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Positive and decreasing solutions for higher order Caputo boundary value problems with sign-changing Green's function

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Abstract: In this paper, Caputo boundary value problems of order $3 < \zeta \leq 4$ are investigated on the interval $[0,1]$. By Guo-Krasnoselskii fixed point theorem, some criteria of existence and multiplicity of positive and decreasing solutions are established. The main novelty of the paper lies in its capability to achieve positive solutions while the corresponding Green's function changes sign. Finally, two examples are provided to illustrate the application of these results.

Key words: Caputo fractional derivatives, sign-changing Green's function, fixed point theorem

1. Introduction

There is currently great interest in fractional differential equations (FDEs), since these equations appear naturally in modelling many real world processes, see [8, 23]. Many interesting works were presented for the study of theoretical knowledge and applications of FDEs, see [5–7, 15, 17, 20, 26, 27], and the references therein.

Boundary value problems (BVPs) for integer or fractional order differential equations with positive solutions arise in many fields of science and engineering, see [2, 13, 24]. Therefore, the solvability of positive solutions constitute a significant class of problems, see [12, 18, 19, 22]. By using the fixed point theorems on cone, Bai and Lü [1] studied the existence of positive solutions for Riemann-Liouville (R-L) two-point BVPs with nonnegative Green's function. In fact, most of the existing papers have been written on positive solutions are based on the condition the corresponding Green's functions are nonnegative, see [25].

Recently, several papers have discussed on the existence of positive solutions while the Green's function changes sign, see [3, 4, 11, 21, 25]. In [14], Ma established some criteria of existence and nonexistence of positive solutions for nonlinear periodic BVPs under the condition the Green's kernel changes sign. In [16], Sun and Zhao discussed the following BVP

$$u'''(\chi) = f(\chi, u(\chi)),$$

$$u'(0) = u(1) = u''(\eta) = 0,$$

where f is a given function, η is a given constant, the corresponding Green's kernel may changes sign on $[0,1] \times [0,1]$. By iterative technique, they gave some existence results of positive solutions for such problem.

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Inspired by the above works, in present paper we deals with the follwing BVP

$$ {}^{C}D_{0+}^{\zeta}u(\chi) = f(\chi, u(\chi)), \quad \chi \in [0,1], \tag{1.1} $$

$$ u'(0) = u'''(0) = 0, \ \ u'''(\eta) + \lambda u''(0) = 0, \ \ u(1) - \gamma u(0) = 0, \tag{1.2} $$

where ${}^{C}D_{0+}^{\zeta}$ is the Caputo FD, $3 < \zeta \leq 4$, $0 < \eta, \lambda, \gamma < 1$ are constants, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

To the best of our knowledge, although the idea on obtaining positive solutions while the Green's function changes sign has been considered by some papers, very little is known on applying such idea on higher order Caputo BVPs in the literature. We undertake this investigation in the present paper. By Guo-Krasnoselskii fixed point theorem, for any positive integer $n (n \geq 2)$, our target is to establish some criteria of existence of at least $n-1$ positive and decreasing solutions for BVP (1.1)–(1.2). The most significant feature is that the present paper capability to achieve positive solutions while the corresponding Green's function changes sign.

The present paper is organized as follows. In Section 2, some useful definitions are introduced, and some lemmas are proved. In Section 3, some sufficient conditions for the existence of positive and decreasing solutions are derived. Section 4 presents some experiments to explain the results. In Section 5, the conclusion is given.

2. Preliminaries

Definition 2.1 *[8] Let $\zeta > 0$. Then the R-L fractional integral is*

$$ I_{0+}^{\zeta} f(\chi) = \frac{1}{\Gamma(\zeta)} \int_0^{\chi} (\chi - \xi)^{\zeta - 1} f(\xi) d\xi. $$

Definition 2.2 *[8] Let $\zeta > 0$. Then the Caputo FD is*

$$ {}^{C}D_{0+}^{\zeta} f(\chi) = \frac{d^n}{d\chi^n} \int_0^{\chi} \frac{(\chi - \xi)^{n-\zeta-1}}{\Gamma(n-\zeta)} \Big(f(\xi) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \xi^k \Big) d\xi, \tag{2.1} $$

where $n = [\zeta] + 1$ for $\zeta \notin \mathbb{N}_0$; $n = \zeta$ for $\zeta \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0, 1, \cdots\}$. If $f \in AC^n[0,1]$, then the Caputo FD is

$$ {}^{C}D_{0+}^{\zeta} f(\chi) = \frac{1}{\Gamma(n-\zeta)} \int_0^{\chi} (\chi - \xi)^{n-\zeta-1} f^{(n)}(\xi) d\xi. \tag{2.2} $$

Lemma 2.3 *[8] Let $\zeta > 0$ and let n be given by Definition 2.2. If $f \in AC^n[0,1]$, then*

$$ I_{0+}^{\zeta} \, {}^{C}D_{0+}^{\zeta} f(\chi) = f(\chi) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \chi^k. $$

