \Big\},$$

where $g_k|_{U_{ijk}} = g_i|_{U_{(U_{ij}}} \cdot (\varphi_{jk} \circ \varphi_{ij})$ and $s_0^0(h) : \{g_i\} \to \{g_i\}$, which is just the identity morphism.

As a remark, the conditions in the definition of the family $\{h\}$ correspond to those in Lemma 2.5 (Eqs. (2.2) and (2.3)). Therefore, **an object of** $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$ is of the form

$$(x, h) = \Big(\{ g_i \in FMet(U_i) \}, \{ \varphi_{ij} \in Aut(Sym^2(T^*U_{ij})) \} \Big), \tag{3.8}$$

where $\{g_i\}$ is an object in $\prod \mathcal{E}(U_i)$, and for each i, j, $\varphi_{ij} := (g_i|_{U_{ij}}, \varphi_{ij})$ is a morphism in $\prod \mathcal{E}(U_{ij})$ satisfying

$$\begin{aligned} &(i) \ \ g_j|_{U_{ij}} = g_i|_{U_{ij}} \cdot \varphi_{ij}, \text{ with } \varphi_{ij} \in Aut(Sym^2(T^*U_{ij})), \\ &(ii) \ \ \textit{On } U_{ijk}, \ \ \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \ \ (\textit{the cocycle condition}), \\ &(iii) \ \ s_0^0(h) : \{g_i\} \to \{g_i\}, \ \textit{the identity morphism.} \end{aligned}$$

In short, an object $\mathbf{g} := \big(\{g_i\}, \{\varphi_{ij}\}\big)$ in $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$ is a collection $\{g_i\}$ of Ricci-flat metrics over covering open subset U_i of U, together with the transition maps $\{\varphi_{ij}\}$ on the overlaps that satisfy the cocycle condition above.

2. A *morphism* $(x, h) \to (x', h')$ in $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$ consists of the following data:

   (a) A morphism $x \xrightarrow{f} x'$ in $\prod \mathcal{E}(U_i)$, such that $\{g_i\} \xrightarrow{\sim} \{g_i'\}$, where $g_i, g_i' \in FMet(U_i)$ with $g_i' = g_i \cdot \varphi_i$ for some $\varphi_i \in Aut(Sym^2(T^*U_i))$.

   (b) For each i, j, a commutative diagram

$$\begin{tikzcd} g_i|_{U_{ij}} \arrow[r, "\varphi_i|_{U_{ij}}"] \arrow[d, "h = \varphi_{ij}"'] & g_i'|_{U_{ij}} \arrow[d, "h' = \varphi_{ij}'"] \\ g_j|_{U_{ij}} \arrow[r, "\varphi_j|_{U_{ij}}"'] & g_j'|_{U_{ij}} \end{tikzcd} \tag{3.9}$$

In fact, it follows from the fact that $g_j|_{U_{ij}} = g_i|_{U_{ij}} \cdot \varphi_{ij}$ and $g'_j|_{U_{ij}} = g'_i|_{U_{ij}} \cdot \varphi'_{ij}$, we have

$$\big(g_i|_{U_{ij}} \cdot \varphi_i|_{U_{ij}}\big) \cdot \varphi'_{ij} = g'_j|_{U_{ij}}.$$

On the other hand, one also has

$$\big(g_i|_{U_{ij}} \cdot \varphi_{ij}\big) \cdot \varphi_j|_{U_{ij}} = g'_j|_{U_{ij}},$$

which imply the commutativity of the diagram; hence, one can also deduce the following relation:

$$\begin{aligned}
&\big(g_i|_{U_{ij}} \cdot \varphi_{ij}\big) \cdot \varphi_j|_{U_{ij}} = \big(g_i|_{U_{ij}} \cdot \varphi_i|_{U_{ij}}\big) \cdot \varphi'_{ij} \quad \forall i, j \\
&\Longleftrightarrow \\
&g_i|_{U_{ij}} \cdot \big(\varphi_j|_{U_{ij}} \circ \varphi_{ij}\big) = g_i|_{U_{ij}} \cdot \big(\varphi'_{ij} \circ \varphi_i|_{U_{ij}}\big) \quad \forall i, j \\
&\Longleftrightarrow \\
&\varphi'_{ij} = \varphi_j|_{U_{ij}} \circ \varphi_{ij} \circ \varphi_i^{-1}|_{U_{ij}} \quad \forall i, j
\end{aligned}$$

Thus, **a morphism** in $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$ from $\mathbf{g} = \big(\{g_i\}, \{\varphi_{ij}\}\big)$ to $\mathbf{g}' = \big(\{g'_i\}, \{\varphi'_{ij}\}\big)$ is a family

$$\Big\{\varphi_i \in Aut(Sym^2(T^*U_i)) : g'_i = g_i \cdot \varphi_i \ \&\ \varphi'_{ij} = \varphi_j|_{U_{ij}} \circ \varphi_{ij} \circ \varphi_i^{-1}|_{U_{ij}}\Big\} \tag{3.10}$$

In short, a morphism $\boldsymbol{\varphi} : \mathbf{g} \to \mathbf{g}'$ in $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$ is a collection $\{\varphi_i\}$ of morphisms, with $\varphi_i \in Mor_{\mathcal{E}(U_i)}(g_i, g'_i)$, such that the action is compatible with the corresponding transition maps in the sense of Diagram 3.9.

