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On Vietoris' hybrid number sequence

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Abstract: This work is intended to establish a relation between Vietoris' sequence, which is a rational sequence, and hybrid numbers. Then it provides some characteristic properties of the hybrid numbers with Vietoris' number coefficients. Some relations between this hybrid number and its norm, the recurrence relations, the generating function, Binet-like formula and Catalan-like identities are also indicated. Furthermore, a determinantal approach is presented to obtain elements of Vietoris' hybrid number sequence.

Key words: Vietoris' number sequence, recurrence relation, hybrid number, tridiagonal matrix

1. Introduction

A hybrid number $z = a + b\mathbf{i} + c\boldsymbol{\epsilon} + d\mathbf{h}$, a, b, c, $d \in \mathbb{R}$ which come to exist by linear combination of complex, dual and hyperbolic numbers introduced by Özdemir, [18]. Here, the hybrid units $\mathbf{i}, \boldsymbol{\epsilon}$ and $\mathbf{h}$ satisfy the conditions:

$$\mathbf{i}^2 = -1, \boldsymbol{\epsilon}^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \boldsymbol{\epsilon} + \mathbf{i}, \tag{1.1}$$

where $1 \leftrightarrow (1,0,0,0)$, $\mathbf{i} \leftrightarrow (0,1,0,0)$, $\boldsymbol{\epsilon} \leftrightarrow (0,0,1,0)$, $\mathbf{h} \leftrightarrow (0,0,0,1)$ represent real, complex, dual, and hyperbolic units, respectively. The set of hybrid numbers is a non-commutative ring according to addition and multiplication (see detailed information in [11, 18].)

Before continuing with the importance of the paper, it seems worthwhile to mention the special number sequences. Fibonacci number sequence is one of the most important special integer sequences with almost limitless applications. Mathematicians have been fascinated by it for almost 800 years. One may refer to the first comprehensive survey of mathematics' most fascinating number sequences Fibonacci and Lucas numbers [13, 17] for details. The study of the relation between special number sequences and multicomponent number systems has attracted much attention. This is largely due to the following problem: Can existing special number sequences be generalized to different number systems? This problem is studied for special hybrid numbers such as Fibonacci, Lucas, Pell, Padovan, etc., and their properties and so on; see [7, 10, 12, 15, 20–23].

As a general case of integer sequences, Vietoris' number sequence is one of the rational sequences. This special rational sequence is firstly defined in a theorem by Vietoris [24] with important applications, in harmonic analysis [1] and in the theory of stable holomorphic functions [19]. The sth element of the Vietoris' number

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sequence $\{v_s\}_{s\geq 0}$ is given by the formula:

$$v_s = \frac{1}{2^s}\binom{s}{\lfloor \frac{s}{2} \rfloor}, \; s \geq 0, \tag{1.2}$$

where $\binom{s}{\lfloor \frac{s}{2} \rfloor}$ is the central binomial coefficient ([14]) and the notion $\lfloor , \rfloor$ represents the floor function. The first several values of this sequence are (related with the sequence A283208 in OEIS*):

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \ldots$$

The detailed information including the properties of Vietoris' number sequence can be found in [2–5, 8, 9, 14, 25].

Our main research is concerned with Vietoris' number sequence with 4-dimensions via noncommutative hybrid numbers by a unified approach. For this target, the study is organized as follows. In Section 2, basic concepts about the hybrid numbers and the Vietoris' number sequence are presented. In Section 3, starting with the definition of Vietoris' hybrid numbers, several properties, the recurrence relations, and Catalan-like identities are discussed. Furthermore, a generating function and Binet-like formula are examined. This section finishes with a tridiagonal matrix approach to Vietoris' hybrid numbers.

2. Basic notions and arguments

To be brief, but a little bit more detailed, in this section we give a discussion of the hybrid numbers ([11, 18]) and then restrict this discussion to the Vietoris' number sequence ([2–5, 8, 9, 14, 24, 25]) also.

Let $z_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\epsilon} + d_1\mathbf{h}$ and $z_2 = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\epsilon} + d_2\mathbf{h}$ be hybrid numbers. For the equality $z_1 = z_2$ is valid if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_2$ and $d_1 = d_2$. The addition (hence subtraction) of the hybrid numbers z_1 and z_2 is $z_1 + z_2 = (a_1 \pm a_2) + (b_1 \pm b_2)\mathbf{i} + (c_1 \pm c_2)\boldsymbol{\epsilon} + (d_1 \pm d_2)\mathbf{h}$. By means of the conditions in

Table 1. The multiplication of the hybrid units, [18]

.	$\mathbf{i}$	$\boldsymbol{\epsilon}$	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\epsilon}+\mathbf{i}$
$\boldsymbol{\epsilon}$	$1+\mathbf{h}$	0	$-\boldsymbol{\epsilon}$
$\mathbf{h}$	$-\boldsymbol{\epsilon}-\mathbf{i}$	$\boldsymbol{\epsilon}$	1

Table 1 which are extended form of the equations (1.1), the multiplication of the hybrid numbers z_1 and z_2 is given as:

$$\begin{aligned} z_1z_2 \;&= (a_1a_2 - b_1b_2 + b_1c_2 + b_2c_1 + d_1d_2) + (a_1b_2 + a_2b_1 + b_1d_2 - d_1b_2)\mathbf{i} \\ &\quad + (a_1c_2 + b_1d_2 + a_2c_1 - c_1d_2 - b_2d_1 + c_2d_1)\boldsymbol{\epsilon} + (a_1d_2 - b_1c_2 + b_2c_1 + a_2d_1)\mathbf{h}. \end{aligned}$$

The multiplication is not commutative but associative. For z_1, $S_{z_1} = a_1$ is called the scalar part and $\mathbf{V}_{z_1} = b_1\mathbf{i} + c_1\boldsymbol{\epsilon} + d_1\mathbf{h}$ is called the vector part. The conjugate of z_1 is:

$$\overline{z_1} = a_1 - b_1\mathbf{i} - c_1\boldsymbol{\epsilon} - d_1\mathbf{h}. \tag{2.1}$$

*The Encyclopedia of Integer Sequences, https://oeis.org/book.

We thus get $\overline{z_1} = S_{z_1} - \mathbf{V}_{z_1}$. Every hybrid number can be written with 2×2 real matrices:

$$M(z_1) = M\left(a_1 + b_1\mathbf{i} + c_1\boldsymbol{\epsilon} + d_1\mathbf{h}\right) = \begin{bmatrix} a_1 + c_1 & b_1 - c_1 + d_1 \\ -b_1 + c_1 + d_1 & a_1 - c_1 \end{bmatrix}, \tag{2.2}$$

with

$$M(1) \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M(\mathbf{i}) \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, M(\boldsymbol{\epsilon}) \leftrightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, M(\mathbf{h}) \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, the character of z_1 is defined by

$$C(z_1) = \overline{z_1}z_1 = z_1\overline{z_1} = a_1^2 + (b_1 - c_1)^2 - c_1^2 - d_1^2 = \det(M(z_1)). \tag{2.3}$$

z_1 is spacelike, timelike, lightlike if $C(z_1) < 0$, $C(z_1) > 0$, $C(z_1) = 0$, respectively. The norm of z_1 is:

$$||z_1|| = \sqrt{|C(z_1)|} = \sqrt{|a_1^2 + (b_1 - c_1)^2 - c_1^2 - d_1^2|}. \tag{2.4}$$

The inverse of z_1 is given by: $z_1^{-1} = \dfrac{\overline{z_1}}{C(z_1)}$, $||z_1|| \neq 0$. The hybrid vector of z_1 is: $\xi_{z_1} = ((b_1 - c_1), c_1, d_1)$ and

$$C_\xi(z_1) = -(b_1 - c_1)^2 + c_1^2 + d_1^2 = \frac{(\operatorname{tr}(M(z_1)))^2 - 4\det(M(z_1))}{4}. \tag{2.5}$$

z_1 is elliptic (complike), hyperbolic (hyperlike), parabolic (duallike) if $C_\xi(z_1) < 0$, $C_\xi(z_1) > 0$, $C_\xi(z_1) = 0$, respectively. The norm of the hybrid vector of z_1 is denoted by $N(z_1) = \sqrt{|C_\xi(z_1)|}$.

