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A study of the scattering analysis of the multiplicative Sturm-Liouville problem

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Abstract: The main purpose of this study is to examine the scattering analysis of a multiplicative Sturm-Liouville boundary value problem. Firstly, we find the Jost solution of this problem and give the Jost function to construct the scattering function of the problem. Secondly, we obtain asymptotic equations for Jost solution and investigate some properties of the Jost function. Moreover, we present other solutions of the problem by giving the relations between these solutions and Jost solution. Finally, we define the scattering function of this problem by using the Jost function and examine the basic properties of the scattering function.

Key words: Multiplicative calculus, multiplicative derivative, multiplicative integral, Jost solution, scattering function

1. Introduction

Multiplicative calculus, also known as non-Newtonian calculus or calculus without limits, is an alternative approach to traditional (classiscal) calculus that offers a unique perspective on mathematical analysis and differentiation. Unlike the familiar differential and integral calculus developed by Sir Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century, multiplicative calculus focuses on the multiplicative aspects of change and growth. Classical analysis built on algebra, trigonometry, and analytical geometry includes concepts such as limit, derivative, integral, and series. Since the basic operation in this analysis is addition, this analysis is also called additive analysis or Newtonian analysis and the fundamental notions of limits as well as derivatives play a central role in understanding rates of change and infinitesimal variations in this analysis. Multiplicative calculus, however, challenges these conventional concepts by introducing alternative notions of differentiation and integration that are based on multiplicative operations. This alternative calculus seeks to provide a more natural framework for describing phenomena involving exponential growth, relative changes, and proportional relationships. The development of multiplicative calculus can be attributed to mathematicians such as Otto Toeplitz, Ladislaus Natanson, and Anatol Rapoport, who explored the possibilities of calculus without limits in the early 20th century. Then it was developed to cover the classical calculus by Grossman and Katz [23, 24]. Today, this nonstandard approach has gained renewed interest and is actively studied by mathematicians and researchers, offering a fresh perspective on problems related to growth, scaling, and nonlinear dynamics. This new calculus provides a potential as mathematical tool for applications in science and engineering. Different

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2010 *AMS Mathematics Subject Classification:* 34N05, 34A30, 34A99, 11U10 applications of this analysis have been studied in many areas [3, 11, 13, 17, 18, 30, 31]. The basic concepts of multiplicative analysis were obtained in detail and its important applications were shown in some studies [10, 12, 27, 34]. It is important to note that while classical analysis is applied to situations or problems involving linear functions, multiplicative analysis is applied to problems consisting of exponential functions. This implies that multiplicative analysis is a useful supplement to classical analysis because many natural events that could directly affect us change exponentially. For example, the populations of countries and the magnitude of earthquakes behave in this manner. Using multiplicative calculus instead of classical calculus allows better physical assessment of such cases. Moreover, multiplicative analysis yields better results in many areas such as finance, economics, biology, engineering, medicine, and population compared to classical analysis [11, 14, 17, 18, 29]. Therefore, in recent years, a significant number of studies have been carried out in many fields, focusing on multiplicative calculus [20–22, 26, 32–35].

On the other hand, it is known that the boundary value problems given on infinite range are used in the mathematical modeling of various applied problems such as the analysis of mass transfer, the study of unstable flow of a gas, and heat transfer. There are many studies examining spectral and scattering analysis of boundary value problems including Schrödinger equation, Sturm-Liouville equation, Hamilton system, and their discrete analogues [4, 5, 25, 28] in classical case. However, few studies have examined the spectral properties of these problems in multiplicative form [20–22, 26, 35]. To the best of our knowledge, there is currently no paper addressing the scattering properties of such multiplicative problems. Therefore, in this paper, we are intrigued by this issue and seek to explore it by constructing the scattering function of a multiplicative boundary value problem based on the Sturm-Liouville equation. Our aim is to investigate some properties of the scattering function for this multiplicative boundary value problem.

The rest of the paper is organized as follows. In Section 2, some basic concepts, definitions, and theorems of multiplicative analysis that we will use in next parts are given. In Section 3, Jost solution of the main problem and the properties of the kernel function of the Jost solution are presented and the asymptotic equations for the Jost solution are given. By identifying the other solutions of the problem, we get relations among solutions in this section. In Section 4, we establish the Jost function and the scattering function of the multiplicative problem and we present some properties of the Jost function. Finally, we investigate the properties of the scattering function.

2. Preliminaries

In this section, we give some well-known fundamental definitions and theorems of the multiplicative calculus used in this study.

Definition 1 *[10] Let $f : A \subseteq \mathbb{R} \to \mathbb{R}^+$ be a differentiable in usual case, where $f(x) > 0$ for all x. If the below limit exists and is positive*

$$f^*(x) = \lim_{h \to 0} \left[\frac{f(x+h)}{f(x)} \right]^{\frac{1}{h}},$$

then $f^*(x)$ is called the multiplicative $(or\ ^*)$ derivative of f at x.