Now, for a covering family $\{U_i \subseteq U\}$ of U, *the canonical morphism*

$$\Psi : \mathcal{E}(U) \longrightarrow \mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet)) \tag{3.11}$$

is defined as a functor of groupoids, where

- for each object g in $FMet(U)$, $g \xrightarrow{\Psi} \Big(\{g|_{U_i}\}, \{\varphi_{ij} = id\}\Big)$, together with the trivial cocyle condition.
- for each morphism $g \xrightarrow[(g,\varphi)]{\sim} g \cdot \varphi$, with $\varphi \in Aut(Sym^2(T^*U))$, it assigns

$$\big(g \xrightarrow[(g,\varphi)]{\sim} g \cdot \varphi\big) \xrightarrow{\Psi} \Big(\{\varphi_i := \varphi|_{U_i}\}\Big),$$

where $\varphi|_{U_i}$ trivially satisfies the desired relation (3.10) for being a morphism in $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$.

Lemma 3.4 Ψ *is a fully faithful and essentially surjective functor.*

Proof $\underline{\Psi \text{ is } \textit{essentially surjective}}$: Let $\mathbf{g} := \big(\{g_i\}, \{\varphi_{ij}\}\big)$ be an object in $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$. Then we have a family of objects $\{g_i\}$, with the family of transition functions $\{\varphi_{ij}\}$ satisfying the cocycle condition $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ on U_{ijk}, such that $g_j|_{U_{ij}} = g_i|_{U_{ij}} \cdot \varphi_{ij}$.

We need to show that these are patched together to form a metric $g \in FMet(U)$. In fact, our site structure on $\mathcal{C}$ consists of good covers for which the intersection of open subsets U_i's in U are either empty or open subsets diffeomorphic to $\mathbb{R}^n$. Also, $Sym^2(T^*U)$ is a locally free sheaf over U. In this regard, the following fact is useful: *All cocycles are trivializable on manifolds diffeomorphic to* $\mathbb{R}^n$. Therefore, we conclude that $\{\varphi_{ij} = id\}$ for all i, j.

Now, we have a trivial cocycle condition with $\varphi_{ij} = id$. It follows that g_i is a section of the sheaf $Sym^2(T^*U_i)$ over U_i satisfying $g_j|_{U_{ij}} = g_i|_{U_{ij}}$ for all i, j. So, g_i's are glued together by transition functions φ_{ij}, along with the trivial cocycle condition, to form $g \in FMet(U)$ so that $g|_{U_i} = g_i$ and $\varphi|_{U_i} = \varphi_i$ for all i. Therefore, Ψ is *essentially surjective.*

Ψ is *fully faithful:* We need to show that the induced map

$$\underline{\Psi} : Hom_{\mathcal{E}(U)}(g, g') \longrightarrow Hom_{\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))}(\Psi(g), \Psi(g'))$$

is a bijection of sets. To this end, we consider the corresponding *sheaf-Hom* $\mathcal{H}om(\mathcal{S}, \mathcal{S}')$, with $\mathcal{S} = \mathcal{S}' = Sym^2(T^*U)$, where $\mathcal{H}om(\mathcal{S}, \mathcal{S}')$ is the collection of the data $\mathcal{H}om(\mathcal{S}, \mathcal{S}')(V) := Mor(\mathcal{S}|_V, \mathcal{S}'|_V)$. Here $\mathcal{S}|_V$ denotes the restriction of the sheaf to the open subset $V \subset U$. Then, both injectivity and surjectivity of $\underline{\Psi}$ follow from the fact that the *sheaf-Hom* $\mathcal{H}om(\mathcal{S}, \mathcal{S}')$ is a sheaf over U. Let us explain the details below.

If we assume $\underline{\Psi}(\varphi) = id$, then it means, by definition, $\varphi_i := \varphi|_{U_i} = id$ for all i. By construction, it implies that each $\varphi|_{U_i} \in \mathcal{H}om(Sym^2(T^*U), Sym^2(T^*U))(U_i)$. Because *sheaf-Hom* is a sheaf over U, we obtain $\varphi = id$, and hence injectivity of $\underline{\Psi}$.

Now, let $\boldsymbol{\varphi} : \Psi(g) \to \Psi(g')$ be a a morphism in $\mathrm{holim}_{Grpds}(\mathcal{E}(U_\bullet))$. Then it can be viewed as a collection $\{\varphi_i\}$ of morphisms such that the action is compatible with the corresponding transition maps in the sense of Diagram 3.9. Here both $\Psi(g)$ and $\Psi(g')$ are collections of Ricci-flat metrics $\{g|_{U_i}\}$ and $\{g'|_{U_i}\}$, respectively, along with the trivial transition maps. Therefore, Diagram 3.9 with $\varphi_{ij} = \varphi'_{ij} = id$ implies that $\varphi_j|_{U_{ij}} = \varphi_i|_{U_{ij}}$, where each $\varphi_i \in \mathcal{H}om(Sym^2(T^*U), Sym^2(T^*U))(U_i)$. Because *sheaf-Hom* is a sheaf over U, we conclude that there exists $\varphi \in \mathcal{H}om(Sym^2(T^*U), Sym^2(T^*U))(U)$ such that $\varphi|_{U_i} = \varphi_i$. Equivalently, it means $\varphi \in Hom_{\mathcal{E}(U)}(g, g')$, with $\underline{\Psi}(\varphi) = \{\varphi_i\}$. This proves the desired surjectivity and completes the proof. $\square$

From Lemma 3.4, we conclude that the canonical morphism Ψ in Equation (3.11) is a weak equivalence in *Grpds*, and this completes the proof of Theorem 1.1. Then we obtain the following definition.