Now, let us discuss the basic properties of Vietoris' number sequence $\{v_s\}_{s\geq 0}$ belonging to the sth element formula given by equation (1.2). Even members of $\{v_s\}_{s\geq 0}$ are given by: $v_{2n} = \frac{1}{2^{2n}}\begin{pmatrix} 2n \\ n \end{pmatrix}$, $n \geq 0$, where $v_{2n} = v_{2n-1}$. The two-term recurrence relation for $\{v_{2n}\}_{n\geq 0}$ is given as:

$$v_{2n+2} = d(2n)v_{2n},\ n \geq 0, \tag{2.6}$$

where $d(k) = \frac{k+1}{k+2}, k \geq 0$. Hence v_{2n} in terms of any v_{2k} or v_0 can be written follows, respectively ([8, 9]):

$$v_{2n} = \prod_{l=1}^{n-k} d(2n - 2l)v_{2k}, \quad n > k, \tag{2.7}$$

$$v_{2n+2} = \prod_{i=0}^{n} d(2i)v_0 = \frac{(2n+1)!!}{(2n+2)!!}. \tag{2.8}$$

The three consecutive term recurrence relation with an even index is:

$$v_{2n+2} = \frac{1}{2}v_{2n+1} + \frac{1}{2}d\,(2n)\,v_{2n}. \tag{2.9}$$

It can also be written as:

$$v_{2n+2} = \frac{1}{2}d\,(2n)\,v_{2n} + \frac{1}{2}d\,(2n)\,d\,(2n-2)\,v_{2n-2}. \tag{2.10}$$

The generating function is (see [5]):

$$g(x) = \frac{\sqrt{1+x}-\sqrt{1-x}}{x\sqrt{1-x}} = \sum_{s=0}^{\infty} v_s x^s, \;\; 0 < |x| < 1. \tag{2.11}$$

The Binet-like formula is given by:

$$v_{2n} = c_1\,(2n)\, r_1^{2n}\,(2n) + c_2\,(2n)\, r_2^{2n}\,(2n)\,, \tag{2.12}$$

where

$$r_1\,(2n) = \frac{1}{4}\left(1-\sqrt{1+8d(2n)}\right), \quad r_2\,(2n) = \frac{1}{4}\left(1+\sqrt{1+8d(2n)}\right) \tag{2.13}$$

and

$$\begin{cases} c_1\,(2n) = \dfrac{r_2^{2n}(2n)-v_2}{r_2^{2n}(2n)-r_1^{2n}(2n)} \displaystyle\prod_{k=1}^{n-1}(2r_1(2k)-1)r_1(2k) \\ c_2\,(2n) = \dfrac{v_2-r_1^{2n}(2n)}{r_2^{2n}(2n)-r_1^{2n}(2n)} \displaystyle\prod_{k=1}^{n-1}(2r_2(2k)-1)r_2(2k). \end{cases} \tag{2.14}$$

Here r_1 and r_2 have the following basic properties (see [8]): $r_2(0) = \frac{1+\sqrt{5}}{4}$ is half of the golden ratio, $r_1(2n)+r_2(2n) = \frac{1}{2}$, $r_1(2n)r_2(2n) = -\frac{1}{2}d(2n)$.

3. Vietoris' hybrid numbers

In this section, we extend the results of Vietoris' numbers to Vietoris' hybrid numbers taking into account the fundamental properties given in previous section.

Definition 3.1 *The sth element of the hybrid sequence with Vietoris' numbers $\{\mathsf{V}_s\}_{s\geq 0}$ is defined by V_s and determined by as follows:*

$$\mathsf{V}_s = v_s + v_{s+1}\mathbf{i} + v_{s+2}\boldsymbol{\epsilon} + v_{s+3}\mathbf{h}, \; \forall s \in \mathbb{N}. \tag{3.1}$$

Here the hybrid units satisfy the conditions in Table 1.

Since $v_{2n-1} = v_{2n}$, the even and odd indexed Vietoris' hybrid numbers are of the following forms:

$$\mathsf{V}_{2n} = v_{2n} + v_{2n+2}\,(\mathbf{i}+\boldsymbol{\epsilon}) + v_{2n+4}\mathbf{h}, \quad \text{for } s = 2n \tag{3.2}$$

and

$$\mathsf{V}_{2n+1} = v_{2n+2}\,(1+\mathbf{i}) + v_{2n+4}\,(\boldsymbol{\epsilon}+\mathbf{h})\,, \quad \text{for } s = 2n+1. \tag{3.3}$$

By using equation (2.6), we can write:

$$\begin{cases} \mathsf{V}_{2n} &= v_{2n}\mathsf{R}(2n), \\ \mathsf{V}_{2n+1} &= v_{2n+2}\mathsf{S}(2n+2), \end{cases} \tag{3.4}$$

where

$$\mathsf{R}(2n) = 1 + d(2n)\,(\mathbf{i}+\boldsymbol{\epsilon}) + d(2n)d(2n+2)\mathbf{h} \tag{3.5}$$

$$\mathsf{S}(2n+2) = 1 + \mathbf{i} + d(2n+2)\,(\boldsymbol{\epsilon}+\mathbf{h})\,. \tag{3.6}$$

Theorem 3.2 *Let $\{\mathsf{V}_s\}_{s\geq 0}$ be Vietoris' hybrid number sequence. $\forall n\in\mathbb{N}$, the following relations between the conjugation and the norm are satisfied:*

(i) $\left.\begin{array}{l} \dagger\,\mathsf{V}_{2n}-\mathbf{i}\mathsf{V}_{2n+1}-\boldsymbol{\epsilon}\mathsf{V}_{2n+2}-\mathbf{h}\mathsf{V}_{2n+3} \\ \ddagger\,\mathsf{V}_{2n}-\mathsf{V}_{2n+1}\mathbf{i}-\mathsf{V}_{2n+2}\boldsymbol{\epsilon}-\mathsf{V}_{2n+3}\mathbf{h} \end{array}\right\} = v_{2n}+v_{2n+2}-2v_{2n+4}-v_{2n+6},$

(ii) $\left.\begin{array}{l} \dagger\,\mathsf{V}_{2n+1}-\mathbf{i}\mathsf{V}_{2n+2}-\boldsymbol{\epsilon}\mathsf{V}_{2n+3}-\mathbf{h}\mathsf{V}_{2n+4} \\ \ddagger\,\mathsf{V}_{2n+1}-\mathsf{V}_{2n+2}\mathbf{i}-\mathsf{V}_{2n+3}\boldsymbol{\epsilon}-\mathsf{V}_{2n+4}\mathbf{h} \end{array}\right\} = v_{2n+2}-v_{2n+4}-v_{2n+8},$