Lemma 1 *[10] Let $f : A \to \mathbb{R}$ be positive and its usual derivative at x exists. Then, there exists the following relation between the classical derivative and the * derivative*

$$f^{*}(x)=e^{(\ln o f)'(x)}.$$

It follows from the last equation that $f'(x) = f(x)\ln f^{*}(x)$. Moreover, the second-order multiplicative derivative of f is obtained by taking multiplicative derivative of the function f^{*} and it is represented by f^{**}. By taking n-times multiplicative derivative of the function f consecutively, we get n-th order multiplicative derivative of the function f at the point x as

$$f^{*(n)}(x)=e^{(\ln\circ f)^{(n)}(x)}.$$

Theorem 1 *[10] Assume that f, g are * differentiable functions and h is a classical differentiable function at the point x. Then the following equations are satisfied for * derivative.*

i. $(cf)^{*}(x)=f^{*}(x), \quad c\in\mathbb{R}^{+}$,

ii. $(fg)^{*}(x)=f^{*}(x)g^{*}(x)$,

iii. $\left(\frac{f}{g}\right)^{*}(x)=\frac{f^{*}(x)}{g^{*}(x)}$,

iv. $\left(f^{h}\right)^{*}(x)=f^{*}(x)^{h(x)}f(x)^{h'(x)}$,

v. $(foh)^{*}(x)=f^{*}(h(x))^{h'(x)}$,

vi. $(f+g)^{*}(x)=f^{*}(x)^{\frac{f(x)}{f(x)+g(x)}}g^{*}(x)^{\frac{g(x)}{f(x)+g(x)}}$.

Definition 2 *[10] Let f be a positive, bounded function on $[a,b]$, where $-\infty<a<b<\infty$. Then, the symbol $\int_a^b f(x)^{dx}$ is called multiplicative intgral or * integral of f on $[a,b]$. By this definition, if f is positive and Riemann integrable on $[a,b]$, then it is * integrable on $[a,b]$ and can be written as*

$$\int_a^b f(x)^{dx}=e^{\int_a^b(\ln o f)(x)dx}.$$

Conversely, one can show that if f is Riemann integrable on $[a,b]$, then

$$\int_a^b f(x)dx=\ln\int_a^b\left(e^{f(x)}\right)^{dx}.$$

Theorem 2 *[10] Let f,g be * integrable functions on $[a,b]$. Then, the items below hold:*

i. $\displaystyle\int_a^b\left[f(x)^{k}\right]^{dx}=\left[\int_a^b f(x)^{dx}\right]^{k}$,

ii. $\displaystyle\int_a^b\left[f(x)g(x)\right]^{dx}=\int_a^b f(x)^{dx}\int_a^b g(x)^{dx}$,

iii. $\int_a^b \left[\frac{f(x)}{g(x)}\right]^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}}$,

iv. $\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} \int_c^b f(x)^{dx}$,

v. $\int_a^b \left[f^*(x)^{g(x)}\right]^{dx} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \left\{ \int_a^b \left[f(x)^{g'(x)}\right]^{dx} \right\}^{-1}$

where $k \in \mathbb{R}$ is a constant and $c \in [a,b]$.

For a better understanding of some multiplicative definitions, we need to know some fundamental algebraic structures for construction of the main problem. Arithmetic operations created with exponential functions are called multiplicative algebraic (* algebraic) operations. Let us present some properties of these operations with *arithmetic table for $f,g \in \mathbb{R}^+$

$$f \oplus g = fg, \quad f \ominus g = \frac{f}{g}, \quad f \odot g = f^{\ln g} = g^{\ln f}.$$

These operations create some algebraic structures. If $\oplus : A \times A \to A$ is an operation, where $A \neq \emptyset$ and $A \subset \mathbb{R}^+$, the algebraic structure $(A, \oplus)$ is called a multiplicative group (* group). Similarly, $(A, \oplus, \odot)$ is a multiplicative ring (* ring) . This situation allows us to define different structures.

Definition 3 *Let $K \subset A$ be a nonempty set and $<,>_*: K \times K \to \mathbb{R}^+$ be a function such that the following axioms are satisfied for each $f,g,h \in K$*

i) $< f,f >_* \geq 1$,

ii) $< f,f >_* = 1$ *if* $f = 1$,

iii) $< f \oplus g, h >_* = < f,h >_* \oplus < g,h >_*$,

iv) $< e^\alpha \odot f, g >_* = e^\alpha \odot < f,g >_*$, $\quad \alpha \in \mathbb{R}$, ***v)*** $< f,g >_* = < g,f >_*$.

Then, this mapping is called multiplicative (*) inner product on K and denoted by $<,>_*$. Also, the space $(K, <,>_*)$ is called the * inner product space [20].

