Turkish Journal of Mathematics

Volume 48 | Number 4 Article 2

7-3-2024

# Properties of gyrogroups induced by groups whose central quotients being 2-Engel

JATURON WATTANAPAN

TEERAPONG SUKSUMRAN

Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2024) 48: 634 – 644
© TÜBİTAK
doi:10.55730/1300-0098.3531

# Properties of gyrogroups induced by groups whose central quotients being 2-Engel

**Jaturon WATTANAPAN**[1,2], **Teerapong SUKSUMRAN**[3,*]
[1]Office of Research Administration, Chiang Mai University, Chiang Mai, Thailand
[2]Faculty of Science, Chiang Mai University, Chiang Mai, Thailand
[3]Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand

**Received:** 22.03.2024 • **Accepted/Published Online:** 07.06.2024 • **Final Version:** 03.07.2024

**Abstract:** A group $\Gamma$ is said to be CCII if the quotient $\Gamma/Z(\Gamma)$ is 2-Engel or, equivalently, commutator-inversion invariant, where $Z(\Gamma)$ is the center of $\Gamma$. In this article, we prove algebraic and topological properties of gyrogroups that are induced by CCII groups. Then, using a classification of non-abelian groups of order $n$ with $n < 32$, we determine all finite CCII groups of order less than $32$.

**Key words:** 2-Engel group, CCII group, gyrogroup, commutator-inversion invariant, topological gyrogroup

## 1. Preliminaries

We follow standard definitions and notations in group theory. For standard definitions and notations in gyrogroup theory, we refer the reader to [5, 8]. We set notations and summarize basic relevant results in this section.

Suppose that $\Gamma$ is a group. Set $Z(\Gamma) = \{z \in \Gamma : gz = zg \text{ for all } g \in \Gamma\}$, known as the center of $\Gamma$. For each $a \in \Gamma$, conjugation by $a$, denoted by $\alpha_a$, is the (inner) automorphism of $\Gamma$ defined by $\alpha_a(g) = aga^{-1}$ for all $g \in \Gamma$. Let $g, h \in \Gamma$. The commutator of $g$ and $h$ is defined as $[g, h] = g^{-1}h^{-1}gh$. Recall that the derived subgroup of $\Gamma$, denoted by $\Gamma'$, is the subgroup of $\Gamma$ generated by all the commutators in $\Gamma$. As in Definition 3.1 of [7], $\Gamma$ is said to be commutator-inversion invariant if $[g, h] = [g^{-1}, h^{-1}]$ for all $g, h \in \Gamma$, and $\Gamma$ is said to be central by a commutator-inversion invariant group if $\Gamma/Z(\Gamma)$ is commutator-inversion invariant. It is shown in Theorem 3.1 of [7] that $\Gamma$ is 2-Engel if and only if $\Gamma$ is commutator-inversion invariant, which gives another characterization of 2-Engel groups. The fundamental importance of commutator-inversion invariant groups lies in the following theorem.

**Theorem 1 (Theorem 4.2 of [7])** *Let $\Gamma$ be a group. Suppose that $\Gamma/Z(\Gamma)$ is commutator-inversion invariant. Then, the underlying set $\Gamma$ is a gyrogroup, denoted by $\Gamma^{\mathrm{gyr}}$, under the binary operation defined by the equation*

$$a \oplus b = aaba^{-1} \qquad \text{for all } a, b \in \Gamma. \tag{1}$$

We remark that Theorem 1 was first proved by Foguel and Ungar in [2] in terms of left gyrogroups (see Theorem 3.4 of [2]) using the notion of 2-Engel groups (see Theorem 3.7 of [2]).

*Correspondence: teerapong.suksumran@cmu.ac.th
2020 *AMS Mathematics Subject Classification:* 20F99, 20F18, 20N05

In the resulting gyrogroup $\Gamma^{\mathrm{gyr}}$ above, the identity of $\Gamma^{\mathrm{gyr}}$ is the same as the identity of $\Gamma$, and the inverse of an element $a$ in $\Gamma^{\mathrm{gyr}}$ is the same as the inverse of $a$ in $\Gamma$. Moreover, if $a$ and $b$ are elements in $\Gamma$, then the gyroautomorphism of $\Gamma^{\mathrm{gyr}}$ generated by $a$ and $b$ is the inner automorphism generated by the commutator $[a^{-1}, b]$. It is shown in Theorem 4.3 of [7] that $\Gamma^{\mathrm{gyr}}$ is associative if and only if $\Gamma$ is nilpotent of class $n$ with $n \le 2$, which gives a characterization for $\Gamma^{\mathrm{gyr}}$ to form a group under the induced gyrogroup operation. The converse of Theorem 1 is also true in the sense that if the underlying set of $\Gamma$ is a gyrogroup under the binary operation defined by (1) with the gyration map sending the pair $(a, b)$ to the inner automorphism that is generated by $[a^{-1}, b]$, then $\Gamma/Z(\Gamma)$ is necessarily commutator-inversion invariant by Theorem 4.4 of [7].