(iii) $\mathsf{V}_s+\overline{\mathsf{V}}_s=2v_s\,,$

(iv) $\mathsf{V}_s^2+||\mathsf{V}_s||^2=2v_s\mathsf{V}_s\,.$

Proof On account of equations (2.1) and (2.4), the proof is obvious. □

Theorem 3.3 *Let $\{\mathsf{V}_s\}_{s\geq 0}$ be Vietoris' hybrid number sequence. Then the following matrices can be written:*

$$M(\mathsf{V}_{2n})=v_{2n}\begin{pmatrix} 1+d(2n) & d(2n)d(2n+2) \\ d(2n)d(2n+2) & 1-d(2n)\end{pmatrix}, \tag{3.7}$$

$$M(\mathsf{V}_{2n+1})=v_{2n}d(2n)\begin{pmatrix} 1+d(2n+2) & 1 \\ -1+2d(2n+2) & 1-d(2n+2)\end{pmatrix}. \tag{3.8}$$

Proof Applying equation (2.2) we have:

$$M(\mathsf{V}_{2n})=\begin{pmatrix} v_{2n}+v_{2n+2} & v_{2n+1}-v_{2n+2}+v_{2n+3} \\ -v_{2n+1}+v_{2n+2}+v_{2n+3} & v_{2n}-v_{2n+2}\end{pmatrix}.$$

Combining $v_{2n}=v_{2n-1}$ and equation (2.6), the above matrix transforms into the matrix given in equation (3.7). Similarly $M(\mathsf{V}_{2n+1})$ can be easily calculated. □

Corollary 3.4 *Let $\{\mathsf{V}_s\}_{s\geq 0}$ be Vietoris' hybrid number sequence. $\forall n\in\mathbb{N}$, the following statements are given for the characters and types of the even and odd indexed Vietoris' hybrid numbers:*

(i)

$$\begin{cases} C(\mathsf{V}_{2n})= & v_{2n}^2\left(1-d^2(2n)-d^2(2n)d^2(2n+2)\right), \\ C(\mathsf{V}_{2n+1})= & v_{2n}^2d^2(2n)\left(2-2d(2n+2)-d^2(2n+2)\right). \end{cases}$$

For $n<2$, V_{2n} is timelike since $C(\mathsf{V}_{2n})>0$. For $n\geq 2$, V_{2n} is spacelike due to the fact that $C(\mathsf{V}_{2n})<0$. V_{2n+1} is certainly spacelike since $C(\mathsf{V}_{2n+1})<0$. The lightlike Vietoris' hybrid number is not valid.

(ii)

$$\begin{cases} C_\xi(\mathsf{V}_{2n})= & v_{2n}^2d^2(2n)\left(1+d^2(2n+2)\right), \\ C_\xi(\mathsf{V}_{2n+1})= & -v_{2n}^2d^2(2n)\left(1-2d(2n+2)-d^2(2n+2)\right). \end{cases}$$

Hence, the Vietoris' hybrid numbers V_{2n} and V_{2n+1} are certainly hyperbolic since $C_\xi(\mathsf{V}_{2n})>0$ and $C_\xi(\mathsf{V}_{2n+1})>0$.

Proof Considering equations (2.3), (2.5) and matrices (3.7), (3.8), the classifications can be obtained quickly. □

Remark 3.5 For $n < 2$, V_{2n} is timelike hyperbolic, $n \geq 2$, V_{2n} is spacelike hyperbolic and $\forall n \in \mathbb{N}$, V_{2n+1} is spacelike hyperbolic.

Proposition 3.6 *Let* $\{\mathsf{V}_s\}_{s\geq 0}$ *be Vietoris' hybrid number sequence. Then the followings hold:*

(i) $\mathsf{V}_{2n} + \mathsf{V}_{2n+1} = v_{2n}\left[1 + d(2n) + 2d(2n)\mathbf{i} + d(2n)\left(1 + d(2n+2)\right)\boldsymbol{\epsilon} + 2d(2n)d(2n+2)\mathbf{h}\right]$,

(ii) $\mathsf{V}_{2n} - \mathsf{V}_{2n+1} = v_{2n}\left[1 - d(2n) + d(2n)\left(1 - d(2n+2)\right)\boldsymbol{\epsilon}\right]$,

(iii) $\mathsf{V}_{2n} + \mathsf{V}_{2n-1} = v_{2n}\left[2 + (d(2n)+1)\mathbf{i} + 2d(2n)\boldsymbol{\epsilon} + d(2n)(d(2n+2)+1)\mathbf{h}\right]$,

(iv) $\mathsf{V}_{2n} - \mathsf{V}_{2n-1} = v_{2n}\left[(d(2n)-1)\mathbf{i} + d(2n)(d(2n+2)-1)\mathbf{h}\right]$,

(v) $\mathsf{V}_{2n+1} + \mathsf{V}_{2n-1} = v_{2n}\left[(d(2n)+1)(1+\mathbf{i}) + d(2n)(d(2n+2)+1)(\boldsymbol{\epsilon}+\mathbf{h})\right]$,

(vi) $\mathsf{V}_{2n+1} - \mathsf{V}_{2n-1} = v_{2n}\left[(d(2n)-1)(1+\mathbf{i}) + d(2n)(d(2n+2)-1)(\boldsymbol{\epsilon}+\mathbf{h})\right]$,

(vii) $\mathsf{V}_{2n} + \mathsf{V}_{2n+2} = v_{2n}\left[(1+d(2n)) + d(2n)(d(2n+2)+1)(\mathbf{i}+\boldsymbol{\epsilon}) + d(2n)d(2n+2)(d(2n+4)+1)\mathbf{h}\right]$,

(viii) $\mathsf{V}_{2n+2} - \mathsf{V}_{2n} = v_{2n}\left[(d(2n)-1) + d(2n)(d(2n+2)-1)(\mathbf{i}+\boldsymbol{\epsilon}) + d(2n)d(2n+2)(d(2n+4)-1)\mathbf{h}\right]$.

Proof By considering equations (2.6), (3.2), and (3.3), the proofs are clear. □

Proposition 3.7 *Let* $\{\mathsf{V}_s\}_{s\geq 0}$ *be Vietoris' hybrid number sequence. Then, the following properties are given:*

(i) $\mathsf{V}_{2n}\mathsf{V}_{2m} - \overline{\mathsf{V}}_{2n}\overline{\mathsf{V}}_{2m} = 2v_{2n}v_{2m}\mathbf{V}_{\mathsf{R}(2n)+\mathsf{R}(2m)}$,

(ii) $\mathsf{V}_{2n}\mathsf{V}_{2m+1} - \overline{\mathsf{V}}_{2n}\overline{\mathsf{V}}_{2m+1} = 2d(2m)v_{2n}v_{2m}\mathbf{V}_{\mathsf{R}(2n)+\mathsf{S}(2m+2)}$,

(iii) $\mathsf{V}_{2n+1}\mathsf{V}_{2m} - \overline{\mathsf{V}}_{2n+1}\overline{\mathsf{V}}_{2m} = 2d(2n)v_{2n}v_{2m}\mathbf{V}_{\mathsf{S}(2n+2)+\mathsf{R}(2m)}$,

(iv) $\mathsf{V}_{2n+1}\mathsf{V}_{2m+1} - \overline{\mathsf{V}}_{2n+1}\overline{\mathsf{V}}_{2m+1} = 2d(2n)d(2m)v_{2n}v_{2m}\mathbf{V}_{\mathsf{S}(2n+2)+\mathsf{S}(2m+2)}$.