Note that $L_2^*[a,b] = \left\{ f : \int_a^b [f(x) \odot f(x)]^{dx} < \infty \right\}$ is an * inner product space with multiplicative inner product

$$<,>_*: L_2^*[a,b] \times L_2^*[a,b] \to \mathbb{R}^+, \quad < f,h >_* = \int_a^b [f(x) \odot h(x)]^{dx},$$

where $f,h \in L_2^*[a,b]$ are positive functions [32]. It is clear that this space is the multiplicative analogue of the well-known $L_2[a,b]$.

Definition 4 *Assume that* $y_1, y_2, ...y_n$ *are positive functions which are multiplicative differentiable at least* $(n-1)$ *times and a matrix* M *with dimension* $n \times n$ *is defined as*

$$M = \begin{pmatrix} \ln y_1 & \ln y_2 & . \quad . \quad . & \ln y_n \\ \ln y_1^* & \ln y_2^* & . \quad . \quad . & \ln y_n^* \\ . & & & . \\ . & & & . \\ . & & & . \\ \ln y_1^{*(n-1)} & \ln y_2^{*(n-1)} & . \quad . \quad . & \ln y_n^{*(n-1)} \end{pmatrix}.$$

Then, the determinant W_n *defined as*

$$W_n\left(y_1, y_2, ...y_n\right) = \det M$$

is called the multiplicative Wronskian determinant of the functions $\{y_i\}_{i=1}^n$ *[33].*

Now, let us consider the following multiplicative boundary value problem generated by the multiplicative Sturm-Liouville equation

$$\left(y^{**}\right)^{-1} y^{q(x)} = y^{\lambda^2}, \quad 0 \leq x < \infty \tag{1}$$

and the boundary condition

$$y(0) = 1, \tag{2}$$

where λ is a spectral parameter and q is a real-valued function satisfying the following condition

$$\int\limits_0^\infty \left(x^{|q(x)|}\right)^{dx} < \infty. \tag{3}$$

Throughout the paper, we refer to the multiplicative boundary value problem given by (1)-(2) as the multiplicative Sturm-Liouville problem, for brevity.

3. Jost solution of the multiplicative Sturm-Liouville problem

In this section, our aim is to obtain the Jost solution of the multiplicative Sturm-Liouville problem (1)-(2) and to examine the properties of the Jost solution by providing asymptotic equations. Furthermore, we explore alternative solutions to the problem and establish relationships between them in this section. We will denote the bounded solution of (1) satisfying the condition

$$\lim_{x \to \infty} y\left(x, \lambda\right)^{e^{-i\lambda x}} = e, \quad \lambda \in \mathbb{R} \tag{4}$$

by $f\left(x, \lambda\right)$. Jost solutions are especially useful for the studies consisting of spectral and scattering analysis of differential and difference operators in classical case [1, 4–9, 19]. This study will show that they are also important for multiplicative form of these problems.

Theorem 3 *The solution* $f\left(x, \lambda\right)$ *of the problem (1)-(2) is also a solution of the following integral equation:*

$$f\left(x, \lambda\right) = e^{e^{i\lambda x}} \int\limits_x^\infty \left[f\left(t, \lambda\right)^{\frac{q(t)\sin\lambda(t-x)}{\lambda}} \right]^{dt}, \quad \lambda \in \mathbb{R} \tag{5}$$

on the other hand, the opposite of this statement is also true.

Proof The first- and second-order multiplicative derivatives of the solution $f(x,\lambda)$ given in (5) are found as

$$f^{*}(x,\lambda)=e^{i\lambda e^{i\lambda x}-\int_{x}^{\infty}q(t)\cos\lambda(t-x)\ln f(t,\lambda)dt}$$

and

$$f^{**}(x,\lambda)=e^{-\lambda^{2}e^{i\lambda x}+q(x)\ln f(x,\lambda)-\int_{x}^{\infty}\lambda q(t)\sin\lambda(t-x)\ln f(t,\lambda)dt},$$

respectively. By making the necessary operations for the equation of $f^{**}(x,\lambda)$, we write

$$f^{**}(x,\lambda)=f(x,\lambda)^{-\lambda^{2}}f(x,\lambda)^{q(x)}.$$

It gives that the $f(x,\lambda)$, which is the solution of the integral equation (5), is also the solution of the problem (1)-(2). Now, let us try to prove the other part of the theorem. We consider the general solution of (1) as

$$y(x,\lambda)=e^{c_{1}(x)e^{i\lambda x}+c_{2}(x)e^{-i\lambda x}}. \tag{6}$$

Using the method of variation of parameters for equation (6), we obtain

$$y^{*}(x,\lambda)=e^{i\lambda c_{1}(x)e^{i\lambda x}-i\lambda c_{2}(x)e^{-i\lambda x}}, \tag{7}$$

$$y^{**}(x,\lambda)=e^{i\lambda c_{1}'(x)e^{i\lambda x}-i\lambda c_{2}'(x)e^{-i\lambda x}}\left[y(x,\lambda)\right]^{-\lambda^{2}}. \tag{8}$$