Definition 3.5 *The stack* $\mathcal{E} : \mathcal{C}^{op} \to Grpds$ *constructed above is called the* moduli stack of solutions to the vacuum Einstein field equations on M*, with* $\Lambda = 0$. *We sometimes call it directly the* stack of Einstein gravity*.*

3.2. Proof of Theorem 1.2

In this section, we provide a sketch of the proof of Theorem 1.2. In fact, after fixing our notation and giving the explicit definitions, the result follows from Theorem 1.1 with some natural modifications.

Note also that in the proof of Theorem 1.1, morphisms in the source category are all canonical inclusions, and hence pullbacks of (Ricci-flat) metrics by these morphisms are just restrictions to some smaller open subsets, and hence still Ricci-flat. Therefore, for a “family version” of this category, (fiberwise) open embeddings can be viewed as suitable substitutes.

Moreover, we require our geometric structure (Lorentzian, Ricci-flat) to vary in families parametrized over cartesian spaces¶. Therefore, throughout this subsection, we work with sheaves on the site Fam_n of families of manifolds with n-dimensional fibers, together with fiberwise open embeddings. More precisely, we have the following definition.

Definition 3.6 *Let Fam_n be the site, where an object, denoted by M/U, is a submersion $\pi : M \to U$ with n-dimensional fibers and U an object in $Cart$, and a morphism $M/U \to M'/U'$ is a smooth bundle map that is a fiberwise open embedding.*

Moreover, the site structure is determine by the covering families that are a collection of morphisms $\{M_i/U_i \to M/U\}$ such that $\{M_i\}$ is an open cover of M.

A sketch of the proof of Theorem 1.2. Denote by $\mathcal{E}^{fam}$ the presheaf on Fam_n

$$Fam_n^{op} \longrightarrow Grpds, \quad M/U \mapsto \mathcal{E}^{fam}(M/U),$$

where $Ob(\mathcal{E}^{fam}(M/U)) := \{g \in \Gamma(Sym^2(T^*(M/U))) : Ric(g) = 0\}$.

Here $T^*(M/S)$ is the relative cotangent bundle $T^*(M/U) = Coker(T^*U \to T^*M)$, which allows us to define fiberwise versions (or "families") of many familiar structures. Indeed, we are currently interested in (pseudo) Riemannian structures.

In this regard, a pseudo-Riemannian metric g on M/U is a section of the relative bundle $Sym^2(T^*(M/U))$. In other words, for an object $\pi : M \to U$ in Fam_n, g is a (Ricci-flat) pseudo-Riemannian metric on the vertical tangent bundle $ker(\pi_*) \subset TM$. Thus, for any parameter $u \in U$ and $p \in M_u := \pi^{-1}(u)$, $g|_p$ is a metric on $ker(\pi_{*,p}) \subset T_pM$.

Using the fact that an object of $\mathcal{E}^{fam}(M/U)$ is a (Ricci-flat) pseudo-Riemannian metric on the vertical tangent bundle $ker(\pi_*) \subset TM$, *morphisms* in the groupoid $\mathcal{E}^{fam}(M/U)$ can be defined via particular automorphisms of $Sym^2(T^*(M/U))$ induced by infinitesimal transformations as in Lemma 3.1. Likewise, composition can be defined by using similar arguments in Lemma 3.1.

Functoriality follows from the fiberwise nature of the current construction. Given a map $F : N/V \to M/U$ in Fam_n, we have a commutative diagram

$$\begin{tikzcd} N \arrow[r, "F"] \arrow[d, "\pi_V"'] & M \arrow[d, "\pi_U"] \\ V \arrow[r, "f"'] & U \end{tikzcd}$$

such that for each $v \in V$, $F_v : N_v \to M_{f(v)}$ is an open embedding. If g is a Ricci-flat metric on $T(M/U)$, so is its pullback under fiberwise open embeddings. Therefore, using the diagram above, F^*g gives a Ricci-flat metric on $T(N/V)$, and hence an object in $\mathcal{E}^{fam}(N/V)$. Likewise, a morphism φ in $\mathcal{E}^{fam}(M/U)$ can be pulled-back via F, and due to the fiberwise action of the morphisms, $F^*\varphi$ gives a morphism in $\mathcal{E}^{fam}(N/V)$. The other compatibility conditions are straightforward to check by following similar arguments in Lemma 3.1.

¶More details on geometric structures via stacks and on geometries in families can be found in [12]

Finally, one can achieve the stackification of the prestack $\mathcal{E}^{fam}$ by following more or less the same arguments in the proof of Theorem 1.1 "fiberwisely", with some modifications (using families, fiberwise open embeddings, and the site structure above, etc.).

3.3. Proof of Theorem 1.3

As discussed in subsection 2.2.3, we say that there is an *equivalence* between 3D gravity and gauge theory if the phase spaces of gravity and the associated gauge theory can be identified (cf. Definition 2.18). In fact, Mess proved in [13] that this equivalence occurs for a particular setup (cf. Remark 2.19, Theorem 2.20).

Now, we would like to show that once it exists, the equivalence induces a morphism between the corresponding stacks. To this end, we shall first revisit [3] and introduce a particular stack similar to BG_{con} given in [3, Example 2.11]. This helps us to extend the moduli space $\mathcal{M}_{flat,M,G}$ of flat G-connections to a nontrivial stack, denoted by $\mathcal{M}$. Later on, we provide the desired natural 2-morphism $\Phi : \mathcal{M} \Rightarrow \mathcal{E}$ in (3.13).