Proof (iii) From equations (3.4), we can write:

$$\mathsf{V}_{2n+1}\mathsf{V}_{2m} - \overline{\mathsf{V}}_{2n+1}\overline{\mathsf{V}}_{2m} = v_{2n+2}v_{2m}\mathsf{S}(2n+2)\mathsf{R}(2m) - v_{2n+2}v_{2m}\overline{\mathsf{S}(2n+2)}\,\overline{\mathsf{R}(2m)}.$$

Considering equations (3.5) and (3.6), we obtain:

$$\begin{aligned}\mathsf{V}_{2n+1}\mathsf{V}_{2m} - \overline{\mathsf{V}}_{2n+1}\overline{\mathsf{V}}_{2m} = &\ (1+d(2m))\,\mathbf{i} + (d(2n+2)+d(2m))\,\boldsymbol{\epsilon}\\ &+ (d(2n+2)+d(2m)d(2m+2))\,\mathbf{h}.\end{aligned}$$

Hence from equation (2.6) and the vector part $\mathbf{V}_{\mathsf{S}(2n+2)+\mathsf{R}(2m)}$ of $\mathsf{S}(2n+2)+\mathsf{R}(2m)$, the proof is obvious.

The other parts are clear using the same manner. □

Additionally, the expression $\mathsf{V}_{2n}\overline{\mathsf{V}}_{2m} - \overline{\mathsf{V}}_{2n}\mathsf{V}_{2m}$ is calculated as $2v_{2n}v_{2m}\mathbf{V}_{\mathsf{R}(2n)-\mathsf{R}(2m)}$. One can see that the only difference between this identity and part (i) of Proposition 3.7 is the sign in vector part. Hence the identities $\mathsf{V}_{2n}\overline{\mathsf{V}}_{2m+1} - \overline{\mathsf{V}}_{2n}\mathsf{V}_{2m+1}$, $\mathsf{V}_{2n+1}\overline{\mathsf{V}}_{2m} - \overline{\mathsf{V}}_{2n+1}\mathsf{V}_{2m}$ and $\mathsf{V}_{2n+1}\overline{\mathsf{V}}_{2m+1} - \overline{\mathsf{V}}_{2n+1}\mathsf{V}_{2m+1}$ can be calculated easily by the same manner considering parts (ii)-(iv) of Proposition 3.7.

3.1. The recurrence relations

In this subsection, the concept of the recurrence relations of Vietoris' number sequence are extended to Vietoris' hybrid number sequence.

Theorem 3.8 *Let* $\{\mathsf{V}_s\}_{s\geq 0}$ *be Vietoris' hybrid number sequence. Then,*

$$\mathsf{V}_{2n+2} = \mathsf{V}_{2n+1}\eta_r(2n+2) = \eta_l(2n+2)\mathsf{V}_{2n+1}, \tag{3.9}$$

where

$$\begin{cases} \eta_r(2n+2) &= \dfrac{\eta_0+\eta_1\mathbf{i}+\eta_2\boldsymbol{\epsilon}+\eta_3\mathbf{h}}{2-d(2n+2)\left(2+d(2n+2)\right)}, \\ \eta_l(2n+2) &= \dfrac{\eta_0+\eta_4\mathbf{i}-\eta_2\boldsymbol{\epsilon}+\eta_5\mathbf{h}}{2-d(2n+2)\left(2+d(2n+2)\right)} \end{cases}$$

with

$$\begin{cases} \eta_0 &= 1-d^2(2n+2)\left(1+d(2n+4)\right), \\ \eta_1 &= -1+d(2n+2)\left(1+d(2n+2)-d(2n+4)\right), \\ \eta_2 &= d(2n+2)d(2n+4)\left(d(2n+2)-1\right), \\ \eta_3 &= d(2n+2)\left(-d(2n+2)+d(2n+4)\right), \\ \eta_4 &= -1+d(2n+2)\left(1-d(2n+2)+d(2n+4)\right), \\ \eta_5 &= d(2n+2)\left(-2+d(2n+2)+d(2n+4)\right). \end{cases} \tag{3.10}$$

Proof Using equations (3.4), we can write:

$$\mathsf{V}_{2n+2} = v_{2n+2}\mathsf{S}(2n+2)\frac{\overline{\mathsf{S}(2n+2)}\mathsf{R}(2n+2)}{\mathsf{S}(2n+2)\overline{\mathsf{S}(2n+2)}}.$$

Considering Table 1, we compute $\overline{\mathsf{S}(2n+2)}\mathsf{R}(2n+2)$ as $\eta_0+\eta_1\mathbf{i}+\eta_2\boldsymbol{\epsilon}+\eta_3\mathbf{h}$ (see equations (3.10)). Also computation of $\mathsf{S}(2n+2)\overline{\mathsf{S}(2n+2)}$ gives $2-d(2n+2)\left(2+d(2n+2)\right)$. Hence we obtain:

$$\mathsf{V}_{2n+2} = \mathsf{V}_{2n+1}\frac{\eta_0+\eta_1\mathbf{i}+\eta_2\boldsymbol{\epsilon}+\eta_3\mathbf{h}}{2-d(2n+2)\left(2+d(2n+2)\right)} = \mathsf{V}_{2n+1}\eta_r(2n+2).$$

Again using equations (3.4), we now turn to the another case:

$$\mathsf{V}_{2n+2} = v_{2n+2}\mathsf{R}(2n+2) = \frac{\mathsf{R}(2n+2)\overline{\mathsf{S}(2n+2)}}{\overline{\mathsf{S}(2n+2)}\mathsf{S}(2n+2)}\mathsf{S}(2n+2)v_{2n+2}.$$

On account of equation Table 1, the computation of $\mathsf{R}(2n+2)\overline{\mathsf{S}(2n+2)}$ gives $\eta_0+\eta_4\mathbf{i}-\eta_2\boldsymbol{\epsilon}+\eta_5\mathbf{h}$ (see equations (3.10)), and so

$$V_{2n+2} = \frac{\eta_0+\eta_4\mathbf{i}-\eta_2\boldsymbol{\epsilon}+\eta_5\mathbf{h}}{2-d(2n+2)\left(2+d(2n+2)\right)}\mathsf{V}_{2n+1} = \eta_l(2n+2)\mathsf{V}_{2n+1}.$$