By writing the equations (7) and (8) in (1), we easily find the following equation:

$$e^{c_{1}'(x)e^{i\lambda x}-c_{2}'(x)e^{-i\lambda x}}=y(x,\lambda)^{\frac{q(x)}{i\lambda}}.$$

By making necessary calculations, the coefficients $c_1(x)$ and $c_2(x)$ are obtained as

$$c_{1}(x)=k-\int_{x}^{\infty}\frac{q(t)e^{-i\lambda t}}{2i\lambda}\ln y(t,\lambda)\,dt \tag{9}$$

and

$$c_{2}(x)=m+\int_{x}^{\infty}\frac{q(t)e^{i\lambda t}}{2i\lambda}\ln y(t,\lambda)\,dt, \tag{10}$$

where $\lim\limits_{t\to\infty}c_1(t)=k$ and $\lim\limits_{t\to\infty}c_2(t)=m$. By substituting the $c_1(x)$ and $c_2(x)$ coefficients in equation (6), we get

$$y(x,\lambda)^{e^{-i\lambda x}}=e^{k+me^{-2i\lambda x}+\int_{x}^{\infty}q(t)\frac{\sin\lambda(t-x)}{\lambda}e^{i\lambda(t-x)}e^{-i\lambda t}\ln y(t,\lambda)dt}. \tag{11}$$

It can be easily seen that for $\lambda\in\mathbb{R}$ and $t>x$,

$$\left|\frac{\sin\lambda\left(t-x\right)}{\lambda}e^{i\lambda(t-x)}\right|\leq 1.$$

From the last inequality and condition (4), we obtain $k=1$ and $m=0$. This implies that the solution $f(x,\lambda)$ of the problem (1)-(2) is also the solution of the integral equation (5). Hence, the proof is concluded.

In the next part of the study, we will assume that the real-valued function q satisfies the condition

$$e^{\int\limits_0^\infty x|q(x)|dx}<\infty. \tag{12}$$

For all $\lambda\in\mathbb{R}$, the Jost solution of problem (1)-(2) has the integral representation

$$f\left(x,\lambda\right)=e^{e^{i\lambda x}}\int\limits_x^\infty\left[K\left(x,t\right)^{e^{i\lambda t}}\right]^{dt}, \tag{13}$$

under the condition (12), where the kernel $K(x,t)$ may be expressed in terms of the potential function q [2]. Note that (13) is the multiplicative form of the classical integral representation of the Jost solution. Because of this, it plays an important role in the solutions of direct and inverse multiplicative problems of quantum scattering theory. Furthermore, we obtain the following inequalities for the kernel of f by using the Fourier transform and successive approximations method

$$|K(x,t)|\leq\sigma\left(\frac{x+t}{2}\right)^c \tag{14}$$

and

$$\left|K_x\left(x,t\right)\right|,\left|K_t\left(x,t\right)\right|\leq e^{\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right|+c\ln\sigma\left(\frac{x+t}{2}\right)}, \tag{15}$$

where

$$\sigma(x)=e^{\int\limits_x^\infty|q(s)|ds},\quad \sigma_1(x)=e^{\int\limits_x^\infty\ln\sigma(t)dt}. \tag{16}$$

Theorem 4 *If (12) holds, then* $K\left(x,.\right)\in L_1\left(x,\infty\right)$, $K_x^*\left(x,.\right)\in L_1\left(x,\infty\right)$ *and* $K_t^*\left(x,.\right)\in L_1\left(x,\infty\right)$.

Proof By using Definition 2 and equations (14) and (16), we get

$$\begin{aligned}\int\limits_x^\infty\left|K\left(x,t\right)\right|^{dt}&\leq e^{c\int\limits_x^\infty\ln\sigma\left(\frac{x+t}{2}\right)dt}\\&=e^{2c\int\limits_0^\infty\int\limits_u^\infty|q(s)|dsdu}\\&=e^{2c\int\limits_0^\infty\int\limits_0^s|q(u)|duds}\\&=e^{2c\int\limits_0^\infty s|q(s)|ds}.\end{aligned} \tag{17}$$

By the help of condition (12) and inequality (17), we find that $K(x,.) \in L_1(x,\infty)$. From (15) and (16), we write

$$\int_x^\infty |K(x,t)|^{dt} \leq e^{\int_x^\infty \left\{\frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + c\ln\sigma\left(\frac{x+t}{2}\right)\right\}dt}$$

$$\leq e^{2\int_x^\infty \left\{\frac{1}{4}|q(u)| + c\ln\sigma(u)\right\}du}. \tag{18}$$

Using (12) and (18), we get $K_x^*(x,.) \in L_1(x,\infty)$. In a similar way, we easily obtain $K_t^*(x,.) \in L_1(x,\infty)$, which completes the proof.