Lemma 2.4 *The BVP*

$$ {}^{C}D_{0+}^{\zeta} u(\chi) = y(\chi), \ \ 3 < \zeta \leq 4, \ \ y \in C[0,1], \tag{2.3} $$

$$ u'(0) = u'''(0) = 0, \ \ u'''(\eta) + \lambda u''(0) = 0, \ \ u(1) - \gamma u(0) = 0, \tag{2.4} $$

has a unique solution

$$ u(\chi) = \int_0^1 G(\chi, \xi) y(\xi) d\xi, \tag{2.5} $$

where

$$G(\chi,\xi)=g_1(\chi,\xi)+g_2(\chi,\xi)+g_3(\chi,\xi), \tag{2.6}$$

and

$$g_1(\chi,\xi)=-\frac{1}{(1-\gamma)\Gamma(\zeta)}(1-\xi)^{\zeta-1},\quad (\chi,\xi)\in[0,1]\times[0,1],$$

$$g_2(\chi,\xi)=\begin{cases}0, & 0\le\chi\le\xi\le 1,\\ \frac{(\chi-\xi)^{\zeta-1}}{\Gamma(\zeta)}, & 0\le\xi\le\chi\le 1,\end{cases}$$

$$g_3(\chi,\xi)=\begin{cases}0, & \xi\ge\eta,\\ \frac{(\frac{1}{1-\gamma}-\chi^2)(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)}, & \xi<\eta.\end{cases}$$

Proof. By Lemma 2.3, we may transfer (2.3) to the integral equation

$$u(\chi)-\sum_{k=0}^{3}\frac{u^{(k)}(0)}{k!}\chi^k=\frac{1}{\Gamma(\zeta)}\int_0^{\chi}(\chi-\xi)^{\zeta-1}y(\xi)d\xi.$$

Using the condition of $u'(0)=u'''(0)=0$, it follows

$$u(\chi)-u(0)-\frac{u''(0)}{2}\chi^2=\frac{1}{\Gamma(\zeta)}\int_0^{\chi}(\chi-\xi)^{\zeta-1}y(\xi)d\xi.$$

According to $u'''(\eta)+\lambda u''(0)=0,\ \ u(1)-\gamma u(0)=0$, we have

$$u''(0)=\frac{-1}{\lambda\Gamma(\zeta-3)}\int_0^{\eta}(\eta-\xi)^{\zeta-4}y(\xi)d\xi,$$

$$u(0)=-\frac{1}{(1-\gamma)\Gamma(\zeta)}\int_0^1(1-\xi)^{\zeta-1}y(\xi)d\xi-\frac{1}{2(1-\gamma)}u''(0).$$

Thus

$$\begin{aligned}u(\chi)&=\tfrac{1}{\Gamma(\zeta)}\textstyle\int_0^{\chi}(\chi-\xi)^{\zeta-1}y(\xi)d\xi-\tfrac{1}{(1-\gamma)\Gamma(\zeta)}\int_0^1(1-\xi)^{\zeta-1}y(\xi)d\xi+\tfrac{\frac{1}{1-\gamma}-\chi^2}{2\lambda\Gamma(\zeta-3)}\int_0^{\eta}(\eta-\xi)^{\zeta-4}y(\xi)d\xi\\ &=\textstyle\int_0^1\Big(g_1(\chi,\xi)+g_2(\chi,\xi)+g_3(\chi,\xi)\Big)y(\xi)d\xi\\ &=\textstyle\int_0^1 G(\chi,\xi)y(\xi)d\xi.\end{aligned}$$

Conversely, if $u(\chi)$ satisfies the integral expression $u(\chi)=\int_0^1 G(\chi,\xi)y(\xi)d\xi$, then

$$u(\chi)=I_{0+}^{\zeta}y(\chi)+c_0+c_1\chi+c_2\chi^2+c_3\chi^3, \tag{2.7}$$

where

$$c_0=-\frac{1}{1-\gamma}I_{0+}^{\zeta}y(1)+\frac{1}{2(1-\gamma)\lambda}I_{0+}^{\zeta-3}y(\eta),\quad c_2=\frac{-1}{2\lambda}I_{0+}^{\zeta-3}y(\eta),\quad c_1=c_3=0. \tag{2.8}$$

Since $\zeta > 3$, we have $u'''(\chi) = I_{0+}^{\zeta-3} y(\chi)$. Thus $u \in C^3[0,1]$. By (2.1), it follows from the equality

$$\begin{aligned} {}^C D_{0+}^{\zeta} u(\chi) &= \tfrac{d^4}{d\chi^4} \int_0^{\chi} \tfrac{(\chi-\xi)^{3-\zeta}}{\Gamma(4-\zeta)} \Big(u(\xi) - u(0) - u'(0)\xi - \tfrac{u''(0)}{2}\xi^2 - \tfrac{u'''(0)}{6}\xi^3 \Big) d\xi \\ &= \tfrac{d^4}{d\chi^4} \int_0^{\chi} \tfrac{(\chi-\xi)^{3-\zeta}}{\Gamma(4-\zeta)} \Big(I_{0+}^{\zeta} y(\xi) + c_0 + c_2\xi^2 - u(0) - u'(0)\xi - \tfrac{u''(0)}{2}\xi^2 - \tfrac{u'''(0)}{6}\xi^3 \Big) d\xi \\ &= \tfrac{d^4}{d\chi^4} I_{0+}^{4-\zeta} I_{0+}^{\zeta} y(\chi) = y(\chi) \end{aligned} \tag{2.9}$$

that ${}^C D_{0+}^{\zeta} u \in C[0,1]$. From the argument used in Remark 2 of [10], we deduce that $u \in AC^4[0,1]$. The equality (2.9) also implies $u(\chi)$ satisfies (2.3). Moreover, through (2.7) and (2.8), we can obtain $u(\chi)$ satisfies (2.4). Thus, $u(\chi)$ is a solution of the BVP (2.3)–(2.4).