This completes the proof. □

Theorem 3.9 *Let* $\{\mathsf{V}_s\}_{s\geq 0}$ *be Vietoris' hybrid number sequence. Then,*

$$\mathsf{V}_{2n+1} = \mathsf{V}_{2n}\zeta_r(2n) = \zeta_l(2n)\mathsf{V}_{2n}, \tag{3.11}$$

where

$$\begin{cases} \zeta_r(2n) &= d(2n)\dfrac{\zeta_0+\zeta_1\mathbf{i}+\zeta_2\boldsymbol{\epsilon}+\zeta_3\mathbf{h}}{1-d^2(2n)\left(1+d^2(2n+2)\right)}, \\ \zeta_l(2n) &= d(2n)\dfrac{\zeta_0+\zeta_1\mathbf{i}+\zeta_4\boldsymbol{\epsilon}+\zeta_5\mathbf{h}}{1-d^2(2n)\left(1+d^2(2n+2)\right)} \end{cases}$$

with

$$\begin{cases} \zeta_0 &= 1-d(2n)d(2n+2)\left(1+d(2n+2)\right), \\ \zeta_1 &= 1-d(2n), \\ \zeta_2 &= -d(2n)+d(2n+2)\left(1+d(2n)-d(2n)d(2n+2)\right), \\ \zeta_3 &= -d(2n)+d(2n+2), \\ \zeta_4 &= -d(2n)+d(2n+2)\left(1-d(2n)+d(2n)d(2n+2)\right), \\ \zeta_5 &= d(2n)\left(1-2d(2n+2)\right)+d(2n+2). \end{cases} \tag{3.12}$$

Proof Using equations (3.4), we obtain:

$$\mathsf{V}_{2n+1} = d(2n)v_{2n}\mathsf{S}(2n+2) = v_{2n}\mathsf{R}(2n)d(2n)\frac{\overline{\mathsf{R}(2n)}\mathsf{S}(2n+2)}{\mathsf{R}(2n)\overline{\mathsf{R}(2n)}}.$$

According to Table 1, we see that $\overline{\mathsf{R}(2n)}\mathsf{S}(2n+2) = \zeta_0+\zeta_1\mathbf{i}+\zeta_2\boldsymbol{\epsilon}+\zeta_3\mathbf{h}$ (see equations (3.12)) and $\mathsf{R}(2n)\overline{\mathsf{R}(2n)} = 1-d^2(2n)\left(1+d^2(2n+2)\right)$. We thus get $\mathsf{V}_{2n+1} = \mathsf{V}_{2n}\zeta_r(2n)$. Similarly, considering Table 1, equations (3.4) and (3.12), we obtain:

$$\begin{aligned} \mathsf{V}_{2n+1} &= d(2n)v_{2n}\mathsf{S}(2n+2) \\ &= d(2n)\frac{\mathsf{S}(2n+2)\overline{\mathsf{R}(2n)}}{\overline{\mathsf{R}(2n)}\mathsf{R}(2n)}\mathsf{R}(2n)v_{2n} \\ &= d(2n)\frac{\zeta_0+\zeta_1\mathbf{i}+\zeta_4\boldsymbol{\epsilon}+\zeta_5\mathbf{h}}{1-d^2(2n)\left(1+d^2(2n+2)\right)}\mathsf{V}_{2n} \\ &= \zeta_l(2n)\mathsf{V}_{2n}. \end{aligned}$$

□

Theorem 3.10 *Let* $\{\mathsf{V}_s\}_{s\geq 0}$ *be Vietoris' hybrid number sequence. Then,*

$$\mathsf{V}_{2n+2} = \mathsf{V}_{2n}\mu_r(2n) = \mu_l(2n)\mathsf{V}_{2n},$$

where

$$\begin{cases} \mu_r(2n) &= d(2n)\dfrac{\mu_0+\mu_1\mathbf{i}+\mu_2\boldsymbol{\epsilon}+\mu_3\mathbf{h}}{1-d^2(2n)\left(1+d^2(2n+2)\right)}, \\ \mu_l(2n) &= d(2n)\dfrac{\mu_0+\mu_4\mathbf{i}+\mu_2\boldsymbol{\epsilon}+\mu_3\mathbf{h}}{1-d^2(2n)\left(1+d^2(2n+2)\right)} \end{cases}$$

with

$$\begin{cases} \mu_0 &= 1-d(2n)d(2n+2)\left(1+d(2n+2)d(2n+4)\right), \\ \mu_1 &= d(2n)\left(-1+d^2(2n+2)-d(2n+2)d(2n+4)\right)+d(2n+2), \\ \mu_2 &= -d(2n)+d(2n+2), \\ \mu_3 &= d(2n+2)\left(-d(2n)+d(2n+4)\right), \\ \mu_4 &= d(2n)\left(-1-d^2(2n+2)+d(2n+2)d(2n+4)\right)+d(2n+2). \end{cases} \tag{3.13}$$

Proof From equations (3.4), we have:

$$\mathsf{V}_{2n+2} = d(2n)v_{2n}\mathsf{R}(2n+2) = v_{2n}\mathsf{R}(2n)d(2n)\frac{\overline{\mathsf{R}(2n)}\mathsf{R}(2n+2)}{\mathsf{R}(2n)\overline{\mathsf{R}(2n)}}.$$

Applying Table 1, we can assert that $\overline{\mathsf{R}(2n)}\mathsf{R}(2n+2) = \mu_0 + \mu_1\mathbf{i} + \mu_2\boldsymbol{\epsilon} + \mu_3\mathbf{h}$ (see equations (3.13)), and $\mathsf{R}(2n)\overline{\mathsf{R}(2n)} = 1 - d^2(2n)\left(1 + d^2(2n+2)\right)$. This gives $\mathsf{V}_{2n+2} = \mathsf{V}_{2n}\mu_r(2n)$. Moreover, we have:

$$\mathsf{V}_{2n+2} = d(2n)v_{2n}\mathsf{R}(2n+2) = d(2n)\frac{\mathsf{R}(2n+2)\overline{\mathsf{R}(2n)}}{\overline{\mathsf{R}(2n)}\mathsf{R}(2n)}\mathsf{R}(2n)v_{2n}.$$

From this we conclude that

$$\mathsf{V}_{2n+2} = d(2n)\frac{\mu_0 + \mu_1\mathbf{i} + \mu_2\boldsymbol{\epsilon} + \mu_3\mathbf{h}}{1 - d^2(2n)\left(1 + d^2(2n+2)\right)}\mathsf{V}_{2n} = \mu_l(2n)\mathsf{V}_{2n},$$

where $\mathsf{R}(2n+2)\overline{\mathsf{R}(2n)} = \mu_0 + \mu_4\mathbf{i} + \mu_2\boldsymbol{\epsilon} + \mu_3\mathbf{h}$. □

Theorem 3.11 *Let* $\{\mathsf{V}_s\}_{s\geq 0}$ *be Vietoris' hybrid number sequence. Then,*

$$\mathsf{V}_{2n+1} = \mathsf{V}_{2n-1}\delta_r(2n) = \delta_l(2n)\mathsf{V}_{2n-1},$$

where

$$\begin{cases} \delta_r(2n) & = & d(2n)\dfrac{\delta_0 + \delta_1\left(\mathbf{i} - 2\mathbf{h}\right)}{2 - d(2n)\left(2 + d(2n)\right)}, \\ \delta_l(2n) & = & d(2n)\dfrac{\delta_0 - \delta_1\left(\mathbf{i} + 2\boldsymbol{\epsilon}\right)}{2 - d(2n)\left(2 + d(2n)\right)} \end{cases}$$

with

$$\begin{cases} \delta_0 & = & 2 - d(2n)\left(1 + d(2n+2)\right) - d(2n+2), \\ \delta_1 & = & d(2n) - d(2n+2). \end{cases} \tag{3.14}$$

Considering the functions η_r, ζ_r, μ_r and δ_r in Theorem 3.8-Theorem 3.11, the following theorem can be given.