Theorem 5 *Assume (12). Then $f(x,\lambda)$ satisfies the following asymptotic equations for $x \in [0,\infty)$ and $\lambda \in \overline{\mathbb{C}}_+$*

a) $\ln f(x,\lambda) = e^{i\lambda x}[1+o(1)], \quad |\lambda| \to \infty,$

b) $\ln f^*(x,\lambda) = e^{i\lambda x}[i\lambda + O(1)], \quad |\lambda| \to \infty.$

Proof **a)** By using (13), we clearly find that

$$f(x,\lambda)^{e^{-i\lambda x}} = e^{1+\int_x^\infty e^{i\lambda(t-x)}\ln K(x,t)dt}.$$

Since $\left|e^{i\lambda(t-x)}\right| \leq 1$ for $\lambda \in \overline{\mathbb{C}}_+$, by using Theorem 4, we obtain

$$\ln f(x,\lambda) = e^{i\lambda x}[1+o(1)], \quad x \in [0,\infty), \quad \lambda \in \overline{\mathbb{C}}_+, \quad |\lambda| \to \infty.$$

b) By taking the multiplicative derivative in equation (13), we get

$$f_x^*(x,\lambda)^{e^{-i\lambda x}} = e^{i\lambda - \ln K(x,x) + \int_x^\infty e^{i\lambda(t-x)}\frac{K_x(x,t)}{K(x,t)}dt}.$$

It can be easily found that $|\ln K(x,x)| < \infty$ and $\left|\frac{K_x(x,t)}{K(x,t)}\right| < \infty$ for $x \in [0,\infty)$ and $x > t$. If we consider these inequalities in the last equation, we write

$$\ln f^*(x,\lambda) = e^{i\lambda x}[i\lambda + O(1)], \quad x \in [0,\infty), \quad \lambda \in \overline{\mathbb{C}}_+, \quad |\lambda| \to \infty.$$

This completes the proof of Theorem 3.3.

Theorem 6 *Under the condition (12), the Jost solution of (1)-(2) satisfies the following asymptotic equations for $x \to \infty$*

a) $\ln f(x,\lambda) = e^{i\lambda x}[1+o(1)], \quad \lambda \in \overline{\mathbb{C}}_+,$

b) $\ln f_x^*(x,\lambda) = e^{i\lambda x}[i\lambda + o(1)], \quad \lambda \in \overline{\mathbb{C}}_+,$

c) $\ln f(x,-\lambda) = e^{-i\lambda x}[1+o(1)], \quad \lambda \in \overline{\mathbb{C}}_-,$

d) $\ln f_x^*(x,-\lambda) = e^{-i\lambda x}[-i\lambda + o(1)], \quad \lambda \in \overline{\mathbb{C}}_-.$

Proof **a)** From (13) and Theorem 4, the following asymptotic equation is obtained whenever $x \to \infty$:

$$e^{1+\int_x^\infty e^{i\lambda(t-x)}\ln K(x,t)dt} = e^{1+o(1)}, \quad \lambda \in \overline{\mathbb{C}}_+.$$

Using (13) and the last equality, we find

$$\ln f(x,\lambda) = e^{i\lambda x}\left[1+o(1)\right],$$

for $\lambda \in \overline{\mathbb{C}}_{+}$ and $x \to \infty$.

b) By taking the multiplicative derivative of (13) according the variable x, we write

$$f_x^{*}(x,\lambda)^{e^{-i\lambda x}} = e^{i\lambda - \ln K(x,x) + \int_x^{\infty} e^{i\lambda(t-x)} \frac{K_x(x,t)}{K(x,t)} dt}.$$

From (14), (15), and the last equation, we obtain

$$\left| e^{i\lambda - \ln K(x,x)^{t} + \int_x^{\infty} e^{i\lambda(t-x)} \frac{K_x(x,t)}{K(x,t)} d} \right| \le e^{i\lambda + o(1)}$$

for $\lambda \in \overline{\mathbb{C}}_{+}$ and $x \to \infty$. This completes the proof of items **(a)** and **(b)** . The proofs of **(c)** and **(d)** can be done similarly.

4. Other solutions of the multiplicative Sturm-Liouville equation

In this section, we construct other solutions of the problem given in (1)-(2) and we present the relations of these solutions to the Jost solution. For $\lambda \in \mathbb{C}$, let us denote the solutions of equation (1) by $S(x,\lambda)$ and $C(x,\lambda)$ satisfying the initial conditions

$$y(0) = 1, \quad y^{*}(0) = e \tag{19}$$

and

$$y(0) = e, \quad y^{*}(0) = 1, \tag{20}$$

respectively. It is known that the solutions $S(x,\lambda)$ and $C(x,\lambda)$ are entire functions of λ.