Remark 2.5 *From (2.2) in Definition 2.2, we can also get the fact $u(\chi)$ satisfies (2.3). Since $u \in AC^4[0,1]$, we have*

$$\begin{aligned} {}^C D_{0+}^{\zeta} y(\chi) &= \tfrac{1}{\Gamma(4-\zeta)} \int_0^{\chi} (\chi-\xi)^{3-\zeta} u^{(4)}(\xi) d\xi = \tfrac{d}{d\chi} \tfrac{1}{\Gamma(5-\zeta)} \int_0^{\chi} (\chi-\xi)^{4-\zeta} u^{(4)}(\xi) d\xi \\ &= \tfrac{d}{d\chi} \tfrac{1}{\Gamma(5-\zeta)} \Big(u'''(\xi)(\chi-\xi)^{4-\zeta}|_0^{\chi} + \int_0^{\chi} (4-\zeta)(\chi-\xi)^{3-\zeta} u'''(\xi) d\xi \Big) \\ &= \tfrac{d}{d\chi} \tfrac{1}{\Gamma(4-\zeta)} \int_0^{\chi} (\chi-\xi)^{3-\zeta} u'''(\xi) d\xi = \tfrac{d}{d\chi} \tfrac{1}{\Gamma(4-\zeta)} \int_0^{\chi} (\chi-\xi)^{3-\zeta} I_{0+}^{\zeta-3} y(\xi) d\xi \\ &= \tfrac{d}{d\chi} I_{0+}^{4-\zeta} I_{0+}^{\zeta-3} y(\chi) = y(\chi). \end{aligned}$$

Remark 2.6 *Let*

$$\lambda \le \gamma \eta^2 (\zeta - 3) \tag{2.10}$$

hold. Then $G(\chi,\xi)$ given by (2.6) changes sign.

Proof. From the expression of $g_2(\chi,\xi)$ and $g_3(\chi,\xi)$, it follows that $g_2(\chi,\xi)$ is increasing with respect to χ, and $g_3(\chi,\xi)$ is decreasing with respect to χ for $\xi < \eta$.

For $\xi \ge \eta$, $0 < \eta < 1$, $(\chi,\xi) \in [0,1] \times [0,1]$, it implies

$$\begin{aligned} G(\chi,\xi) &= g_1(\chi,\xi) + g_2(\chi,\xi) + g_3(\chi,\xi) = -\tfrac{(1-\xi)^{\zeta-1}}{(1-\gamma)\Gamma(\zeta)} + g_2(\chi,\xi) \\ &\le -\tfrac{(1-\xi)^{\zeta-1}}{(1-\gamma)\Gamma(\zeta)} + g_2(1,\xi) = -\tfrac{(1-\xi)^{\zeta-1}}{(1-\gamma)\Gamma(\zeta)} + \tfrac{(1-\xi)^{\zeta-1}}{\Gamma(\zeta)}. \end{aligned}$$

Since $3 < \zeta \le 4$, $0 < \gamma < 1$, we have

$$G(\chi,\xi) \le \frac{-(1-\xi)^{\zeta-1} + (1-\gamma)(\chi-\xi)^{\zeta-1}}{(1-\gamma)\Gamma(\zeta)} < \frac{-(1-\xi)^{\zeta-1} + (\chi-\xi)^{\zeta-1}}{(1-\gamma)\Gamma(\zeta)} \le 0.$$

Next, we consider the case $\xi < \eta$, $0 < \eta < 1$, $(\chi,\xi) \in [0,1] \times [0,1]$, we have

$$\begin{aligned} G(\chi,\xi) &\ge -\tfrac{(1-\xi)^{\zeta-1}}{(1-\gamma)\Gamma(\zeta)} + g_2(\chi,\xi) + g_3(1,\xi) \ge -\tfrac{(1-\xi)^{\zeta-1}}{(1-\gamma)\Gamma(\zeta)} + \tfrac{(\frac{1}{1-\gamma}-1)(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)} \\ &= \tfrac{1}{\Gamma(\zeta)}\big(-\tfrac{(1-\xi)^{\zeta-1}}{(1-\gamma)} + \tfrac{(\frac{1}{1-\gamma}-1)(\zeta-1)(\zeta-2)(\zeta-3)(\eta-\xi)^{\zeta-4}}{2\lambda}\big) \\ &\ge \tfrac{1}{\Gamma(\zeta)}\big(-\tfrac{1}{1-\gamma} + \tfrac{(\frac{1}{1-\gamma}-1)(\zeta-1)(\zeta-2)(\zeta-3)(\eta-\xi)^{\zeta-4}}{2\lambda}\big). \end{aligned}$$

It follows from (2.10) that

$$G(\chi,\xi) \geq \frac{1}{\Gamma(\zeta)}(-\frac{1}{1-\gamma} + \frac{(\zeta-1)(\zeta-2)(\eta-\xi)^{\zeta-4}}{2(1-\gamma)\eta^2}) \geq \frac{-\eta^2+(\eta-\xi)^{\zeta-4}}{(1-\gamma)\eta^2\Gamma(\zeta)} > 0.$$

Thus, it gets $G(\chi,\xi) > 0,\ 0 \leq \xi < \eta$, and $G(\chi,\xi) \leq 0,\ \eta \leq \xi \leq 1$.