A (finite or infinite) group $\Gamma$ is said to be *3-residual* provided the map $T$ defined on $\Gamma$ by $T(x) = x^3$ is surjective or, equivalently, if any element of $\Gamma$ can be written as a cube element in $\Gamma$. A gyrogroup is *degenerate* if its operation is associative. Therefore, every degenerate gyrogroup forms a group under the gyrogroup operation. Suppose that $G$ is a gyrogroup. We say that elements $a, b$, and $c$ of $G$ are *associative* if $a \oplus (b \oplus c) = (a \oplus b) \oplus c$; that elements $a$ and $b$ of $G$ *gyrocommute* if $a \oplus b = \mathrm{gyr}\,[a, b](b \oplus a)$; that $G$ is *gyrocommutative* if $a$ and $b$ gyrocommute for all $a, b \in G$.

## 2. Algebraic properties of gyrogroups induced by CCII groups

In view of Theorem 1, any group such that its quotient by the center is a commutator-inversion invariant group is crucial to construction of a gyrogroup. Therefore, we introduce the notion of a CCII group as follows.

**Definition 1** *A group $\Gamma$ is said to be CCII if the central quotient $\Gamma/Z(\Gamma)$ is commutator-inversion invariant.*

We give a few remarks here. It is clear that any abelian group is CCII; moreover, $\Gamma/Z(\Gamma)$ is obviously commutator-inversion invariant. Throughout the remainder of this article, if $\Gamma$ is a CCII group, then the gyrogroup $\Gamma^{\mathrm{gyr}}$ constructed via Theorem 1 will be employed. Suppose that $\Gamma$ and $\Pi$ are CCII groups. According to Proposition 4.2 of [7], $\Gamma \cong \Pi$ implies $\Gamma^{\mathrm{gyr}} \cong \Pi^{\mathrm{gyr}}$. The converse does not in general hold. However, the converse holds under certain conditions. For instance, if $\Gamma$ is 3-residual, then the converse holds by Theorem 4.5 of [7]. According to Proposition 4.3 of [7], the order of an element $g$ in $\Gamma$ equals the order of $g$ in $\Gamma^{\mathrm{gyr}}$. Furthermore, every subgroup of $\Gamma$ forms a subgyrogroup of $\Gamma^{\mathrm{gyr}}$. In contrast, a subgyrogroup $H$ of $\Gamma^{\mathrm{gyr}}$ forms a subgroup of $\Gamma$ if and only if $hHh^{-1} \subseteq H$ for all $h \in H$.

The main goal of this section is to investigate further properties of the gyrogroup induced by a CCII group. It turns out that algebraic properties of CCII groups and their corresponding gyrogroups are intertwined and have strong connections. We begin by proving that the process of constructing a gyrogroup as in Theorem 1 and the process of taking finite direct products are interchangeable.

**Proposition 1** *Let $\Gamma_1$ and $\Gamma_2$ be CCII groups. Then, $\Gamma_1 \times \Gamma_2$ is a CCII group, and*

$$(\Gamma_1 \times \Gamma_2)^{\mathrm{gyr}} = {\Gamma_1}^{\mathrm{gyr}} \times {\Gamma_2}^{\mathrm{gyr}}.$$

**Proof** The first part of the proposition was proved in Proposition 3.4 of [7]. Note that the underlying sets of $(\Gamma_1 \times \Gamma_2)^{\mathrm{gyr}}$ and ${\Gamma_1}^{\mathrm{gyr}} \times {\Gamma_2}^{\mathrm{gyr}}$ are equal. It remains to show that the gyrogroup operations on these two sets coincide. We temporarily denote the gyrogroup operations in $(\Gamma_1 \times \Gamma_2)^{\mathrm{gyr}}, {\Gamma_1}^{\mathrm{gyr}}$, and ${\Gamma_2}^{\mathrm{gyr}}$ by $\oplus_c$. Then, we obtain that

$$\begin{aligned}(a,b)\oplus_c(b,d) &= (a,b)(a,b)(c,d)(a,b)^{-1}\\ &= (aaca^{-1}, bbdb^{-1})\\ &= (a\oplus_c c, b\oplus_c d)\\ &= (a,b)\oplus(c,d),\end{aligned}$$

which completes the proof. □

Note that Proposition 1 is easily extended to the case of a finite number of CCII groups. Moreover, the proposition is not, in general, true for semidirect products. In fact, the symmetric group $S_3$ of degree 3 is not commutator-inversion invariant since $[(1\ 2\ 3),(2\ 3)] = (1\ 2\ 3)$, whereas $[(1\ 2\ 3)^{-1},(2\ 3)^{-1}] = (1\ 3\ 2)$. This also implies that $S_3$ is not CCII since the center of $S_3$ is trivial. It is known that $S_3$ can be recognized as the semidirect product of the cyclic groups of order 2 and order 3 and that every cyclic group is CCII. Next, we show that the structures of $\Gamma$ and $\Gamma^{\rm gyr}$ are identical whenever $\Gamma$ is an abelian group. Therefore, we may pay attention to non-abelian CCII groups.