Theorem 7 *The solutions $S(x,\lambda)$ and $C(x,\lambda)$, which are the solutions of (1), have the integral represantations*

$$S(x,\lambda) = e^{\frac{\sin \lambda x}{\lambda}} \int_0^x \left[S(t,\lambda)^{\frac{q(t)\sin\lambda(x-t)}{\lambda}} \right]^{dt} \tag{21}$$

and

$$C(x,\lambda) = e^{\cos\lambda x} \int_0^x \left[C(t,\lambda)^{\frac{q(t)\sin\lambda(x-t)}{\lambda}} \right]^{dt}, \tag{22}$$

satisfying the initial conditions (19) and (20), respectively.

Proof The first and second order multiplicative derivatives of the solution $S(x,\lambda)$ given in (21) can be found as

$$S^{*}(x,\lambda) = e^{\cos\lambda x + \int_0^x q(t)\cos\lambda(x-t)\ln S(t,\lambda)dt},$$

$$S^{**}(x,\lambda) = e^{-\lambda\sin\lambda x + q(x)\ln S(x,\lambda) - \lambda\int_0^x q(t)\sin\lambda(x-t)\ln S(t,\lambda)dt}.$$

It is easily seen that the solution $S(x,\lambda)$ satisfies the boundary condition (19). By making the necessary operations in the second equation given by $S^{**}(x,\lambda)$, we get

$$S^{**}(x,\lambda)=S(x,\lambda)^{-\lambda^2}S(x,\lambda)^{q(x)}.$$

It means that $S(x,\lambda)$ given in (21) is the solution of (1) and satisfies the initial condition (19). Similarly, it can be obtained that the solution $C(x,\lambda)$ given in (22) is the solution of (1) and satisfies the initial condition (20). It gives the proof of the theorem.

Lemma 2 *The following equations are satisfied for the Wronskians of the solutions of (1).*
a) $W[S(x,\lambda),C(x,\lambda)]=-1,\quad \lambda\in\mathbb{C},$
b) $W[f(x,\lambda),S(x,\lambda)]=\ln f(0,\lambda),\quad \lambda\in\overline{\mathbb{C}}_+.$

Proof **a)** With the definition of Wronskian given in Definition 4, we write

$$W[S(x,\lambda),C(x,\lambda)]=\ln\left[C^*(x,\lambda)\right]^{\ln S(x,\lambda)}-\ln\left[S^*(x,\lambda)\right]^{\ln C(x,\lambda)}.$$

Since the Wronskian is independent of x, by using the conditions (19) and (20) whenever $x\to 0$, we have

$$W[S(x,\lambda),C(x,\lambda)]=-1,\quad \lambda\in\mathbb{C}$$

following the last equation.
b) From Definition 4, we obtain the Wronskian of the solutions $f(x,\lambda)$ and $S(x,\lambda)$ as

$$W[f(x,\lambda),S(x,\lambda)]=\ln\left[S^*(x,\lambda)\right]^{\ln f(x,\lambda)}-\ln\left[f^*(x,\lambda)\right]^{\ln S(x,\lambda)}.$$

By taking the limit for $x\to 0$ in the last equation and using the condition (19), we obtain

$$W[f(x,\lambda),S(x,\lambda)]=\ln f(0,\lambda)$$

for $\lambda\in\overline{\mathbb{C}}_+$. This gives the proof of the lemma.

5. Jost function and scattering function of the multiplicative Sturm-Liouville problem

In this section, our aim is to obtain the Jost function of (1)-(2) and then to examine the properties of the scattering function of the main problem. By considering the boundary condition (2) and the Jost solution given in (13), we obtain the Jost function of the problem by

$$f(\lambda):=f(0,\lambda)=e\int_0^\infty\left[K(0,t)^{e^{i\lambda t}}\right]^{dt}. \tag{23}$$

It is evident that for $\lambda\in\mathbb{R}/\{0\}$

$$f(-\lambda)=\overline{f(\lambda)}. \tag{24}$$

Lemma 3 *The following Wronskian equation holds for* $\lambda\in\mathbb{R}/\{0\}$

$$W[f(x,\lambda),f(x,-\lambda)]=-2i\lambda.$$

Theorem 8 *For all* $\lambda \in \mathbb{R}/\{0\}$, $f(\lambda) \neq 1$.

Proof By using Lemma 3 and equation (24), we write

$$\frac{f^*(-\lambda)^{\ln f(\lambda)}}{f^*(\lambda)^{\ln f(-\lambda)}} = e^{-2i\lambda}. \tag{25}$$

Assume that there exists a $\lambda_0 \in \mathbb{R}/\{0\}$ such that $f(\lambda_0) = 1$. From the equations (24) and (25), we find that $e^{-2i\lambda_0} = 1$. Since this result gives a contradiction, $f(\lambda) \neq 1$, for all $\lambda \in \mathbb{R}/\{0\}$.