Lemma 2.7 *Let* (2.10) *hold and*

$$K_0 = \{y \in C[0,1] : y \geq 0 \text{ and } y \text{ is decreasing on } [0,1]\}.$$

Suppose that $y \in K_0$. *Then* u *given by* (2.5) *satisfies* $u \in K_0$, *and* u *is concave on* $[0,\eta]$.

Proof. According to Lemma 2.4, we have $u(\chi) = \int_0^1 G(\chi,\xi)y(\xi)d\xi$. Since $y(\chi) \geq 0, \chi \in [0,1]$, we get

$$\begin{aligned} u''(\chi) &= \tfrac{1}{\Gamma(\zeta-2)} \int_0^\chi (\chi-\xi)^{\zeta-3} y(\xi)d\xi - \tfrac{1}{\lambda\Gamma(\zeta-3)} \int_0^\eta (\eta-\xi)^{\zeta-4} y(\xi)d\xi \\ &\leq \tfrac{1}{\Gamma(\zeta-2)} \int_0^\chi (\chi-\xi)^{\zeta-3} y(\xi)d\xi - \tfrac{1}{\gamma\eta^2(\zeta-3)\Gamma(\zeta-3)} \int_0^\eta (\eta-\xi)^{\zeta-4} y(\xi)d\xi \\ &\leq \tfrac{1}{\Gamma(\zeta-2)} \Big(\int_0^\chi \Big((\chi-\xi)^{\zeta-3} - (\eta-\xi)^{\zeta-4}\Big) y(\xi)d\xi - \int_\chi^\eta (\eta-\xi)^{\zeta-4} y(\xi)d\xi \Big) \\ &\leq 0, \quad \chi \in [0,\eta]. \end{aligned}$$

This shows u is concave on $[0,\eta]$.

Next we will prove $u \in K_0$. For $\chi \in [0,\eta]$, $u'(\chi)$ is decreasing, then

$$u'(\chi) \leq u'(0) = 0. \tag{2.11}$$

For $\chi \in (\eta,1]$, in view of $y \in K_0$ and (2.10), it gets

$$\begin{aligned} u'(\chi) &= \tfrac{1}{\Gamma(\zeta-1)} \int_0^\chi (\chi-\xi)^{\zeta-2} y(\xi)d\xi - \tfrac{\chi}{\lambda\Gamma(\zeta-3)} \int_0^\eta (\eta-\xi)^{\zeta-4} y(\xi)d\xi \\ &\leq \tfrac{1}{\Gamma(\zeta-1)} \int_0^1 y(\xi)d\xi - \tfrac{\eta^2}{\lambda\Gamma(\zeta-3)} \int_0^1 y(\eta\xi)d\xi \\ &\leq \tfrac{1}{\Gamma(\zeta-1)} \int_0^1 y(\xi)d\xi - \tfrac{1}{\gamma\Gamma(\zeta-2)} \int_0^1 y(\eta\xi)d\xi \leq 0. \end{aligned} \tag{2.12}$$

As a consequence of (2.11) and (2.12), we have $u'(\chi) \leq 0,\ \chi \in [0,1]$. This shows that $u(\chi)$ is decreasing on $[0,1]$. Considering

$$\begin{aligned} u(1) &= (\tfrac{1}{1-\gamma} - 1)\Big(\tfrac{1}{2\lambda\Gamma(\zeta-3)} \int_0^\eta (\eta-\xi)^{\zeta-4} y(\xi)d\xi - \tfrac{1}{\Gamma(\zeta)} \int_0^1 (1-\xi)^{\zeta-1} y(\xi)d\xi \Big) \\ &\geq (\tfrac{1}{1-\gamma} - 1)\Big(\tfrac{\eta}{2\lambda\Gamma(\zeta-3)} \int_0^1 y(\eta\xi)d\xi - \tfrac{1}{\Gamma(\zeta)} \int_0^1 y(\xi)d\xi \Big) \\ &\geq (\tfrac{1}{1-\gamma} - 1)\Big(\tfrac{(\zeta-2)(\zeta-1)}{2\gamma\eta\Gamma(\zeta)} \int_0^1 y(\eta\xi)d\xi - \tfrac{1}{\Gamma(\zeta)} \int_0^1 y(\xi)d\xi \Big) \geq 0, \end{aligned}$$

we get $u(\chi) \geq 0,\ \chi \in [0,1]$. Hence $u \in K_0$.

Lemma 2.8 *Let (2.10) hold and $y \in K_0$. Then u given by (2.5) satisfies*

$$\min_{\chi\in[0,\theta]} u(\chi) \geq \theta^* \|u\|,$$

where $\theta \in (0, \eta)$, $\theta^* = \frac{\eta-\theta}{\eta}$, $\|u\| = \max_{\chi\in[0,1]} |u(\chi)|$.