Theorem 3.12 *Let* $\{\mathsf{V}_s\}_{s\geq 0}$ *be Vietoris' hybrid number sequence. Then, the three term recurrence relations are given by:*

(i) $\mathsf{V}_{s+1} = \mathsf{V}_s\mathsf{F}_1(s) + \mathsf{V}_{s-1}\mathsf{F}_0(s-1)$, *where*

$$\mathsf{F}_1(s) = \begin{cases} \frac{1}{2}\zeta_r(s), & s = 2n \\ \frac{1}{2}\eta_r(s+1), & s = 2n+1 \end{cases} \quad \text{and} \quad \mathsf{F}_0(s-1) = \begin{cases} \frac{1}{2}\delta_r(s), & s = 2n \\ \frac{1}{2}\mu_r(s-1) & s = 2n+1. \end{cases}$$

(ii) $\mathsf{V}_{s+2} = \mathsf{V}_s\mathsf{G}_1(s) + \mathsf{V}_{s-2}\mathsf{G}_0(s-2)$, *where*

$$\mathsf{G}_1(s) = \begin{cases} \frac{1}{2}\mu_r(s), & s = 2n \\ \frac{1}{2}\delta_r(s+1), & s = 2n+1 \end{cases} \quad \text{and} \quad \mathsf{G}_0(s-2) = \begin{cases} \mu_r(s-2)\mathsf{G}_1(s), & s = 2n \\ \delta_r(s-1)\mathsf{G}_1(s) & s = 2n+1. \end{cases}$$

Proof (i) The proof is a simple calculation by using Theorem 3.8-Theorem 3.11, and a relation $\mathsf{V}_{s+1} = \frac{1}{2}\mathsf{V}_{s+1} + \frac{1}{2}\mathsf{V}_{s+1}$.
(ii) Let us consider the case $s = 2n+1$. From equations (3.4), we obtain $\mathsf{V}_{2n+3} = v_{2n+4}\mathsf{S}(2n+4)$. From equation (2.10), we have:

$$\begin{aligned}
\mathsf{V}_{2n+3} = & \left(\tfrac{1}{2}d(2n+2)v_{2n+2} + \tfrac{1}{2}d(2n+2)d(2n)v_{2n}\right)\mathsf{S}(2n+4) \\
= & \tfrac{1}{2}\mathsf{V}_{2n+1}d(2n+2)\tfrac{\overline{\mathsf{S}}(2n+2)}{\mathsf{S}(2n+2)\overline{\mathsf{S}}(2n+2)}\mathsf{S}(2n+4) + \tfrac{1}{2}\mathsf{V}_{2n-1}\left[d(2n)\tfrac{\overline{\mathsf{S}}(2n)}{\mathsf{S}(2n)\overline{\mathsf{S}}(2n)}\mathsf{S}(2n+2)\right] \\
& \left[d(2n+2)\tfrac{\overline{\mathsf{S}}(2n+2)}{\mathsf{S}(2n+2)\overline{\mathsf{S}}(2n+2)}\mathsf{S}(2n+4)\right] \\
= & \tfrac{1}{2}\mathsf{V}_{2n+1}\delta_R(2n+2) + \tfrac{1}{2}\mathsf{V}_{2n-1}\delta_R(2n)\delta_R(2n+2) \\
= & \mathsf{V}_{2n+1}\mathsf{G}_1(2n+1) + \mathsf{V}_{2n-1}\mathsf{G}_0(2n-1).
\end{aligned}$$

The case $s = 2n$ can also be proved by the same manner. □

Theorem 3.13 *Consider even values of* p *and* $\mathsf{R}(p)\overline{\mathsf{R}}(p) \neq 0$. *A Catalan-like identity for* $\{\mathsf{V}_s\}_{s\geq 0}$ *is given by:*

$$\mathsf{V}_s^2 - \mathsf{V}_{s-p}\mathsf{V}_{s+p} = \mathsf{V}_p^2\mathcal{T}(s,p), \quad s > p,$$

where

$$\mathcal{T}(s,p) = \left[\frac{\overline{\mathsf{R}}(p)}{\mathsf{R}(p)\overline{\mathsf{R}}(p)}\prod_{l=1}^{\lfloor\frac{s+1-p}{2}\rfloor} d\left(2\left\lfloor\frac{s+1}{2}\right\rfloor - 2l\right)\right]^2 \mathcal{K}(s,p)$$

with

$$\mathcal{K}(s,p) = \begin{cases} \mathsf{R}^2(s) - \mathcal{X}(s,p)\mathsf{R}(s-p)\mathsf{R}(s+p), & s = 2n, \\ \mathsf{S}^2(s+1) - \mathcal{X}(s,p)\mathsf{S}(s+1-p)\mathsf{S}(s+1+p), & s = 2n+1, \end{cases}$$

and

$$\mathcal{X}(s,p) = \begin{cases} \prod_{l=1}^{\lfloor\frac{p}{2}\rfloor} \frac{d(s+p-2l)}{d(s-p+2l-2)}, & s+p \text{ even} \\ \prod_{l=1}^{\lfloor\frac{p}{2}\rfloor} \frac{d(s+1+p-2l)}{d(s-1-p+2l)}, & s+p \text{ odd.} \end{cases} \tag{3.15}$$

Proof Let us consider $p = 2k$ and conduct the proof by taking $s = 2n$ and $s = 2n+1$, respectively.
- For $s = 2n$, from equations (2.7) and (3.4), we obtain:

$$\mathsf{V}_{2n+2k} = v_{2n+2k}\mathsf{R}(2n+2k) = \prod_{l=1}^{k} d(2n+2k-2l)v_{2n}\mathsf{R}(2n+2k), \quad n > k, \tag{3.16}$$

and

$$\mathsf{V}_{2n-2k} = v_{2n-2k}\mathsf{R}(2n-2k) = \prod_{l=1}^{k} \frac{1}{d(2n-2k+2l-2)}v_{2n}\mathsf{R}(2n-2k), \quad n > k. \tag{3.17}$$

From equations (3.4), (3.16), and (3.17), we get:

$$\begin{aligned}\mathsf{V}_{2n}^2-\mathsf{V}_{2n-2k}\mathsf{V}_{2n+2k} &= v_{2n}^2\mathsf{R}^2(2n)-v_{2n-2k}v_{2n+2k}\mathsf{R}(2n-2k)\mathsf{R}(2n+2k)\\ &= v_{2k}^2\prod_{l=1}^{n-k}d^2(2n-2l)\left(\mathsf{R}^2(2n)\ -\mathcal{X}(2n,2k)\mathsf{R}(2n-2k)\mathsf{R}(2n+2k)\right)\\ &= v_{2k}^2\prod_{l=1}^{n-k}d^2(2n-2l)\mathcal{K}(2n,2k)\\ &= \mathsf{V}_{2k}^2\mathcal{T}(2n,2k).\end{aligned}$$