The stack of flat connections. We first need to introduce the "flat" counterpart of the classifying stack BG_{con}. Just for simplicity, we again use $\mathcal{M}$ for the flat case whose construction is the same as that of BG_{con}. Keep also in mind that for the gravitational interpretation (with $\Lambda = 0$), one requires to consider the case of $G = ISO(2,1)$. In this regard, we start with the following lemma.

Lemma 3.7 *Let $\mathcal{C}$ be the category in Lemma 3.1 such that M is a Lorentzian 3-manifold of the form $\Sigma \times \mathbb{R}$ with Σ a closed Riemann surface of genus $g > 1$. The functor $\mathcal{M} : \mathcal{C}^{op} \to$ Grpds defined below is a stack.*

1. For each object U of $\mathcal{C}$, $\mathcal{M}(U)$ *is a groupoid of flat G-connections on U, where* **objects** *are the elements of the set $\Omega^1(U, \mathfrak{g})_{flat}$ of Lie algebra-valued 1-forms on U, with $F_A = 0$, and* **morphisms** *form the set $Hom_{\mathcal{M}(U)}(A, A') = \{\rho \in \mathcal{G} : A' = A \bullet \rho\}$, where the action of the gauge group on $\Omega^1(U, \mathfrak{g})_{flat}$ is defined as follows: Over U, for all $\rho \in \mathcal{G} = C^\infty(U, G)$ and $A \in \Omega^1(U, \mathfrak{g})_{flat}$, we set $A \bullet \rho := \rho^{-1} \cdot A \cdot \rho + \rho^{-1} \cdot \mathrm{d}\rho$. We denote a morphism $A \xrightarrow{\sim} A' = A \bullet \rho$ in $Hom_{\mathcal{M}(U)}(A, A')$ by (A, ρ), or just by ρ when the meaning is clear from the context.*

   The composition $A \xrightarrow{(A,\rho)} A \bullet \rho \xrightarrow{(A\bullet\rho,\sigma)} A \bullet \rho \bullet \sigma = A \bullet (\rho\sigma)$ *is given by $(A, \rho\sigma)$, with $\sigma \circ \rho := \rho\sigma$.*

2. To each morphism $U \xrightarrow{f} V$ *in $\mathcal{C}$, i.e. $f : U \hookrightarrow V$ with $U \subset V$, one assigns $\mathcal{M}(V) \xrightarrow{\mathcal{M}(f)} \mathcal{M}(U)$. Here $\mathcal{M}(f)$ is a functor of categories whose action on objects and on morphisms of $\mathcal{M}(V)$ is given as follows.*

   (a) *For any object $A \in \mathcal{M}(V) = \Omega^1(V, \mathfrak{g})_{flat}$, we have $\mathcal{M}(f) : A \mapsto f^*A$ $(= A|_U)$. Here we use the fact that the pullback under f (indeed the restriction to an open subset U in our case) of a flat connection is also flat.*

   (b) *For any morphism $(A, \rho) \in Hom_{\mathcal{M}(V)}(A, A')$ with $\rho \in \mathcal{G} = C^\infty(V, G)$ such that $A' = A \bullet \rho$, it follows from the fact that*
   $$f^*(A \bullet \rho) = f^*A \bullet f^*\rho, \tag{3.12}$$
   where $f^*\rho = \rho \circ f \in C^\infty(U, G)$, we conclude that $f^*(A \bullet \rho)$ lies in the orbit space of f^*A. Hence we get
   $$\left(A \xrightarrow[(A,\rho)]{\sim} A' = A \bullet \rho\right) \xrightarrow{\mathcal{M}(f)} \left(f^*A \xrightarrow[(f^*A,\rho\circ f)]{\sim} f^*(A \bullet \rho) = f^*A \bullet f^*\rho\right).$$

That is, $\mathcal{M}(f)(A,\rho) := (f^*A, f^*\rho)$ is a morphism in $\mathcal{M}(U)$. Note that Equation (3.12) can be proven via the local computations of the pullback of a connection A together with the action $A \bullet \rho$.

Proof This is similar to the proofs of Lemma 3.1 and Theorem 1.1, with the special setup, where $n = 3$ and M as above. For a complete treatment to the generic case (i.e. without flatness requirement), see [3, Examples 2.10 and 2.11]. For the flat case, on the other hand, one has exactly the same proof with $\Omega^1(U,\mathfrak{g})_{flat}$ instead of $\Omega^1(U,\mathfrak{g})$ thanks to the fact that the pullback of a flat connection by a canonical inclusion $U \hookrightarrow V$ between open subsets is also flat. We leave details to the reader. □

Let us summarize our progress so far.

1. Before (nontrivial) stacky constructions, we already have an isomorphism $\mathcal{EH}(M) \xrightarrow{\sim} \mathcal{M}_{flat,\Sigma,G}$ between the phase spaces of gravity and the associated gauge theory for the case of vacuum 3D gravity on $M = \Sigma \times (0,\infty)$, with $\Lambda = 0$ and $G = ISO(2,1)$, where Σ is a closed Riemann surface of genus $g > 1$ (cf. Remark 2.19, Theorem 2.20). Notice that both spaces $\mathcal{EH}(M), \mathcal{M}_{flat,\Sigma,G}$ are essentially the set of equivalence classes. Since any set can be seen as a groupoid with the elements as objects and identity morphisms, we have a trivial stacky structure only.