**Proposition 2** *Let $\Gamma$ be a CCII group, and let $a,b\in\Gamma$. Then, $a\oplus b = ab$ if and only if $ab = ba$.*

**Proof** This proposition follows directly from the fact that $a\oplus b = ab$ if and only if $aaba^{-1} = ab$ if and only if $ab = ba$. □

In fact, according to Theorem 4.3 of [7], the gyrogroup induced by any nilpotent group of class $n$, where $n\le 2$, is degenerate. The next proposition shows some relationships between (group) homomorphisms of $\Gamma$ and (gyrogroup) homomorphisms of $\Gamma^{\rm gyr}$. In particular, this gives a characterization for a homomorphism of $\Gamma^{\rm gyr}$ to be a homomorphism of $\Gamma$.

**Proposition 3** *Let $\Gamma$ be a CCII group, and let $\phi:\Gamma\to\Gamma$ be a map.*

1. *If $\phi$ is a homomorphism of $\Gamma$, then $\phi$ is a homomorphism of $\Gamma^{\rm gyr}$.*

2. *Let $\phi$ be a homomorphism of $\Gamma^{\rm gyr}$. Then, $\phi$ is a homomorphism of $\Gamma$ if and only if $\phi\circ\alpha_a = \alpha_{\phi(a)}\circ\phi$ for all $a\in\Gamma$, where $\alpha_a$ denotes the inner automorphism generated by $a$.*

**Proof** In view of (1), part 1 holds trivially. The proof of the forward implication of part 2 is straightforward. To prove the converse of part 2, note that $\phi(a^{-1}) = \phi(\ominus a) = \ominus\phi(a) = \phi(a)^{-1}$ for all $a\in\Gamma$ since $\phi$ preserves taking gyrogroup inverses. Suppose that $\phi \circ \alpha_a = \alpha_{\phi(a)} \circ \phi$ for all $a \in \Gamma$. Let $a, b \in \Gamma$. Then,

$$\begin{aligned}
\phi(ab) &= \phi(a^{-1}aaba^{-1}a) \\
&= \phi(a^{-1}(a \oplus b)a) \\
&= \phi \circ \alpha_{a^{-1}}(a \oplus b) \\
&= \alpha_{\phi(a^{-1})} \circ \phi(a \oplus b) \\
&= \alpha_{\phi(a^{-1})}(\phi(a) \oplus \phi(b)) \\
&= \phi(a^{-1})\phi(a)\phi(a)\phi(b)\phi(a)^{-1}\phi(a^{-1})^{-1} \\
&= \phi(a)^{-1}\phi(a)\phi(a)\phi(b)\phi(a)^{-1}\phi(a) \\
&= \phi(a)\phi(b),
\end{aligned}$$

which completes the proof of part 2. □

In light of Proposition 3, we gain the following corollary immediately.

**Corollary 1** *Let $\Gamma$ be a CCII group, and let $\mathcal{A}_{\mathrm{gyr}}$ be the subgroup of $\mathrm{Aut}\,(\Gamma^{\mathrm{gyr}})$ generated by all the gyroautomorphisms of $\Gamma^{\mathrm{gyr}}$.*

1. *Then, $\mathcal{A}_{\mathrm{gyr}} \subseteq \mathrm{Aut}\,(\Gamma) \subseteq \mathrm{Aut}\,(\Gamma^{\mathrm{gyr}})$.*
2. *Let $\phi$ be an automorphism of $\Gamma^{\mathrm{gyr}}$. Then, $\phi$ is an automorphism of $\Gamma$ if and only if $\phi \circ \alpha_a = \alpha_{\phi(a)} \circ \phi$ for all $a \in \Gamma$.*

The following proposition gives a characterization for three elements in the induced gyrogroup being associative. In particular, it shows that the induced gyrogroup $\Gamma^{\mathrm{gyr}}$ satisfies the associative law (that is, $\Gamma^{\mathrm{gyr}}$ is degenerate) if and only if all the commutators of $\Gamma$ lie in the center of $\Gamma$.