Proof. By Lemma 2.7, we have

$$u(\chi) \geq \frac{\eta-\chi}{\eta} u(0) + \frac{\chi}{\eta} u(\eta) \geq \frac{\eta-\chi}{\eta} u(0) = \frac{\eta-\chi}{\eta} \|u\|, \quad \chi \in [0, \eta].$$

Consequently, for $\theta \in (0, \eta)$, it gets

$$\min_{\chi\in[0,\theta]} u(\chi) = u(\theta) \geq \frac{\eta-\theta}{\eta} \|u\| = \theta^* \|u\|.$$

Lemma 2.9 *[9] (Guo-Krasnoselskii fixed point theorem) Let $\mathbb{E}$ be a Banach space and $K \subseteq \mathbb{E}$ be a cone. Suppose $\Omega_1 \subseteq \mathbb{E}$ and $\Omega_2 \subseteq \mathbb{E}$ are bounded open sets such that $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator, and T satisfies either*
(1) $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$, or
(2) $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$.
Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

Let $0 < \theta \leq \eta - \frac{2\gamma(\zeta-3)}{\zeta(\zeta-1)(\zeta-2)}\eta^2 < \eta$ and

$$K = \{u \in K_0 : \min_{\chi\in[0,\theta]} u(\chi) \geq \theta^* \|u\|\}.$$

Lemma 3.1 *Let (2.10) hold. Define $T : K \to C[0,1]$ by*

$$Tu(\chi) = \int_0^1 G(\chi, \xi) f(\xi, u(\xi)) d\xi, \quad \chi \in [0, 1].$$

Assume that f satisfies
(H) $f(\chi, u)$ is decreasing with respect to χ, and increasing with respect to u.
Then $T : K \to K$, and T is completely continuous.

Proof. As $u \in K$ and (H) holds, we have $f(\cdot, u(\cdot)) \in K_0$. Thus, we determine that $T : K \to K$ via Lemmas 2.7 and 2.8.

Next, we will use the Arzela-Ascoli theorem to testify the consequence. Here, (a) Let $u_n \to u$ in K. Then

$$\begin{aligned} |Tu_n(\chi) - Tu(\chi)| &\leq \tfrac{1}{\Gamma(\zeta)} \int_0^\chi (\chi-\xi)^{\zeta-1} |f(\xi, u_n(\xi)) - f(\xi, u(\xi))| d\xi \\ &\quad + \tfrac{1}{(1-\gamma)\Gamma(\zeta)} \int_0^1 (1-\xi)^{\zeta-1} |f(\xi, u_n(\xi)) - f(\xi, u(\xi))| d\xi \\ &\quad + \tfrac{\frac{1}{1-\gamma} - \chi^2}{2\lambda\Gamma(\zeta-3)} \int_0^\eta (\eta-\xi)^{\zeta-4} |f(\xi, u_n(\xi)) - f(\xi, u(\xi))| d\xi. \end{aligned}$$

Note that f is continuous, and thus, we obtain

$$\|Tu_n - Tu\| \to 0, \ \ n \to \infty.$$

Thus, T is continuous.

(b) Let $\Omega \subset K$ be bounded. Then, for any $u \in \Omega$, we have

$$\begin{aligned} |Tu(\chi)| &\leq \tfrac{L}{\Gamma(\zeta)} \int_0^{\chi} (\chi-\xi)^{\zeta-1} d\xi + \tfrac{L}{(1-\gamma)\Gamma(\zeta)} \int_0^1 (1-\xi)^{\zeta-1} d\xi + \tfrac{\frac{1}{1-\gamma} L}{2\lambda\Gamma(\zeta-3)} \int_0^{\eta} (\eta-\xi)^{\zeta-4} d\xi \\ &\leq (1+\tfrac{1}{1-\gamma}) \tfrac{L}{\Gamma(\zeta)} + \tfrac{\frac{1}{1-\gamma} L}{2\lambda\Gamma(\zeta-2)}, \end{aligned}$$

where $L = \max\limits_{0\leq\chi\leq 1, 0\leq u\leq M} |f(\chi,u)| + 1$. Hence, T is uniformly bounded.

(c) For any $u \in \Omega$, $\chi_1, \chi_2 \in [0,1]$, $\chi_1 < \chi_2$, we get

$$\begin{aligned} |Tu(\chi_2) - Tu(\chi_1)| &\leq \left| \tfrac{1}{\Gamma(\zeta)} \int_0^{\chi_2} (\chi_2-\xi)^{\zeta-1} f(\xi,u(\xi)) d\xi - \tfrac{1}{\Gamma(\zeta)} \int_0^{\chi_1} (\chi_1-\xi)^{\zeta-1} f(\xi,u(\xi)) d\xi \right| \\ &\quad + \tfrac{\chi_2^2-\chi_1^2}{2\lambda\Gamma(\zeta-3)} \int_0^{\eta} (\eta-\xi)^{\zeta-4} f(\xi,u(\xi)) d\xi \\ &\leq \tfrac{L}{\Gamma(\zeta+1)} (\chi_2^{\zeta} - \chi_1^{\zeta}) + \tfrac{(\chi_2^2-\chi_1^2)L}{2\lambda\Gamma(\zeta-2)}, \end{aligned} \tag{3.1}$$

which implies that the left-hand side of (3.1)$\to 0$ if $\chi_1 \to \chi_2$. Thus, T is equicontinuous in K.

Combining the above three steps (a), (b), (c) with the Arzela-Ascoli theorem, we discern that T is completely continuous.

Lemma 3.2 *Let (2.10) and (H) hold. Suppose there is a number $r_1 > 0$ such that*

$$f(0, r_1) \leq \frac{r_1}{A},$$

where $A = \frac{1}{(1-\gamma)\lambda\Gamma(\zeta-2)} > 0$ is a constant. Then we have

$$\|Tu\| \leq \|u\|, \ \ u \in K \cap \partial\Omega_{r_1}, \tag{3.2}$$

with $\Omega_{r_1} = \{u \in C[0,1] : \|u\| < r_1\}$.