2. We define the *stack $\mathcal{E}$ of Einstein gravity* (cf. Theorem 1.1, Definition. 3.5) providing a nontrivial stacky structure on top of the naïve moduli space $\mathcal{EH}(M)$.

3. From Lemma 3.7, we give the *stack $\mathcal{M}$ of G-bundles with flat connections* on $M := \Sigma \times (0,\infty)$. Likewise, $\mathcal{M}$ gives a nontrivial stacky structure on top of the moduli space $\mathcal{M}_{flat,M,G}$.

What is next: Let $\mathcal{E}, \mathcal{M}$, and the category $\mathcal{C}$ be as above and $G = ISO(2,1)$. Suppose that the underlying manifold M is of the form $\Sigma \times (0,\infty)$, with Σ a closed Riemann surface of genus $g > 1$. Then we prove the following.

Claim. There exists a natural transformation

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[rr, bend left, "\mathcal{M}"] \arrow[rr, bend right, "\mathcal{E}"'] \arrow[rr, phantom, "\Phi \Downarrow"] & & Grpds, \end{tikzcd} \tag{3.13}$$

between the stacks $\mathcal{E}$ and $\mathcal{M}$. It thus provides a stacky extension of the isomorphism $\mathcal{M}_{flat,\Sigma,G} \xrightarrow{\sim} \mathcal{EH}(M)$.

Construction of the natural 2-morphism Φ in (3.13). Recall from §2.2.2 that given a flat $ISO(2,1)$-connection A with a unique decomposition $A = \omega + e$ in terms of the spin connection and triad, we can construct a flat metric $g(e)$ with $g(e)_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, where η denotes the Minkowski metric. Note also that this construction naturally relates infinitesimal gauge transformations to infinitesimal diffeomorphisms (cf. Remark 2.16). Thus, for any object U in $\mathcal{C}$, we have a natural map

$$\Phi_U : \mathcal{M}(U) \longrightarrow \mathcal{E}(U), \tag{3.14}$$

which is indeed a *functor of groupoids* and defined as follows:

1. To each flat $ISO(2,1)$-connection $A = \omega_A + e_A$ in $\Omega^1(U, \mathfrak{iso}(2,1))_{flat}$, it assigns the corresponding flat metric $g \in FMet(U)$, described by the triad over U. That is, on objects of $\mathcal{M}(U)$, it maps

$$\Phi_U : A \mapsto g_A := g(e_A) = \eta_{ab} e^a_A \otimes e^b_A. \tag{3.15}$$

2. Due to the *equivalence of gravity with gauge theory* (cf. Theorem 2.20), the gauge equivalence classes of connections $[A]$ correspond to the equivalence classes of the associated flat Lorentzian metrics $[g_A]$, and vice versa. This will allow us to form the following diagrams in (3.16) and (3.20).

   Note that, from Remark 2.16, we only consider diffeomorphisms in the connected component of the identity to ensure the desired equivalence. In brief, for any $A' \in [A]$ over an open subset U, i.e. $A' = A \bullet \rho$ for some $\rho \in \mathcal{G} = C^\infty(U, ISO(2,1))$, the corresponding metrics, say g_A and $g_{A\bullet\rho}$, are also equivalent, and hence lie in the same equivalence class (and vice versa). That is, there exists an automorphism φ_ρ of $Sym^2(T^*U)$, an infinitesimal diffeomorphism associated to ρ (cf. Definitions 3.2 and 3.3), such that $g_{A\bullet\rho} = g_A \cdot \varphi_\rho$. In other words, such a correspondence can also be expressed as the commutative diagram

$$\begin{tikzcd} g_A \arrow[r, "\exists\, \varphi_\rho"] & g_A \cdot \varphi_\rho \\ A \arrow[u, "\Phi_U"] \arrow[r, "\rho"'] & A \bullet \rho \arrow[u, "\Phi_U"'] \end{tikzcd} \tag{3.16}$$

   together with the maps (relating infinitesimal diffeomorphisms to infinitesimal gauge transformations)

$$Aut(Sym^2(T^*U)) \longrightarrow C^\infty(U, G), \quad \varphi \mapsto \rho_\varphi, \tag{3.17}$$

$$C^\infty(U, G) \longrightarrow Aut(Sym^2(T^*U)), \quad \rho \mapsto \varphi_\rho. \tag{3.18}$$

   Note that $Aut(Sym^2(T^*U))$ is endowed with the usual composition, and the group operation on $C^\infty(U, G)$ is given by the pointwise multiplication.

3. To each morphism $(A, \rho) : A \longrightarrow A'$ in $Hom_{\mathcal{M}(U)}(A, A')$, Φ_U assigns a morphism

$$g_A \xrightarrow[(g_A, \varphi_\rho)]{\sim} g_A \cdot \varphi_\rho \ (= g_{A'}), \tag{3.19}$$

   where $\varphi_\rho \in L(U)$ is an infinitesimal diffeomorphism corresponding to ρ in accordance with Diagram 3.16. Therefore, for any morphism $f : U \hookrightarrow V$ in $\mathcal{C}$, using the map in (3.18), one also has the following commutative diagram.

$$\begin{tikzcd} Aut(Sym^2(T^*V)) \arrow[r, "f^*"] & Aut(Sym^2(T^*U)) \\ C^\infty(V, ISO(2,1)) \arrow[u] \arrow[r, "f^*"'] & C^\infty(U, ISO(2,1)) \arrow[u] \end{tikzcd} \tag{3.20}$$