**Proposition 4** *Let $\Gamma$ be a CCII group, and let $a, b, c \in \Gamma$. Then,*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

*in $\Gamma^{\mathrm{gyr}}$ if and only if $c$ is a fixed point of the inner automorphism $\alpha_{[a^{-1},b]}$.*

**Proof** Recall that $\mathrm{gyr}\,[a, b] = \alpha_{[a^{-1},b]}$ for all $a, b \in \Gamma$. Then, for all $a, b, c \in \Gamma$, we obtain by the left cancellation law in $\Gamma^{\mathrm{gyr}}$ that

$$\begin{aligned}
a \oplus (b \oplus c) = (a \oplus b) \oplus c &\Leftrightarrow (a \oplus b) \oplus \mathrm{gyr}\,[a, b](c) = (a \oplus b) \oplus c \\
&\Leftrightarrow \mathrm{gyr}\,[a, b](c) = c \\
&\Leftrightarrow \alpha_{[a^{-1},b]}(c) = c,
\end{aligned}$$

and the proof is complete. □

**Corollary 2** *Let $\Gamma$ be a CCII group. Then, $\Gamma$ is nilpotent of class at most 2 if and only if $\Gamma^{\mathrm{gyr}}$ is degenerate.*

**Proof** By Proposition 4, $\Gamma^{\rm gyr}$ is degenerate if and only if $[a,b]$ lies in $Z(\Gamma)$ for all $a,b \in \Gamma$ if and only if $\Gamma$ is nilpotent of class $n$ with $n \le 2$. □

We remark that Corollary 2 was first proved by Foguel and Ungar in [2] in terms of left gyrogroups (see Theorem 3.6 of [2]). Next, we give a characterization for two elements in $\Gamma^{\rm gyr}$ to gyrocommute. In particular, we prove that the induced gyrogroup $\Gamma^{\rm gyr}$ is gyrocommutative if and only if any element of $\Gamma$ commutes with any cube element of $\Gamma$.

**Proposition 5** *Let $\Gamma$ be a CCII group, and let $a,b \in \Gamma$. Then,*

$$a \oplus b = \mathrm{gyr}\,[a,b](b \oplus a)$$

*in $\Gamma^{\rm gyr}$ if and only if $ab^3 = b^3 a$ in $\Gamma$.*

**Proof** A direct computation shows that

$$\begin{aligned} a \oplus b = \mathrm{gyr}\,[a,b](b \oplus a) \quad &\Leftrightarrow \quad aaba^{-1} = [a^{-1},b]bbab^{-1}[a^{-1},b]^{-1} \\ &\Leftrightarrow \quad aaba^{-1} = ab^{-1}a^{-1}bbbab^{-1}b^{-1}aba^{-1} \\ &\Leftrightarrow \quad ab^3 = b^3 a, \end{aligned}$$

and the proof completes. □

**Corollary 3** *Let $\Gamma$ be a CCII group. Then, $\Gamma^{\rm gyr}$ is gyrocommutative if and only if $ab^3 = b^3 a$ for all $a,b \in \Gamma$.*

**Proof** The corollary follows immediately from Proposition 5. □

In view of Corollary 3, we have a sufficient condition for $\Gamma^{\rm gyr}$ being gyrocommutative and $\Gamma$ being abelian to be equivalent, which is given below.

**Proposition 6** *Let $\Gamma$ be a CCII group. If $\Gamma$ is 3-residual, then $\Gamma^{\rm gyr}$ is gyrocommutative if and only if $\Gamma$ is abelian.*

**Proof** Suppose that $\Gamma^{\rm gyr}$ is gyrocommutative. Let $a,b \in \Gamma$. Since $\Gamma$ is 3-residual, there exists an element $c \in \Gamma$ such that $c^3 = b$. By Corollary 3, $ab = ac^3 = c^3 a = ba$. Conversely, if $\Gamma$ is abelian, then $ab^3 = b^3 a$ for all $a,b \in \Gamma$, and so $\Gamma^{\rm gyr}$ is gyrocommutative by Corollary 3. □

As shown in the proof of Corollary 4.5 of [7], on the class of finite groups, the order of a group $\Gamma$ is not divisible by 3 implies that $\Gamma$ is 3-residual. Thus, we have the following corollary immediately.

**Corollary 4** *If $\Gamma$ is a finite CCII group such that $|\Gamma|$ is not divisible by 3, then $\Gamma^{\rm gyr}$ is gyrocommutative if and only if $\Gamma$ is abelian.*

It turns out that the property of being 3-residual is equivalent to the condition that the order of the corresponding group is not divisible by 3 on the class of finite groups. Moreover, this fact is true not only for the prime 3. Therefore, we state the following result for any prime $p$. Let $p$ be a fixed prime. A (finite or infinite) group $\Gamma$ is said to be $p$-residual if the power map $P$ defined on $\Gamma$ by $P(x) = x^p$ is surjective or, equivalently, $g \in \Gamma$ implies $g = h^p$ for some element $h \in \Gamma$.