- For $s=2n+1$, by using equations (2.7) and (3.4), we find:

$$\begin{aligned}\mathsf{V}_{2n-2k+1} =&\ v_{2n-2k+2}\mathsf{S}(2n-2k+2)\\ =&\ \prod_{l=1}^{k}\tfrac{1}{d(2n-2k+2l)}v_{2n+2}\mathsf{S}(2n-2k+2),\quad n>k,\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}\mathsf{V}_{2n+2k+1} =&\ v_{2n+2k+2}\mathsf{S}(2n+2k+2)\\ =&\ \prod_{l=1}^{k}d(2n+2k+2-2l)v_{2n+2}\mathsf{S}(2n+2k+2),\quad n>k.\end{aligned}\tag{3.19}$$

By utilizing equations (3.4), (3.18), and (3.19), we have:

$$\begin{aligned}&\mathsf{V}_{2n+1}^2-\mathsf{V}_{2n-2k+1}\mathsf{V}_{2n+2k+1}\\ &= v_{2n+2}^2\mathsf{S}^2(2n+2)-v_{2n-2k+2}v_{2n+2k+2}\mathsf{S}(2n-2k+2)\mathsf{S}(2n+2k+2)\\ &= v_{2k}^2\prod_{l=1}^{n-k}d^2(2n-2l)d^2(2n)\left(\mathsf{S}^2(2n+2)-\mathcal{X}(2n+1,2k)\mathsf{S}(2n-2k+2)\mathsf{S}(2n+2k+2)\right)\\ &= v_{2k}^2\prod_{l=1}^{n-k}d^2(2n-2l)d^2(2n)\mathcal{K}(2n+1,2k)=\mathsf{V}_{2k}^2\mathcal{T}(2n+1,2k).\end{aligned}$$

The proof is completed. □

Theorem 3.14 *Consider odd values of* p *and* $\mathsf{S}(p+1)\overline{\mathsf{S}}(p+1)\neq 0$*. A Catalan-like identity for Vietoris' hybrid number sequence* $\{\mathsf{V}_s\}_{s\geq 0}$ *is given by:*

$$\mathsf{V}_s^2-\mathsf{V}_{s-p}\mathsf{V}_{s+p}=\mathsf{V}_p^2\mathcal{T}^*(s,p),\quad s>p,$$

where

$$\mathcal{T}^*(s,p)=\left(\frac{\overline{\mathsf{S}}(p+1)}{\mathsf{S}(p+1)\overline{\mathsf{S}}(p+1)}\prod_{l=1}^{\left\lfloor\frac{s-1-p}{2}\right\rfloor}d\left(2\left\lfloor\frac{s}{2}\right\rfloor-2l\right)\right)^2\mathcal{K}^*(s,p)\tag{3.20}$$

with

$$\mathcal{K}^*(s,p)=\begin{cases}\mathsf{R}^2(s)-\mathcal{X}(s,p)d(s)\mathsf{S}(s+1-p)\mathsf{S}(s+1+p), & s=2n,\\ d^2(s-1)\mathsf{S}^2(s+1)-\mathcal{X}(s,p)d(s-1)\mathsf{R}(s-p)\mathsf{R}(s+p), & s=2n+1.\end{cases}\tag{3.21}$$

Here $\mathcal{X}(s,p)$ *is the function defined in equation* (3.15).

Proof Let us consider $p=2k+1$ and conduct the proof by taking $s=2n$ and $s=2n+1$, respectively. From equations (2.7) and (3.4), we have the followings

- For $s=2n$:

$$\mathsf{V}_{2n-2k-1} = v_{2n-2k}\mathsf{S}(2n-2k) = \prod_{l=1}^{k} \frac{1}{d(2n-2k+2l-2)} v_{2n}\mathsf{S}(2n-2k), \quad n>k. \tag{3.22}$$

- For $s=2n+1$:

$$\begin{aligned} \mathsf{V}_{2n+2k+2} &= v_{2n+2k+2}\mathsf{R}(2n+2k+2) \\ &= \prod_{l=1}^{k} d(2n+2k+2-2l) v_{2n+2}\mathsf{R}(2n-2k+2), \quad n>k. \end{aligned} \tag{3.23}$$

With reference to equations (3.4), (3.20), and (3.21), substituting equations (3.19), (3.22), (3.23), and (3.17) into $\mathsf{V}_s^2 - \mathsf{V}_{s-p}\mathsf{V}_{s+p} = \mathsf{V}_p^2\mathcal{T}^*(s,p)$ in order, completes the proof. □

Theorem 3.15 *A generating function for* $\{\mathsf{V}_s\}_{s\geq 0}$ *is:*

$$\mathcal{G}(x) = \frac{1}{x^3}\left(g(x)(x^3+x^2\mathbf{i}+x\boldsymbol{\epsilon}+\mathbf{h}) - \mathsf{g}(x)\right), \quad 0<|x|<1,$$

where $\mathsf{g}(x) = \frac{1}{2}\left(2x^2\mathbf{i}+(2x+x^2)\boldsymbol{\epsilon}+(2+x+x^2)\mathbf{h}\right)$.

Proof Assume that $\mathcal{G}(x) = \sum_{s=0}^{\infty} \mathsf{V}_s\, x^s$ is generating function of $\{\mathsf{V}_s\}_{s\geq 0}$. Multiplying with x^3 and considering equation (2.11), we have:

$$\begin{aligned} x^3\mathcal{G}(x) &= \sum_{s=0}^{\infty} \mathsf{V}_s\, x^{s+3} \\ &= x^3\sum_{s=0}^{\infty} v_s x^s + x^2\sum_{s=0}^{\infty} v_{s+1}x^{s+1}\mathbf{i} + x\sum_{s=0}^{\infty} v_{s+2}x^{s+2}\boldsymbol{\epsilon} + \sum_{s=0}^{\infty} v_{s+3}x^{s+3}\mathbf{h} \\ &= g(x)\left(x^3+x^2\mathbf{i}+x\boldsymbol{\epsilon}+\mathbf{h}\right) - \frac{1}{2}\left(2x^2\mathbf{i}+(2x+x^2)\boldsymbol{\epsilon}+(2+x+x^2)\mathbf{h}\right) \\ &= g(x)\left(x^3+x^2\mathbf{i}+x\boldsymbol{\epsilon}+\mathbf{h}\right) - \mathsf{g}(x). \end{aligned}$$

□

Theorem 3.16 *A Binet-like formula for* $\{\mathsf{V}_s\}_{s\geq 0}$ *is given by:*

$$\mathsf{V}_s = \mathcal{P}_1(s) r_1^{2\left\lfloor \frac{s+1}{2}\right\rfloor}\left(2\left\lfloor\frac{s+1}{2}\right\rfloor\right) + \mathcal{P}_2(s) r_2^{2\left\lfloor \frac{s+1}{2}\right\rfloor}\left(2\left\lfloor\frac{s+1}{2}\right\rfloor\right),$$

where $\mathcal{P}_i(s) = \begin{cases} c_i(s)\mathsf{R}(s), & s=2n \\ c_i(s+1)\mathsf{S}(s+1), & s=2n+1. \end{cases} \quad (i=1,2)$

Here, $r_i(s), c_i(s)$ *are the functions defined in equations* (2.13) *and* (2.14).