4. *Functoriality.* Given a composition of morphisms in $\mathcal{M}(U)$

$$\begin{tikzcd} A \arrow[r, "(A,\rho)"'] \arrow[rr, bend left, "A \bullet (\rho\sigma) =: A \bullet (\sigma \circ \rho)"] & A \bullet \rho \arrow[r, "(A \bullet \rho, \sigma)"'] & (A \bullet \rho) \bullet \sigma, \end{tikzcd}$$

we have the commutative diagram

$$\begin{tikzcd} g_A \arrow[r, "\varphi_\rho"] & g_A \cdot \varphi_\rho \arrow[r, "\varphi_\sigma"] & (g \cdot \varphi_\rho) \cdot \varphi_\sigma =: g \cdot (\varphi_\sigma \circ \varphi_\rho) \\ A \arrow[u] \arrow[r, "\rho"] & A \bullet \rho \arrow[u] \arrow[r, "\sigma"] & (A \bullet \rho) \bullet \sigma = A \bullet (\rho\sigma), \arrow[u] \end{tikzcd}$$

where the vertical maps are Φ_U. Using the commutativity, observe

$$g_A \cdot \varphi_{\rho\sigma} = g_{A\bullet(\rho\sigma)} = g_{A\bullet\rho} \cdot \varphi_\sigma = (g_A \cdot \varphi_\rho) \cdot \varphi_\sigma.$$

Then we obtain $g_A \cdot \varphi_{\rho\sigma} = (g_A \cdot \varphi_\rho) \cdot \varphi_\sigma$ for any A, and hence $\varphi_{\rho\sigma} = \varphi_\rho \cdot \varphi_\sigma$. This gives the desired functoriality on compositions:

$$\Phi_U(A, \rho\sigma) = \varphi_{\rho\sigma} = \varphi_\rho \cdot \varphi_\sigma = \Phi_U(A, \rho) \cdot \Phi_U(A, \sigma).$$

Now, we need to show that for each morphism $f : U \to V$ in $\mathcal{C}$, i.e. $f : U \hookrightarrow V$ with $U \subset V$, we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{M}(V) \arrow[r, "\Phi_V"] \arrow[d, "\mathcal{M}(f)"'] & \mathcal{E}(V) \arrow[d, "\mathcal{E}(f)"] \\ \mathcal{M}(U) \arrow[r, "\Phi_U"'] & \mathcal{E}(U) \end{tikzcd} \tag{3.21}$$

In fact, the commutativity follows from the definition of Φ_U: Let $A \in \Omega^1(V, \mathfrak{iso}(2,1))_{flat}$, then we get, from the construction and from the restriction functor $\cdot|_U$, the natural diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & g_A \arrow[d] \\ A|_U \arrow[r] & g_{A|_U} = (g_A)|_U. \end{tikzcd} \tag{3.22}$$

Hence, for an object $A \in \mathcal{M}(V)$, a direct computation yields

$$\begin{aligned}\big(\mathcal{E}(f) \circ \Phi_V\big)(A) &= f^* g_A \\ &= (g_A)|_U \\ &= g_{A|_U} \quad \text{from (3.22)} \\ &= g_{f^*A} \\ &= \Phi_U(f^*A) \\ &= \big(\Phi_U \circ \mathcal{M}(f)\big)(A), \end{aligned} \tag{3.23}$$

which gives an "objectwise" commutativity of Diagram (3.21). Similarly, for any morphism

$$(A, \rho) : A \longrightarrow A \bullet \rho = A' \text{ in } Hom_{\mathcal{M}(V)}(A, A'),$$

and for each morphism $f : U \hookrightarrow V$, one has another natural diagram again from the definition and the restriction functor as above:

$$\begin{tikzcd} \rho \arrow[r] \arrow[d] & \varphi_\rho \arrow[d] \\ \rho|_U \arrow[r] & \varphi_{\rho|_U} = (\varphi_\rho)|_U \end{tikzcd} \tag{3.24}$$

Therefore, for a morphism (A, ρ) in $\mathcal{M}(V)$, we obtain

$$\begin{aligned}\big(\mathcal{E}(f) \circ \Phi_V\big)(A, \rho) &= (f^* g_A, f^* \varphi_\rho) \quad \text{where } f^* g_A = (g_A)|_U \\ &= (g_{A|_U}, (\varphi_\rho)|_U) \quad \text{from (3.22)} \\ &= (g_{A|_U}, \varphi_{\rho|_U}) \quad \text{from (3.24)} \\ &= (g_{f^*A}, \varphi_{f^*\rho}) \\ &= \Phi_U(f^*A, f^*\rho) \\ &= \big(\Phi_U \circ \mathcal{M}(f)\big)(A, \rho), \end{aligned} \tag{3.25}$$

which implies the desired "morphismwise" commutativity in Diagram (3.21).

Conclusion. We then conclude that Φ in (3.13) defines a *natural transformation* between $\mathcal{M}$ and $\mathcal{E}$ via the collection $\{\Phi_U\}_{U \in \mathcal{C}}$ of the maps described in (3.14). This completes the proof of Theorem 1.3.

Acknowledgment

I would like to thank Alberto Cattaneo, Ödül Tetik, and Neeraj Deshmukh for their helpful discussions and valuable comments. I am also grateful to Ödül Tetik for numerous discussions on geometric structures in the context of stacks.