Proof. From Remark 2.6, it gets

$$0 < G(\chi,\xi) = -\frac{(1-\xi)^{\zeta-1}}{(1-\gamma)\Gamma(\zeta)} + g_2(\chi,\xi) + \frac{(\frac{1}{1-\gamma} - \chi^2)(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)}, \ \ 0 \leq \xi < \eta,$$

and $G(\chi,\xi) \leq 0$, $\eta \leq \xi \leq 1$. As $3 < \zeta \leq 4$, $0 < \eta, \gamma < 1$ and (2.10) holds, for $0 \leq \xi < \eta$, we can deduce

$$\begin{aligned} G(\chi,\xi) &\leq \tfrac{(\chi-\xi)^{\zeta-1}}{\Gamma(\zeta)} + \tfrac{(\frac{1}{1-\gamma}-\chi^2)(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)} \\ &\leq \tfrac{2\gamma\eta^2}{(\zeta-1)(\zeta-2)} \cdot \tfrac{\chi^{\zeta-1}(1-\frac{\xi}{\chi})^{\zeta-1}}{2\lambda\Gamma(\zeta-3)} + \tfrac{(\frac{1}{1-\gamma}-\chi^2)(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)} \\ &\leq \tfrac{2}{(\zeta-1)(\zeta-2)} \cdot \tfrac{\chi^2(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)} + \tfrac{(\frac{1}{1-\gamma}-\chi^2)(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)} \\ &\leq \tfrac{\frac{1}{1-\gamma}(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)} = g_3(0,\xi), \ \ \xi \leq \chi, \end{aligned} \tag{3.3}$$

and

$$G(\chi,\xi) \le \frac{(\frac{1}{1-\gamma}-\chi^2)(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)} \le \frac{\frac{1}{1-\gamma}(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)} = g_3(0,\xi), \;\; \chi \le \xi. \tag{3.4}$$

Thus, for $u \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned} 0 \le Tu(\chi) \;\; &= \textstyle\int_0^1 G(\chi,\xi)f(\xi,u(\xi))d\xi \\ &= \textstyle\int_0^\eta G(\chi,\xi)f(\xi,u(\xi))d\xi + \int_\eta^1 G(\chi,\xi)f(\xi,u(\xi))d\xi \\ &\le \textstyle\int_0^\eta G(\chi,\xi)f(\xi,u(\xi))d\xi \\ &\le f(0,u(0))\cdot \textstyle\int_0^\eta g_3(0,\xi)d\xi \\ &\le f(0,r_1)\tfrac{\frac{1}{1-\gamma}}{2\lambda\Gamma(\zeta-2)} \le f(0,r_1)A \le r_1 = \|u\|, \;\; \chi \in [0,1]. \end{aligned}$$

This shows that (3.2) holds.

Lemma 3.3 *Let (2.10) and (H) hold. Suppose there is a number $r_2 > 0$ such that*

$$f(\theta,\theta^* r_2) \ge \frac{r_2}{B}.$$

where $B = \int_0^\theta G(0,\xi)d\xi$. Then we have

$$\|Tu\| \ge \|u\|, \;\; u \in K \cap \partial\Omega_{r_2}, \tag{3.5}$$

with $\Omega_{r_2} = \{u \in C[0,1] : \|u\| < r_2\}$.

Proof. First, we will prove $0 < B < A$. For $\xi < \theta < \eta$, it gets $G(0,\xi) > 0$ via Remark 2.6. Hence, $B > 0$. As (3.3) and (3.4) hold, we have

$$B = \int_0^\theta G(0,\xi)d\xi \le \int_0^\theta g_3(0,\xi)d\xi < \frac{1}{2\lambda(1-\gamma)\Gamma(\zeta-2)} \le A.$$

Thus, B is a constant, and satisfies $0 < B < A$. Next, we shall show that $\int_\theta^1 G(0,\xi)f(\xi,u(\xi))d\xi \ge 0$. Using the condition (H) and $0 < \theta \le \eta - \frac{2\gamma(\zeta-3)}{\zeta(\zeta-1)(\zeta-2)}\eta^2$, we have

$$\begin{aligned} \textstyle\int_\theta^1 G(0,\xi)f(\xi,u(\xi))d\xi \;\; &= \textstyle\int_\theta^\eta G(0,\xi)f(\xi,u(\xi))d\xi + \int_\eta^1 G(0,\xi)f(\xi,u(\xi))d\xi \\ &\ge f(\eta,u(\eta))\Big(\textstyle\int_\theta^\eta G(0,\xi)d\xi + \int_\eta^1 G(0,\xi)d\xi\Big) \\ &= f(\eta,u(\eta))\Big(\textstyle\int_\theta^\eta \frac{\frac{1}{1-\gamma}(\eta-\xi)^{\zeta-4}}{2\lambda\Gamma(\zeta-3)}d\xi - \int_\theta^1 \frac{1}{(1-\gamma)\Gamma(\zeta)}(1-\xi)^{\zeta-1}d\xi\Big) \\ &= f(\eta,u(\eta))\Big(\tfrac{(\eta-\theta)^{\zeta-3}}{2(1-\gamma)\lambda\Gamma(\zeta-2)} - \tfrac{(1-\theta)^{\zeta}}{(1-\gamma)\Gamma(\zeta+1)}\Big) \\ &\ge \tfrac{f(\eta,u(\eta))}{(1-\gamma)\Gamma(\zeta+1)}\Big(\tfrac{\zeta(\zeta-1)(\zeta-2)(\eta-\theta)^{\zeta-3}}{2\gamma\eta^2(\zeta-3)} - 1\Big) \ge 0. \end{aligned}$$