Theorem 9 *For all* $\lambda \in \mathbb{R}/\{0\}$, *the following equation holds*

$$S(x,\lambda)^{2i\lambda} = e^{\overline{f(\lambda)}f(x,\lambda) - f(\lambda)\overline{f(x,\lambda)}}.$$

Proof Since $f(x,\lambda)$ and $f(x,-\lambda)$ are multiplicative linearly independent solutions of (1), we write

$$S(x,\lambda) = e^{c_1 f(x,\lambda) + c_2 \overline{f(x,\lambda)}}.$$

By taking the multiplicative derivative of the last equation and using (19), we find

$$\begin{aligned} e^{c_1 f(\lambda) + c_2 \overline{f(\lambda)}} &= 1 \\ e^{c_1 f'(\lambda) + c_2 \overline{f'(\lambda)}} &= e. \end{aligned} \tag{26}$$

By making necessary calculations in (26), we obtain the coefficients c_1 and c_2 for $\lambda \in \mathbb{R}/\{0\}$ as follows:

$$c_1 = \frac{\overline{f(\lambda)}}{2i\lambda} \quad \text{and} \quad c_2 = -\frac{f(\lambda)}{2i\lambda}.$$

This completes the proof.

Theorem 10 *The Jost function of (1)-(2) has a finite number of zeros in* $\mathbb{C}_+$.

Proof From Theorem 5 **a** , it can be easily shown that the set of all zeros of the Jost function of (1)-(2) in $\mathbb{C}_+$ is bounded. On the other hand, since the Jost function of (1)-(2) is analytic in $\mathbb{C}_+$, the limit points of the zeros in $\mathbb{C}_+$ are on reel axis. By the help of Theorem 8 and Bolzano Weirstrass Theorem [16], the set of the zeros of Jost function in $\mathbb{C}_+$ of this problem is finite.

Theorem 11 *If* $\lambda_0 \in \mathbb{C}_+$ *and* $f(\lambda_0) = 1$, *then* $f(x,\lambda_0) = S(x,\lambda_0)^{f_x(0,\lambda_0)}$.

Proof By means of Lemma 2 **b**, it can be easily written that

$$W[f(x,\lambda_0), S(x,\lambda_0)] = 0, \quad \lambda_0 \in \mathbb{C}_+.$$

It follows from that $f(x,\lambda_0)$ and $S(x,\lambda_0)$ are multiplicative linear dependent solutions of (1). As a result of this, we get

$$f(x,\lambda_0) = S(x,\lambda_0)^c, \quad c \neq 0.$$

If $c = f_x(0,\lambda_0)$ is arbitrarily chosen for the last equation, the proof is completed.

Theorem 12 *For* $\lambda_1, \lambda_2 \in \mathbb{C}_+$*, if* $f(\lambda_1) = f(\lambda_2) = 1,$ *then the following expression is satisfied*

$$\lim_{x\to 0} W\left[f(x,\lambda_1), \overline{f(x,\lambda_2)}\right] = 0.$$

Proof The following equalities can be easily seen from Theorem 11

$$f(x,\lambda_1) = S(x,\lambda_1)^{f_x(0,\lambda_1)} \quad \text{and} \quad f(x,\lambda_2) = S(x,\lambda_2)^{f_x(0,\lambda_2)}. \tag{27}$$

By using the Definition 4 and equation (27), we write the Wronskian of the solutions $f(x,\lambda_1)$ and $\overline{f(x,\lambda_2)}$ as

$$\begin{aligned} W\left[f(x,\lambda_1), \overline{f(x,\lambda_2)}\right] &= \ln\left[\overline{f^*(x,\lambda_2)}\right]^{\ln f(x,\lambda_1)} - \ln\left[f^*(x,\lambda_1)\right]^{\ln \overline{f(x,\lambda_2)}} \\ &= \overline{f_x(0,\lambda_2)} \ln\left[\overline{S^*(x,\lambda_2)}\right]^{f_x(0,\lambda_1)\ln S(x,\lambda_1)} \\ &\quad -f_x(0,\lambda_1)\ln\left[S^*(x,\lambda_1)\right]^{\overline{f_x(0,\lambda_2)}\ln \overline{S(x,\lambda_2)}}. \end{aligned}$$

In the last equation, by taking the limit whenever $x \to 0$ and using the initial condition (19), we get

$$\lim_{x\to 0} W\left[f(x,\lambda_1), \overline{f(x,\lambda_2)}\right] = 0.$$

Theorem 13 *All zeros of the Jost function of (1)-(2) in* $\mathbb{C}_+$ *are on the imaginary axis.*