I would like to thank the Institute of Mathematics, University of Zurich, where this paper was revisited and improved. Additionally, I would acknowledge the support provided by The Scientific and Technological Research Council of Türkiye (TÜBİTAK) under 2219-International Postdoctoral Research Fellowship Programme (2021-1).

I am grateful to the anonymous referees for their comprehensive review. I would like to express my gratitude for their valuable comments and suggestions, which improved the quality of the manuscript.

A. Appendix A: symmetries in the context of Lagrangian formalism

In what follows, we summarize basic ideas about symmetries of a theory. For more details, we refer to [4].

In brief, Hamilton's action principle allows us to study identities and conserved quantities from the symmetries of the corresponding Lagrangian, and hence invariance properties of the action under certain transformations. This approach applies not only to the trajectories of individual particles in classical mechanics, but also works for continuous fields like $g_{\mu\nu}$. For the case of Einstein-Hilbert action, we consider its change under transformations of the form

$$g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) + \delta g_{\mu\nu}(x). \tag{A.1}$$

The Lagrangian in this case is chosen so that the action $\mathcal{I}_{EH}[g]$ is invariant under the transformation above for the *metrics satisfying Einstein field equations.*

It should be noted that the variations above are not necessarily generated by diffeomorphisms. However, to capture the diffeomorphis-invariant nature of GR, we consider certain types of variations induced by *infinitesimally generated diffeomorphisms*, by which we mean diffeomorphisms that are generated by a vector field X. In that case, we call X the *infinitesimal generator* of the corresponding transformation.

Remark A.1 *Recall that any vector field defines a one-parameter group of diffeomorphisms via its local flow. Using an infinitesimal diffeomorphism φ^X (and hence the corresponding flow), one can examine how the metric tensor field $g_{\mu\nu}$ changes when it is pulled back along the integral curves of X. Notice that this is exactly what the Lie derivative $\mathcal{L}_X g_{\mu\nu}$ measures. We then introduce the following definition.*

Definition A.2 *By a* variation induced from an infinitesimal diffeomorphism φ^X*, we actually mean*

$$\delta g_{\mu\nu} := \mathcal{L}_X g_{\mu\nu}, \tag{A.2}$$

with the transformation $g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) + \mathcal{L}_X g_{\mu\nu}(x)$.

B. Appendix B: vacuum solutions vs. flat connections

Given a 3D gravity theory on M in the sense of Definition 2.14 with $G := G_\Lambda$ in (2.10), we define the *space of holonomies* by

$$\mathcal{H}_M := Hom(\pi_1(M), G)/\sim, \tag{B.1}$$

where the quotient is given by the conjugate action of G. It follows that a (flat or constant curvature) spacetime structure s defines a holonomy $\rho_s \in \mathcal{H}_M$. Therefore, we have a well-defined map

$$Hol : \big\{(G_\Lambda, X_\Lambda) \text{ structures on } M\big\} \longrightarrow \mathcal{H}_M, \ s \mapsto Hol(s) := \rho_s. \tag{B.2}$$

Remark B.1 *The converse is not true in general, meaning that the holonomy may not enough to determine the whole geometry. However, there are important results for some special cases. For instance, when M has* *a topology of the form $\Sigma \times \mathbb{R}$, with $\Lambda = 0$ (corresponding to $G = ISO(2,1)$ with $X = \mathbb{R}^{2,1}$), and Σ a closed Riemann surface of genus $g > 1$, then it has been shown by Mess [13] that the holonomy group determines a unique "maximal" spacetime M. This result will also be relevant for our purposes. In brief, we have the following theorem.*

Theorem B.2 *([13], Prop. 2) Given a Fuchsian representation[‖] $\rho : \pi_1(\Sigma) \to PSL(2;\mathbb{R})$ with Σ a closed Riemann surface of genus $g > 1$, there exists a flat Lorentzian manifold M of the form $\Sigma \times (0, \infty)$ and holonomy $\psi : \pi_1(\Sigma) \to ISO(2,1)$ such that $\psi = \rho$.*

Notice that Theorem B.2 above implies the desired map $\mathcal{M}_{flat,\Sigma,G} \xrightarrow{\simeq} \mathcal{EH}(M)$ between the corresponding classical phase spaces, and hence the equivalence of gravity with gauge theory in the sense of Definition 2.18 (cf. Theorem 2.20).

Holonomy representation vs. flat G-bundles. There is an important interpretation of the elements of $Hom(\pi_1(M), G)/\sim$, which leads to defining the induced map $\mathcal{EH}(M) \to \mathcal{M}_{flat,\Sigma,G}$ between the corresponding classical phase spaces (cf. the map in (2.17)).

Let Σ be a Riemann surface. It has been shown in [5] that there is a one-to-one correspondence between the moduli space $\mathcal{M}_{flat,\Sigma,G}$ of (gauge equivalence classes of) flat G-connections on Σ and the moduli space $Hom(\pi_1(\Sigma), G)/G$ of (holonomy) representations of the surface group $\pi_1(\Sigma)$ in G, where G acts on $Hom(\pi_1(\Sigma), G)$ by conjugation. That is, we have

$$\mathcal{M}_{flat,\Sigma,G} \simeq Hom(\pi_1(\Sigma), G)/G, \tag{B.3}$$

which means that flat connections can be equivalently seen as representations of $\pi_1(\Sigma)$.