Now, it remains to verify (3.5). For $u \in K \cap \partial\Omega_{r_2}$, it gets

$$\begin{aligned}\|Tu\| = Tu(0) &= \int_0^1 G(0,\xi) f(\xi, u(\xi)) d\xi \\ &= \int_0^\theta G(0,\xi) f(\xi, u(\xi)) d\xi + \int_\theta^1 G(0,\xi) f(\xi, u(\xi)) d\xi \\ &\geq \int_0^\theta G(0,\xi) f(\xi, u(\xi)) d\xi \\ &\geq f(\theta, u(\theta)) \int_0^\theta G(0,\xi) d\xi \\ &\geq f(\theta, \theta^* r_2) B \geq r_2 = \|u\|.\end{aligned}$$

Thus, (3.5) holds.

Theorem 3.4 *Let (2.10) and (H) hold. Suppose that: there are two numbers $r_1 > 0$ and $r_2 > 0$ with $r_1 \neq r_2$, and r_1, r_2 satisfy*

$$f(0, r_1) \leq \frac{r_1}{A}, \;\; f(\theta, \theta^* r_2) \geq \frac{r_2}{B}.$$

Then the BVP (1.1)–(1.2) has at least one positive and decreasing solution u, and

$$r_1 \leq \|u\| \leq r_2 \;\; (or \;\; r_2 \leq \|u\| \leq r_1).$$

Furthermore, $u(\chi)$ is concave on $[0, \eta]$.

Proof. Let $r_1 < r_2$. In view of Lemmas 3.2 and 3.3, we get

$$\|Tu\| \leq \|u\|, \;\; u \in K \cap \partial\Omega_{r_1},$$

$$\|Tu\| \geq \|u\|, \;\; u \in K \cap \partial\Omega_{r_2}.$$

Hence, it follows the Guo-Krasnoselskii fixed point theorem that T has a fixed point $u \in K \cap (\overline{\Omega}_{r_2} \backslash \Omega_{r_1})$, which is a desired solution of the BVP (1.1)–(1.2).

Theorem 3.5 *Let (2.10) and (H) hold. Assume that there exist three positive numbers r_1, r_2, r_3 with $r_1 < r_2 < r_3$, and meet one of the following conditions*

$$f(0, r_1) \leq \frac{r_1}{A}, \;\; f(\theta, \theta^* r_2) > \frac{r_2}{B}, \;\; f(0, r_3) \leq \frac{r_3}{A}, \tag{3.6}$$

$$f(\theta, \theta^* r_1) \geq \frac{r_1}{B}, \;\; f(0, r_2) < \frac{r_2}{A}, \;\; f(\theta, \theta^* r_3) \geq \frac{r_3}{B}. \tag{3.7}$$

Then the BVP (1.1)–(1.2) has at least two positive and decreasing solutions u_1 and u_2, and

$$r_1 \leq \|u_1\| < r_2 < \|u_2\| \leq r_3.$$

Proof. We only consider the case when (3.6) is satisfied, as the proof for the case (3.7) is similar. By Lemmas 3.2 and 3.3, we get

$$\|Tu\| \leq \|u\|, \;\; u \in K \cap \partial\Omega_{r_1},$$

$$\|Tu\| > \|u\|, \ \ u \in K \cap \partial\Omega_{r_2},$$

and

$$\|Tu\| \leq \|u\|, \ \ u \in K \cap \partial\Omega_{r_3}.$$

Hence, it follows Guo-Krasnoselskii fixed point theorem that T has two fixed points $u_1 \in K \cap (\overline{\Omega}_{r_2} \backslash \Omega_{r_1})$ and $u_2 \in K \cap (\overline{\Omega}_{r_3} \backslash \Omega_{r_2})$, which are two desired solutions of the BVP (1.1)–(1.2).

Remark 3.6 *Similar to Theorem 3.5, for any positive integer $n (n \geq 2)$, we can obtain the existence of at least $n-1$ positive and decreasing solutions of the BVP (1.1)–(1.2), where n is the number of $r_i, i = 1, 2, \cdots, n$.*

4. Examples

Example 4.1 Consider

$$^{C}D_{0+}^{3.5} u(\chi) = \frac{u(\chi)}{625} + 1 - \chi, \quad \chi \in [0,1], \tag{4.1}$$

$$u'(0) = u'''(0) = 0, \ \ u'''(0.5) + \frac{1}{32} u''(0) = 0, \ \ u(1) - 0.5u(0) = 0, \tag{4.2}$$

where $\zeta = 3.5, \ \eta = 0.5, \ \lambda = \frac{1}{32}, \ \gamma = 0.5, \ f(\chi, u) = \frac{u}{625} + 1 - \chi$. Then (2.10) and (H) are satisfied.