**Theorem 2** *Let $\Gamma$ be a finite group with identity $1$, and let $p$ be a prime. Then, $\Gamma$ is $p$-residual if and only if $p$ does not divide the order of $\Gamma$.*

**Proof** Suppose that $\Gamma$ is $p$-residual. Assume, to the contrary, that $p$ divides $|\Gamma|$. By Cauchy's theorem, there is an element $1 \neq g_1 \in \Gamma$ such that $|g_1| = p$. By assumption, $g_1 = g_2^p$ for some $g_2 \in \Gamma$. Note that $g_2 \neq 1$ since otherwise $g_1 = 1$. Since $g_2^{p^2} = g_1^p = 1$, $|g_2|$ divides $p^2$, and so $|g_2| \in \{1, p, p^2\}$. If $|g_2| = 1$, then $g_2 = 1$, which would imply $g_1 = 1$, a contradiction. If $|g_2| = p$, then $g_1 = g_2^p = 1$, a contradiction. Hence, $|g_2| = p^2$. Next, we prove that if there are elements $g_1, g_2, \ldots, g_n, g_{n+1}$ of $\Gamma$ such that $g_i \neq 1$, $|g_i| = p^i$, $g_i = g_{i+1}^p$ for all $i$ with $1 \leq i \leq n$, then $|g_{n+1}| = p^{n+1}$. Since $g_{n+1}^{p^{n+1}} = g_n^{p^n} = 1$, it follows that $|g_{n+1}|$ divides $p^{n+1}$. If $|g_{n+1}| = p^i$ with $0 \leq i \leq n$, then we would have $g_{n+1-i} = g_{n+2-i}^p = g_{n+3-i}^{p^2} = \cdots = g_n^{p^{i-1}} = g_{n+1}^{p^i} = 1$, a contradiction. Thus, $|g_{n+1}| = p^{n+1}$. By induction, we gain an infinite sequence $\{g_i\}_{i=1}^{\infty}$ of elements in $\Gamma$ such that $|g_i| = p^i$ for all $i \in \mathbb{N}$, which contradicts the fact that $\Gamma$ is finite. Thus, $p$ does not divide $|\Gamma|$. The converse can be proved as in Corollary 4.5 of [7] with appropriate modifications. □

## 3. Topological properties of gyrogroups induced by CCII groups

The goal of this section is to prove some topological properties of the gyrogroup induced by a CCII topological group. It turns out that topological properties of CCII topological groups and their corresponding gyrogroups have some nice connections. Throughout this section, by a CCII topological group we mean a topological group that is also a CCII group. A few concrete examples of CCII topological groups are given below.

**Example 1** *One concrete example of an infinite CCII topological group is the classical Heisenberg group, denoted by $\mathcal{H}$, which is the group of matrices of the form*

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

*where $x, y$, and $z$ are real numbers. It fact, $\mathcal{H}$ is nilpotent of class $2$ and hence is CCII by Proposition 3.1 and Corollary 3.1 of [7]. Furthermore, $\mathcal{H}$ is a topological group since it forms a subgroup of the general linear group of degree $3$ over $\mathbb{R}$.*

**Example 2** *As in Example 5.1 of [7], the dihedral group*

$$D_{16} = \langle r, s : r^8 = s^2 = 1, rs = sr^{-1} \rangle$$

*is a finite CCII group. We can topologize $D_{16}$ to obtain a finite CCII topological group as follows. Recall that the center of $D_{16}$ is $Z(D_{16}) = \{1, r^4\}$. Now, we put a non-discrete topology on $D_{16}$ so that it becomes a topological group. Let $\tau$ be the topology on $D_{16}$ generated by $D_{16}/Z(D_{16})$. Observe that each pair of elements of $D_{16}/Z(D_{16})$ is disjoint, and so the singletons in $D_{16}$ are not open. Let $a, b \in D_{16}$, and let $U$ be an open set containing $ab$. Then, $aZ(D_{16})bZ(D_{16}) = abZ(D_{16}) \subseteq U$. This shows that the multiplication of $D_{16}$ is continuous. Next, let $a \in D_{16}$, and let $V$ be an open set containing $a^{-1}$. Then, $(aZ(D_{16}))^{-1} = a^{-1}Z(D_{16}) \subseteq V$. This shows that the inversion function of $D_{16}$ is continuous.*

We begin with the following proposition, which shows that a CCII topological group induces a topological gyrogroup.

**Proposition 7** *Let $\Gamma$ be a CCII topological group. Then, $\Gamma^{\mathrm{gyr}}$ forms a topological gyrogroup.*

**Proof** Note that the inversion function of $\Gamma^{\mathrm{gyr}}$ is automatically continuous. Moreover, being a composition of continuous functions, $\oplus$ is continuous. Therefore, $\Gamma^{\mathrm{gyr}}$ is a topological gyrogroup. □

Let $G$ be a topological gyrogroup. Recall that $G$ is said to be *strongly topological gyrogroup* if there exists a neighborhood base $\mathcal{U}$ at the identity $e$ of $G$ such that $\mathrm{gyr}[x,y](U) = U$ for all $x, y \in G, U \in \mathcal{U}$ (cf. Section 3 of [1]). For the relevant definitions such as an invariant metric, a gyronorm, we refer the reader to [6].