For simplicity, we now assume that M has a topology of the form $\Sigma \times \mathbb{R}$, with Σ a closed oriented surface. Consider Lorentzian 3D gravity on M for $\Lambda = 0$ described as a $(ISO(2,1), \mathbb{R}^{2,1})$ structure on M. Then we obtain the composition

$$\mathcal{EH}(M) \xrightarrow{\simeq} \big\{(ISO(2,1), \mathbb{R}^{2,1}) \text{ strucs. on } M\big\} \xrightarrow{Hol} \mathcal{H}_M \xrightarrow{\simeq} Hom\big(\pi_1(\Sigma), ISO(2,1)\big)/\sim \xrightarrow{\simeq} \mathcal{M}_{flat,\Sigma,G}, \tag{B.4}$$

where the first map is equal to (2.8); the second map is the one defined in (B.2); the third one is induced by the isomorphism $\pi_1(M) \simeq \pi_1(\Sigma)$ as $M \simeq \Sigma \times \mathbb{R}$; and the last map is (B.3).

In general, for a 3D gravity theory on a generic M in the sense of Definition 2.14 with $G := G_\Lambda$ in (2.10), it is also possible to obtain the induced map (see [6, 7])

$$\mathcal{EH}(M, \Lambda) \simeq \big\{(G_\Lambda, X_\Lambda) \text{ strucs. on } M\big\} \longrightarrow \mathcal{M}_{flat,M,G_\Lambda}, \tag{B.5}$$

which assigns to an equivalence class of (flat or constant curvature) vacuum solution to the field equations a gauge equivalence class of flat G_Λ-connection of the corresponding G_Λ-bundle, where $\mathcal{M}_{flat,M,G_\Lambda}$ denotes the moduli space of flat G_Λ-connections over M.

[‖] By *a Fuchsian representation*, we mean the representations arising from the holonomy of a hyperbolic structure on Σ.

References

[1] Anel M. The geometry of ambiguity: an introduction to the ideas of derived geometry. In: Anel M, Catren G, (editors). New Spaces in Mathematics Formal and Conceptual Reflections. Cambridge, UK: Cambridge University Press, 2021, pp. 505-553.

[2] Benini M, Schenkel A. Higher structures in algebraic quantum field theory. In: Higher Structures in M-Theory - LMS/EPSRC Durham Symposium Proceedings; Durham, UK; 2018. 1910015. https://doi.org/10.1002/prop.201910015

[3] Benini M, Schenkel A, Schreiber U. The stack of Yang–Mills fields on Lorentzian manifolds. Communications in Mathematical Physics 2018; 359: 765-820. https://doi.org/10.1007/s00220-018-3120-1

[4] Bertschinger E. Symmetry transformations, the Einstein-Hilbert action, and gauge invariance. Physics 8.962 notes, Massachusetts Institute of Technology Department of Physics, Cambridge, USA, 2002.

[5] Bradlow, S, Garcia-Prada O, Goldman W, Wienhard A. Representations of Surface Groups: background material for AIM Workshop. In: Representations of Surface Groups - American Institute of Mathematics Workshop; Palo Alto, CA; 2007.

[6] Carlip S. Lectures on (2+1)-Dimensional Gravity. arXiv preprint 1995; gr-qc/9503024. https://doi.org/10.48550/arXiv.gr-qc/9503024

[7] Carlip, S. Geometric structures and loop variables in (2+ 1)-dimensional gravity. arXiv preprint 1993; gr-qc/9309020. https://doi.org/10.48550/arXiv.gr-qc/9309020

[8] Costello K, Gwilliam O. Factorization Algebras in Quamtum Field Theory Vol. 2. Cambridge, UK: Cambridge University Press, 2021.

[9] Hollander S. A homotopy theory for stacks. Israel Journal of Mathematics 2008; 163: 93-124. https://doi.org/10.1007/s11856-008-0006-5

[10] Hovey M. Model Categories. Providence, RI, USA: American Mathematical Society, 2008.

[11] Kiran K.S, Krishnan C, Raju A. 3D gravity, Chern–Simons and higher spins: a mini introduction. Modern Physics Letters A 2015; 30 (32): 1530023. https://doi.org/10.1142/S0217732315300232

[12] Ludewig M, Stoffel A. A framework for geometric field theories and their classification in dimension one. Symmetry, Integrability and Geometry: Methods and Applications 2021; 17: 072. https://doi.org/10.3842/SIGMA.2021.072

[13] Mess G. Lorentz spacetimes of constant curvature. Geometriae Dedicata 2007; 126: 3-45. https://doi.org/10.1007/s10711-007-9155-7

[14] Meusburger, C. Geometrical (2+ 1)-gravity and the Chern-Simons formulation: grafting, Dehn twists, Wilson loop observables and the cosmological constant. Communications in Mathematical Physics 2007; 273.3: 705-754. https://doi.org/10.1007/s00220-007-0255-x

[15] Toën B. Derived algebraic geometry. EMS Surveys in Mathematical Sciences 2014; 1 (2): 153-240. https://doi.org/10.4171/emss/4

[16] Wise DK. MacDowell–Mansouri gravity and Cartan geometry. Classical and Quantum Gravity 2010; 27 (15): 155010. https://doi.org/10.1088/0264-9381/27/15/155010

[17] Witten E. 2+1 dimensional gravity as an exactly soluble system. Nuclear Physics B 1988; 311 (1): 46-78. https://doi.org/10.1016/0550-3213(88)90143-5

[18] Witten E. Quantization of Chern-Simons gauge theory with complex gauge group. Communications in Mathematical Physics 1991; 137: 29-66. https://doi.org/10.1007/BF02099116