Proof Let us assume that λ_1 and λ_2 are arbitrary zeros of the Jost function in $\mathbb{C}_+$. Then it implies $f(\lambda_1) = f(\lambda_2) = 1$. From Theorem 12, it can be written

$$\lim_{x\to 0} W\left[f(x,\lambda_1), \overline{f(x,\lambda_2)}\right] = 0.$$

Since $f(x,\lambda_1)$ and $\overline{f(x,\lambda_2)}$ are the solutions of (1)-(2), they satisfy (1). This yields

$$\begin{aligned} \left[f^{**}(x,\lambda_1)\right]^{-1} f(x,\lambda_1)^{q(x)} &= f(x,\lambda_1)^{\lambda_1^2} \\ \left[\overline{f^{**}(x,\lambda_2)}\right]^{-1}\left[\overline{f(x,\lambda_2)}\right]^{q(x)} &= \left[\overline{f(x,\lambda_2)}\right]^{\overline{\lambda_2}^2}. \end{aligned} \tag{28}$$

By making the necessary calculations in (28), we obtain

$$\frac{f^{**}(x,\lambda_1)^{\ln \overline{f(x,\lambda_2)}}}{\left[\overline{f^{**}(x,\lambda_2)}\right]^{\ln f(x,\lambda_1)}} = \frac{\left[\overline{f(x,\lambda_2)}\right]^{\overline{\lambda_2}^2 \ln f(x,\lambda_1)}}{f(x,\lambda_1)^{\lambda_1^2 \ln \overline{f(x,\lambda_2)}}}. \tag{29}$$

On the other hand, from Definition 4, we write

$$e^{W\left[f(x,\lambda_1),\overline{f(x,\lambda_2)}\right]} = \frac{\left[\overline{f^*(x,\lambda_2)}\right]^{\ln f(x,\lambda_1)}}{f^*(x,\lambda_1)^{\ln \overline{f(x,\lambda_2)}}}. \tag{30}$$

By taking the multiplicative derivative of both sides of the equation (30), we find

$$\frac{d^*}{dx}e^{W\left[f(x,\lambda_1),\overline{f(x,\lambda_2)}\right]}=\frac{\left[\overline{f^{**}(x,\lambda_2)}\right]^{\ln f(x,\lambda_1)}\left[\overline{f^{*}(x,\lambda_2)}\right]^{(\ln f(x,\lambda_1))'}}{f^{**}(x,\lambda_1)^{\ln\overline{f(x,\lambda_2)}}\left[\overline{f^{*}(x,\lambda_1)}\right]^{\left(\ln\overline{f(x,\lambda_2)}\right)'}}.$$

If we consider (29) and the last equality together, we have

$$\frac{d^*}{dx}e^{W\left[f(x,\lambda_1),\overline{f(x,\lambda_2)}\right]}=\frac{\left[\overline{f(x,\lambda_2)}\right]^{\overline{\lambda_2}^2\ln f(x,\lambda_1)}}{f(x,\lambda_1)^{\lambda_1^2\ln\overline{f(x,\lambda_2)}}}. \tag{31}$$

By taking the multiplicative integral from 0 to ∞ on both sides of the equation (31), we get

$$e^{W\left[f(x,\lambda_1),\overline{f(x,\lambda_2)}\right]}\Big|_0^\infty=\int_0^\infty\left(\frac{\left[\overline{f(x,\lambda_2)}\right]^{\overline{\lambda_2}^2\ln f(x,\lambda_1)}}{f(x,\lambda_1)^{\lambda_1^2\ln\overline{f(x,\lambda_2)}}}\right)^{dx}.$$

If we choose $\lambda_1=\lambda_2=\lambda_0$ in the last equation, since $||f(x,\lambda_0)||\neq 0$, we obtain $\lambda_0=ia_0$, $a_0>0$, which gives the result.
Considering the classical definition of the scattering function, we provide the definition of the multiplicative scattering as follows:

Definition 5 *The function*

$$S(\lambda)=\left[\overline{f(\lambda)}\right]^{\frac{1}{\ln f(\lambda)}},\quad \lambda\in\mathbb{R}/\{0\} \tag{32}$$

is called the scattering function of (1)-(2).

Theorem 14 *For all $\lambda\in\mathbb{R}/\{0\}$, the scattering function satisfies*

$$S(-\lambda)=\overline{S(\lambda)}.$$

Proof Let us assume that $\lambda\in\mathbb{R}/\{0\}$. It is clear from (32) that

$$\begin{aligned} S(-\lambda) &= \left[\overline{f(-\lambda)}\right]^{\frac{1}{\ln f(-\lambda)}} \\ &= f(\lambda)^{\frac{1}{\overline{\ln f(\lambda)}}} \\ &= \overline{S(\lambda)}. \end{aligned}$$

This completes the proof.

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Conflict of interest
The authors declare that they have no conflict of interest.

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