**Proposition 8** *Let $\Gamma$ be a CCII group with an invariant metric $d$. Then, the function $\|\cdot\|: \Gamma^{\mathrm{gyr}} \to \mathbb{R}$ defined by $\|x\| = d(e,x)$ for all $x \in \Gamma^{\mathrm{gyr}}$ is a gyronorm on $\Gamma^{\mathrm{gyr}}$. In particular, $\Gamma^{\mathrm{gyr}}$ is a strongly topological gyrogroup with respect to the topology induced by $d$.*

**Proof** Let $a, x, y \in \Gamma^{\mathrm{gyr}}$. By definition,

$$d(a \oplus x, a \oplus y) = d(aaxa^{-1}, aaya^{-1}) = d(x,y).$$

Hence, by Theorem 9 of [6], the function $\|\cdot\|$ is a gyronorm on $\Gamma^{\mathrm{gyr}}$. The last assertion follows from Proposition 7 and Theorem 16 of [9]. □

For basic knowledge of topology used here, we refer the reader to [3], for instance. Recall that a subset $A$ of a topological space $X$ is called a neighborhood of a point $x$ in $X$ if it contains an open set containing $x$. Recall also that a nonempty subset $A$ of a topological group $\Gamma$ is said to be *thin* in $\Gamma$ if for every neighborhood $U$ of the identity $e$ in $\Gamma$, there exists a neighborhood $V$ of $e$ such that $aVa^{-1} \subseteq U$ for all $a \in A$. If $\Gamma$ is thin in itself, then $\Gamma$ is said to be *balanced* (also known as a *SIN-group*). Then, we obtain the following proposition.

**Proposition 9** *Let $\Gamma$ be a CCII topological group. Then, the derived subgroup $\Gamma'$ is thin in $\Gamma$ if and only if $\Gamma^{\mathrm{gyr}}$ is a strongly topological gyrogroup.*

**Proof** Suppose that $\Gamma'$ is thin in $\Gamma$. Let $U$ be a neighborhood of $e$. Then, there exists a neighborhood $V$ of $e$ such that $aVa^{-1} \subseteq U$ for all $a \in \Gamma'$. Then, the set

$$W = \bigcup_{a \in \Gamma'} aVa^{-1} \subseteq U$$

is a neighborhood of $e$. Moreover,

$$\begin{aligned} \mathrm{gyr}\,[x,y](W) &= [x^{-1},y]W[x^{-1},y]^{-1} \\ &= \bigcup_{a \in \Gamma'} [x^{-1},y]aVa^{-1}[x^{-1},y]^{-1} \\ &= \bigcup_{a \in \Gamma'} ([x^{-1},y]a)V([x^{-1},y]a)^{-1} \\ &= W \end{aligned}$$

for all $x, y \in \Gamma^{\mathrm{gyr}}$. Thus, $\Gamma^{\mathrm{gyr}}$ is a strongly topological gyrogroup.

Conversely, suppose that $\Gamma^{\mathrm{gyr}}$ is a strongly topological gyrogroup. Let $U$ be a neighborhood of $e$. Then, there exists a neighborhood $V \subseteq U$ of $e$ such that $[x,y]V[x,y]^{-1} = \mathrm{gyr}\,[x^{-1}, y](V) = V$ for all $x, y \in \Gamma^{\mathrm{gyr}}$. Hence, $\Gamma'$ is thin. □

**Corollary 5** *If $\Gamma$ is a balanced CCII topological group, then $\Gamma^{\mathrm{gyr}}$ is a strongly topological gyrogroup.*

**Proposition 10** *If $\Gamma$ is a CCII topological group, then $\Gamma^{\mathrm{gyr}}$ is uniformizable.*

**Proof** The proposition follows from the fact that $\Gamma$ is uniformizable. □

Let $\Gamma$ be a topological group, and let $\mathcal{N}_s(e)$ be a symmetric open base at $e$. For each $V \in \mathcal{N}_s(e)$, define

$$
\begin{aligned}
E_V^l &= \{(x,y) \in \Gamma \times \Gamma : x^{-1}y \in V\},\\
E_V^r &= \{(x,y) \in \Gamma \times \Gamma : xy^{-1} \in V\},\\
E_V &= E_V^l \cap E_V^r.
\end{aligned}
$$

Denote by $\mathcal{D}_\Gamma$ the set of all symmetric subsets of $\Gamma \times \Gamma$. Then, the following uniformities:

$$
\begin{aligned}
\mathcal{U}^l &= \{U \in \mathcal{D}_\Gamma : E_V^l \subseteq U \text{ for some } V \in \mathcal{N}_s(e)\},\\
\mathcal{U}^r &= \{U \in \mathcal{D}_\Gamma : E_V^r \subseteq U \text{ for some } V \in \mathcal{N}_s(e)\},\\
\mathcal{U} &= \{U \in \mathcal{D}_\Gamma : E_V \subseteq U \text{ for some } V \in \mathcal{N}_s(e)\}
\end{aligned}
$$

are called the left uniformity, right uniformity, and two-sided uniformity on $\Gamma$, respectively. It is well known in the literature that these three uniformities coincide if and only if the group $\Gamma$ is balanced. For each $U \in \mathcal{U}^l$ and for each $x \in \Gamma$, define

$$
U[x] = \{y \in \Gamma : (x,y) \in U\}.
$$

Note that if $V \in \mathcal{N}_s(e)$, then

$$
E_V^l[x] = \{y \in \Gamma : (x,y) \in E_V^l\} = \{y \in \Gamma : x^{-1}y \in V\} = xV.
$$