The Green's function $G(\chi,\xi) = g_1(\chi,\xi) + g_2(\chi,\xi) + g_3(\chi,\xi), \ (\chi,\xi) \in [0,1] \times [0,1]$, where

$$g_1(\chi,\xi) = -\frac{2}{\Gamma(3.5)}(1-\xi)^{2.5}, \ \ (\chi,\xi) \in [0,1] \times [0,1],$$

$$g_2(\chi,\xi) = \begin{cases} 0, & 0 \leq \chi \leq \xi \leq 1, \\ \frac{(\chi-\xi)^{2.5}}{\Gamma(3.5)}, & 0 \leq \xi \leq \chi \leq 1, \end{cases} \qquad g_3(\chi,\xi) = \begin{cases} 0, & \xi \geq 0.5, \\ \frac{16(2-\chi^2)(0.5-\xi)^{-0.5}}{\Gamma(0.5)}, & \xi < 0.5. \end{cases}$$

This shows

$$\begin{aligned} G(\chi,\xi) &= -\tfrac{2}{\Gamma(3.5)}(1-\xi)^{2.5} + g_2(\chi,\xi) + \tfrac{16(2-\chi^2)(0.5-\xi)^{-0.5}}{\Gamma(0.5)} \\ &\geq -\tfrac{2}{\Gamma(3.5)}(1-\xi)^{2.5} + \tfrac{16(2-\chi^2)(0.5-\xi)^{-0.5}}{\Gamma(0.5)} \\ &\geq -\tfrac{2}{\Gamma(3.5)} + \tfrac{16}{\Gamma(0.5)} = 8.4252 > 0, \ \ 0 \leq \xi < 0.5, \end{aligned}$$

and

$$\begin{aligned} G(\chi,\xi) &= -\tfrac{2}{\Gamma(3.5)}(1-\xi)^{2.5} + g_2(\chi,\xi) + g_3(\chi,\xi) \\ &\leq -\tfrac{2}{\Gamma(3.5)}(1-\xi)^{2.5} + \tfrac{(1-\xi)^{2.5}}{\Gamma(3.5)} = -\tfrac{1}{\Gamma(3.5)}(1-\xi)^{2.5} \leq 0, \ \ 0.5 \leq \xi \leq 1. \end{aligned}$$

Thus, Remark 2.6 holds.

Next, we choose $\theta = 0.4$. Through computing, we have

$$\theta^* = \frac{\eta - \theta}{\eta} = \frac{1}{5}, \ \ A = \frac{1}{(1-\gamma)\lambda\Gamma(\zeta-2)} = \frac{64}{\Gamma(1.5)} = 72.2163,$$

$$B=\int_0^{\theta}G(0,\xi)d\xi=\int_0^{0.4}\Big(-\frac{2}{\Gamma(3.5)}(1-\xi)^{2.5}+\frac{32(\eta-\xi)^{-0.5}}{\Gamma(0.5)}\Big)d\xi=13.9707.$$

Moreover, it is easy to achieve

$$f(0,r_1)=1+\frac{r_1}{625}\le\frac{r_1}{A},\ \text{ if }\ r_1\ge 81.6507,$$

$$f(0.4,0.2*r_2)=0.6+\frac{r_2}{3125}\ge\frac{r_2}{B},\ \text{ if }\ r_2\le 8.4201.$$

Thus, we can choose $r_1=82,\ r_2=8$ in Theorem 3.4. Then the conditions of Theorem 3.4 are satisfied. Consequently the BVP (4.1)–(4.2) has a positive and decreasing solution u satisfying

$$8\le\|u\|\le 82.$$

Example 4.2 Consider

$$^{C}D_{0+}^{3.5}u(\chi)=\frac{u^2(\chi)}{20861}+1-\chi,\quad \chi\in[0,1], \tag{4.3}$$

$$u'(0)=u'''(0)=0,\ \ u'''(0.5)+\frac{1}{32}u''(0)=0,\ \ u(1)-0.5u(0)=0, \tag{4.4}$$

where $\zeta=3.5,\ \eta=0.5,\ \lambda=\frac{1}{32},\ \gamma=0.5,\ f(\chi,u)=\frac{u^2}{20861}+1-\chi$. Then (2.10) and (H) are satisfied, and Remark 2.6 holds.

From Example 4.1, we get $\theta=0.4$ and $\theta^*=\frac{1}{5},\ A=72.2163,\ B=13.9707$. By computing, we have

$$f(0.4,0.2*r_1)=0.6+\frac{0.04*r_1^2}{20861}\ge\frac{r_1}{B},\ \text{ if }\ r_1\ge 37322 \text{ or } r_1\le 8.3843,$$

$$f(0,r_2)=1+\frac{r_2^2}{20861}<\frac{r_2}{A},\ \text{ if }\ 143.9608<r_2<144.9075.$$

Thus, we can choose $r_1=8,\ r_2=144,\ r_3=37323$ in Theorem 3.5. Then the conditions of Theorem 3.5 are satisfied. Consequently the BVP (4.3)–(4.4) has two positive and decreasing solution u_1,u_2, and

$$8\le\|u_1\|<144<\|u_2\|\le 37323.$$

5. Conclusion

In the paper, for any positive integer $n(n\ge 2)$, the existence of at least $n-1$ positive and decreasing solutions for Caputo three-point BVPs are studied. The obtained solutions are also proved to be concave on $[0,\eta]$. The main novelty of the paper lies in obtaining positive solutions while the corresponding Green's function changes sign. For future work, one can discuss the positive solutions for other types of FDEs, and can also consider much more difficult research on fractional differential systems.

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