Proof For $s=2n+1$, applying equations (2.12) and (3.4), we assert:

$$\begin{aligned} \mathsf{V}_{2n+1} &= v_{2n+2}\mathsf{S}_{2n+2} \\ &= \left(c_1(2n+2)\, r_1^{2n+2}(2n+2) + c_2(2n+2)\, r_2^{2n+2}(2n+2)\right)\mathsf{S}(2n+2) \\ &= c_1(2n+2)\,\mathsf{S}(2n+2) r_1^{2n+2}(2n+2) + c_2(2n+2)\,\mathsf{S}(2n+2) r_2^{2n+2}(2n+2) \\ &= \mathcal{P}_1(2n+1) r_1^{2n+2}(2n+2) + \mathcal{P}_2(2n+1) r_2^{2n+2}(2n+2). \end{aligned}$$

A similar proof works for the case $s = 2n$. □

In the following theorem, inspired by the studies [6, 8, 16], let us discuss some special tridiagonal matrices that generate the elements of Vietoris' hybrid number sequence $\{\mathsf{V}_s\}_{s\geq 0}$.

Theorem 3.17 *The following tridiagonal matrices of the order $n+1$ with Vietoris' hybrid number entries can be given by:*

$$\text{(i)}\quad \mathsf{K}_{n+1} = \begin{bmatrix} \mathsf{V}_1 & -\mathsf{V}_0 & & & & & \\ \mathsf{F}_0(0) & \mathsf{F}_1(1) & -1 & & & & \\ & \mathsf{F}_0(1) & \mathsf{F}_1(2) & -1 & & & \\ & & \mathsf{F}_0(2) & \mathsf{F}_1(3) & -1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \mathsf{F}_0(n-2) & \mathsf{F}_1(n-1) & -1 \\ & & & & & \mathsf{F}_0(n-1) & \mathsf{F}_1(n) \end{bmatrix},$$

$$\text{(ii)}\quad \mathsf{L}_{n+1} = \begin{bmatrix} \mathsf{V}_0 & & & & & & \\ -1 & 2\mathsf{F}_1(0) & \mathsf{F}_0(0) & & & & \\ & -1 & \mathsf{F}_1(1) & \mathsf{F}_0(1) & & & \\ & & -1 & \mathsf{F}_1(2) & \mathsf{F}_0(2) & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & \mathsf{F}_1(n-2) & \mathsf{F}_0(n-2) \\ & & & & & -1 & \mathsf{F}_1(n-1) \end{bmatrix},$$

$$\text{(iii)}\quad \mathsf{M}_{n+1} = \begin{bmatrix} \mathsf{V}_0 & & & & & & \\ -1 & 2\mathsf{G}_1(0) & \mathsf{G}_0(0) & & & & \\ & -1 & \mathsf{G}_1(2) & \mathsf{G}_0(2) & & & \\ & & -1 & \mathsf{G}_1(4) & \mathsf{G}_0(4) & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & \mathsf{G}_1(2n-4) & \mathsf{G}_0(2n-4) \\ & & & & & -1 & \mathsf{G}_1(2n-2) \end{bmatrix},$$

$$\text{(iv)}\quad \mathsf{N}_{n+1} = \begin{bmatrix} \mathsf{V}_1 & & & & & & \\ -1 & 2\mathsf{G}_1(1) & \mathsf{G}_0(1) & & & & \\ & -1 & \mathsf{G}_1(3) & \mathsf{G}_0(3) & & & \\ & & -1 & \mathsf{G}_1(5) & \mathsf{G}_0(5) & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & \mathsf{G}_1(2n-3) & \mathsf{G}_0(2n-3) \\ & & & & & -1 & \mathsf{G}_1(2n-1) \end{bmatrix},$$

where $\det(\mathsf{K}_{n+1}) = \mathsf{V}_{n+1}$, $\det(\mathsf{L}_{n+1}) = \mathsf{V}_n$, $\det(\mathsf{M}_{n+1}) = \mathsf{V}_{2n}$, $\det(\mathsf{N}_{n+1}) = \mathsf{V}_{2n+1}$. *It is worth noting that the determinants start always with all entries of the last column using Laplace expansion*[†].

Proof (i) The proof is conducted by mathematical induction. For $n = 0$:

$$\det(\mathsf{K}_1) = \mathsf{V}_1 = \frac{1}{2} + \frac{1}{2}\mathbf{i} + \frac{3}{8}\boldsymbol{\epsilon} + \frac{3}{8}\mathbf{h}.$$

[†]For any $[m_{ij}]_{n\times n}$, $\det([m_{ij}]_{n\times n}) = \sum_{i=1}^{n} c_{in} m_{in}$, with $c_{in} = (-1)^{i+n}\det Y_{in}$ where $\det Y_{in}$ is the i,n minor of M, [8].

For $n = 1$, considering Theorem 3.12, we get:

$$\det\left(\begin{bmatrix} \mathsf{V}_1 & -\mathsf{V}_0 \\ \mathsf{F}_0(0) & \mathsf{F}_1(1) \end{bmatrix}\right) = \mathsf{V}_1\mathsf{F}_1(1) + \mathsf{V}_0\mathsf{F}_0(0) = \mathsf{V}_2 = \frac{1}{2} + \frac{3}{8}\mathbf{i} + \frac{3}{8}\boldsymbol{\epsilon} + \frac{5}{16}\mathbf{h}.$$

For $n = 2$, we have:

$$\begin{aligned} \det\left(\begin{bmatrix} \mathsf{V}_1 & -\mathsf{V}_0 & 0 \\ \mathsf{F}_0(0) & \mathsf{F}_1(1) & -1 \\ 0 & \mathsf{F}_0(1) & \mathsf{F}_1(2) \end{bmatrix}\right) &= (-1)^{3+3}\det\left(\begin{bmatrix} \mathsf{V}_1 & -\mathsf{V}_0 \\ \mathsf{F}_0(0) & \mathsf{F}_1(1) \end{bmatrix}\right)\mathsf{F}_1(2) \\ &\quad -(-1)^{3+2}(\mathsf{V}_1)\,\mathsf{F}_0(1) \\ &= \mathsf{V}_2\mathsf{F}_1(2) + \mathsf{V}_1\mathsf{F}_0(1) \\ &= \mathsf{V}_3 \\ &= \frac{3}{8} + \frac{3}{8}\mathbf{i} + \frac{5}{16}\boldsymbol{\epsilon} + \frac{5}{16}\mathbf{h}. \end{aligned}$$

For $n - 1$, assume that $\det(\mathsf{K}_n) = \mathsf{V}_n$. By applying Laplace expansion and Theorem 3.12 for n, we get:

$$\begin{aligned} \det(\mathsf{K}_{n+1}) &= (-1)^{2n+2}\det(\mathsf{K}_n)\,\mathsf{F}_1(n) - (-1)^{2n+1}\det(\mathsf{K}_{n-1})\,\mathsf{F}_0(n-1) \\ &= \mathsf{V}_n\mathsf{F}_1(n) + \mathsf{V}_{n-1}\mathsf{F}_0(n-1) \\ &= \mathsf{V}_{n+1}. \end{aligned}$$

The other parts can be proved in a similar manner by mathematical induction. □

4. Conclusions

Motivated by properties of Vietoris' sequence and hybrid numbers, this work aims to bring together Vietoris' sequence and hybrid numbers to construct Vietoris' hybrid number sequence. Then some properties of the hybrid numbers with Vietoris' number coefficients are examined. For this approach, it is natural to seek the recurrence relations, the generating function, Binet-like formula, and Catalan-like identities. The results are also extended by the tridiagonal matrix approach for Vietoris' hybrid numbers. We hope the concept of Vietoris' hybrid number sequence will prove fruitful for other number sequences.

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