Now, suppose that $\Gamma$ is a CCII topological group. Let ${L_a}^{\mathrm{gyr}}$ be the gyrotranslation by $a$ defined on $\Gamma^{\mathrm{gyr}}$, and let $L_a$ be the left multiplication by $a$ defined on $\Gamma$. Since ${L_a}^{\mathrm{gyr}} = L_a \circ \alpha_a$ for all $a \in \Gamma$, the following proposition is immediate.

**Proposition 11** *Let $\Gamma$ be a CCII topological group, and let $\mathcal{V} \in \{\mathcal{U}^l, \mathcal{U}^r, \mathcal{U}\}$. Then, the left gyrotranslation ${L_a}^{\mathrm{gyr}} : (\Gamma, \mathcal{V}) \to (\Gamma, \mathcal{V})$ is uniformly continuous for all $a \in \Gamma$.*

**Proposition 12** *Let $\Gamma$ be a CII topological group, and let $\mathcal{V} \in \{\mathcal{U}^l, \mathcal{U}^r, \mathcal{U}\}$. Then, the right gyrotranslation ${R_a}^{\mathrm{gyr}} : (\Gamma, \mathcal{V}) \to (\Gamma, \mathcal{V})$ is uniformly continuous for all $a \in \Gamma$.*

**Proof** Let $a \in \Gamma^{\text{gyr}}$. Then, for each $x \in \Gamma$, a direct computation shows that

$$\begin{aligned} {R_a}^{\text{gyr}}(x) &= x \oplus a \\ &= (a^{-1} \oplus x^{-1})^{-1} \\ &= (a^{-1}a^{-1}x^{-1}a)^{-1} \\ &= a^{-1}xaa \\ &= R_a \circ \alpha_{a^{-1}}(x). \end{aligned}$$

Thus, ${R_a}^{\text{gyr}} = R_a \circ \alpha_{a^{-1}}$, and so ${R_a}^{\text{gyr}}$ is uniformly continuous. □

**Theorem 3** *Let $\Gamma$ be a CCII topological group. Then, the collection*

$$\widehat{\Gamma^{\text{gyr}}} = \{{L_a}^{\text{gyr}} : a \in \Gamma^{\text{gyr}}\}$$

*is equicontinuous with respect to $\mathcal{U}^l$ if and only if $\Gamma$ is balanced.*

**Proof** Suppose that $\widehat{\Gamma^{\text{gyr}}}$ is equicontinuous with respect to $\mathcal{U}^l$. Let $U \in \mathcal{N}_s(e)$. By assumption, there is an open set $V$ containing $e$ such that

$${L_a}^{\text{gyr}}(V) \subseteq E^l_U[{L_a}^{\text{gyr}}(e)] = {L_a}^{\text{gyr}}(e)U = aU$$

for all $a \in \Gamma^{\text{gyr}}$. That is, $\alpha_a(V) \subseteq L_{a^{-1}}(aU) = U$ for all $a \in \Gamma^{\text{gyr}}$. Thus, $\Gamma$ is balanced. Conversely, suppose that $\Gamma$ is balanced. Let $x \in \Gamma$, and let $U \in \mathcal{N}_s(e)$. Since $\Gamma$ is balanced, there is an open set $V$ containing $e$ such that $V \subseteq a^{-1}Ua$ for all $a \in \Gamma$. It follows that

$${L_a}^{\text{gyr}}(xV) = aaxVa^{-1} \subseteq aaxa^{-1}Uaa^{-1} = (a \oplus x)U = {L_a}^{\text{gyr}}(x)U = E^l_U[{L_a}^{\text{gyr}}(x)]$$

for all $a \in \Gamma$. Hence, $\widehat{\Gamma^{\text{gyr}}}$ is equicontinuous with respect to $\mathcal{U}^l$. □

## 4. CCII groups of order less than 32

In this section, we collect finite CCII groups of order less than 32, using a classification of finite non-abelian groups, up to isomorphism, as in Appendix B of [4]. Recall that any group of order $1, 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 29$, and $31$ is abelian. In Section 5 of [7], the author collects all finite CCII groups of order less than 24. We continue to determine all finite CCII groups of order $n$ with $24 \leq n < 32$. In Table 1, $\mathbb{Z}_n$ denotes the cyclic group of order $n$, $D_n$ denotes the dihedral group of order $n$, $Q_n$ denotes the generalized quaternion of order $n$, $A_4$ denotes the alternating group of degree 4, $S_4$ denotes the symmetric group of degree 4, $SL(2,3)$ is the special linear group of $2 \times 2$ matrices with entries from a field of order 3, $B(2,3)$ is the Burnside group, and the group $\Gamma_{m,n}$ has order $m$ and is defined by the following presentations:

$$\begin{aligned} \Gamma_{24,1} &= \langle x, y : x^3 = y^8 = 1, yxy^{-1} = x^{-1}\rangle, \\ \Gamma_{24,t} &= \langle x, y, z : x^4 = y^2 = z^3 = 1, yxy^{-1} = x^{-1}, xzx^{-1} = z^{-1}, yzy^{-1} = z\rangle, \\ \Gamma_{27,1} &= \langle x, y : x^9 = y^3 = 1, yxy^{-1} = x^4\rangle. \end{aligned}$$

It turns out that only four groups are CCII: $\mathbb{Z}_3 \times D_8, \mathbb{Z}_3 \times Q_8, \Gamma_{27,1}$, and $B(2,3)$. Unfortunately, their corresponding gyrogroups are degenerate, and so they give no new examples of gyrogroup structures.

**Table 1**. The non-abelian groups of order $n$ with $24 \leq n < 32$, up to isomorphism.

| Order | Group $\Gamma$ | CCII group | Structure of $\Gamma^{\text{gyr}}$ |
|---|---|---|---|
| 24 | $\mathbb{Z}_4 \times D_6$ | no | n/a |
| 24 | $\mathbb{Z}_2 \times Q_{12}$ | no | n/a |
| 24 | $\mathbb{Z}_2 \times D_{12}$ | no | n/a |
| 24 | $\mathbb{Z}_2 \times A_4$ | no | n/a |
| 24 | $\mathbb{Z}_3 \times D_8$ | yes | degenerate gyrogroup |
| 24 | $D_{24}$ | no | n/a |
| 24 | $S_4$ | no | n/a |
| 24 | $Q_{24}$ | no | n/a |
| 24 | $SL(2,3)$ | no | n/a |
| 24 | $\mathbb{Z}_3 \times Q_8$ | yes | degenerate gyrogroup |
| 24 | $\Gamma_{24,1}$ | no | n/a |
| 24 | $\Gamma_{24,t}$ | no | n/a |
| 26 | $D_{26}$ | no | n/a |
| 27 | $\Gamma_{27,1}$ | yes | degenerate gyrogroup |
| 27 | $B(2,3)$ | yes | degenerate gyrogroup |
| 28 | $D_{28}$ | no | n/a |
| 28 | $Q_{28}$ | no | n/a |
| 30 | $D_{30}$ | no | n/a |
| 30 | $\mathbb{Z}_3 \times D_{10}$ | no | n/a |
| 30 | $\mathbb{Z}_5 \times D_6$ | no | n/a |

**Acknowledgment**

This research is part of the project "Algebraic and Topological Properties of Gyrogroups Induced by CCII Groups", led by the first author and supported by the National Research Council of Thailand and Chiang Mai University via the Grant for a New Researcher Career Path, Year 2023, under Grant No. N42A670294. The authors would like to thank the referees for their insightful comments, which helped strengthen the manuscript.

## References

[1] Bao M, Lin F. Feathered gyrogroups and gyrogroups with countable pseudocharacter. Filomat 2019; 33 (16): 5113-5124. https://doi.org/10.2298/FIL1916113B

[2] Foguel T, Ungar AA. Gyrogroups and the decomposition of groups into twisted subgroups and subgroups. Pacific Journal of Mathematics 2001; 197 (1): 1-11. https://doi.org/10.2140/pjm.2001.197.1

[3] Gaal SA. Point Set Topology. London, England: Academic Press, 1964.

[4] Humphreys JF. A Course in Group Theory. Oxford, England: Oxford University Press, 1996.

[5] Suksumran T. The algebra of gyrogroups: Cayley's theorem, Lagrange's theorem, and isomorphism theorems. In: Rassias TM, Pardalos PM (editors). Essays in Mathematics and its Applications: In Honor of Vladimir Arnold. Switzerland: Springer, 2016, pp. 369-437. https://doi.org/10.1007/978-3-319-31338-2_15

[6] Suksumran T. On metric structures of normed gyrogroups. In: Rassias TM and Pardalos PM (editors). Springer Optimization and its Applications, vol. 154. Cham, Germany: Springer, 2019, pp. 529-542. https://doi.org/10.1007/978-3-030-31339-5_20

[7] Suksumran T. On commutator-inversion invariant groups and gyrogroup construction. Journal of Algebra and its Applications 2024; 23 (3): Article 2450042. https://doi.org/10.1142/S0219498824500427

[8] Ungar AA. Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity. Hackensack, NJ, USA: World Scientific, 2008.

[9] Wattanapan J, Atiponrat W, Suksumran T. Embedding of strongly topological gyrogroups in path-connected and locally path-connected gyrogroups. Symmetry 2020; 12 (11): Article 1817. https://doi.org/10.3390/sym12111817