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KHALED BEN AMARA
khaled.benamara.etud@fss.usf.tn

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# Existence and uniqueness of continuous solutions for iterative functional differential equations in Banach algebras

**Khaled BEN AMARA***
Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Sfax, Tunisia

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**Abstract:** This paper is devoted to studying the existence and uniqueness of continuous solutions of the following iterative functional differential equation

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)-h\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}{f\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}\right)=g\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right),t\in J,\\ x(0)=x_0. \end{cases}$$

By using of Boyd-Wong's fixed point theorem and under suitable conditions, we establish the existence and uniqueness of a continuous solution.

**Key words:** Banach algebras, continuous solutions, fixed point theory, iterative differential equation

## 1. Introduction

Iterative differential equations provide powerful tools for describing many phenomena in various fields of science and engineering. There has been a great deal of research published on the existence of solutions for iterative differential equations, see [8, 11, 13, 14, 18–20]. Fixed point theory plays a great role in this context and generally in nonlinear integro-differential equations, see [1–4, 7, 8, 15].

A number of authors focused on studying the existence of solutions of iterative differential equations such as A. Bouakkaz, A. Ardjouni And A. Djoudi in [5], S. Staněk in [19], P. Zhang and X. Gong in [20] and H. Y. Zhao and J. Liu in [21]. In [19], S. Staněk, using fixed point theory, studied existence of continuous solutions considering differential iterative equations

$$x'(t)=x(x(t))-bx(t). \tag{1.1}$$

As a generalization, in 2014 P. Zhang and X. Gong [20] considered the differential iterative equation

$$x'(t)=g\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right), \tag{1.2}$$

and developed the existence of an analytic solution of this problem by using the Schauder fixed point theorem. In 2019, by virtue of Schauder and Banach fixed point theorems, H. Y. Zhao and J. Liu [21] investigated the

*Correspondence: khaled.benamara.etud@fss.usf.tn

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existence, uniqueness and stability solutions for nonhomogeneous iterative functional differential equations of the form

$$x'(t) = \lambda_1 x^{[1]}(t) + \lambda_2 x^{[2]}(t) + \ldots + \lambda_n x^{[n]}(t) + f(t). \tag{1.3}$$

Recently, A. Bouakkaz, A. Ardjouni, and A. Djoudi [5] discussed the problem of the existence of periodic solution for the iterative functional differential equation

$$\frac{d}{dt}(x(t)) = \frac{d}{dt}\left(h\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right)\right) - a(t)x^{[1]}(t) + f\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right). \tag{1.4}$$

Since the Banach algebras represent a practical framework for several functional integro-differential equations, there has been increasing attention by scholars in studying the existence of solutions of different nonlinear equations in Banach algebras, see [1, 2, 4, 7, 9, 12, 15, 16].

Motivated by the above-mentioned works and theoretical investigations, in this paper, we establish a new existence and uniqueness approach for a large class of iterative differential equations in Banach algebras by mixing the properties of the Banach algebras with the Boyd Wong fixed point theorem.

More precisely, by using fixed point theory for the product and the sum of nonlinear operators defined in Banach algebras, we consider the problem of existence and uniqueness results for the iterative differential equations of type

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t) - h\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right)}{f\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right)}\right) = g\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right), t \in J, \\ \\ x(0) = x_0, \end{cases} \tag{1.5}$$

where $J = [0,1]$, $f : J \times \mathbb{R}^n \to \mathbb{R} \setminus \{0\}$, $g, h : J \times \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$, are given functions, and $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[n]}(t) = x^{[n-1]}(x(t))$.

The problem (1.5) has not been studied in the literature before, so the results of this paper are new to the theory of iterative differential equations and it significantly extends and generalizes some works in the literature. For example, if $n = 2$, $f = 1$, $g(t, x_1, x_2) = x_2 - bx_1$ and $h = 0$. Then the problem (1.5) reduces to problem (1.1), which has been studied by S. Staněk in [19].

In the special case when $f = 1$ and $h = 0$ in the iterative differential problem (1.5), it reduces to problem (1.2), which has been studied by P. Zhang and X. Gong [20] via Schauder's fixed point theorem.

If $f = 1$, $h = 0$ and $g(t, x_1, \ldots, x_2, x_n) = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n + f(t)$. Then the problem (1.5) reduces to the iterative differential problem (1.3), which has been studied in [21] via the principle of contraction mappings.

In other when $g\left(t, x_1, \ldots, x_n\right) = -a(t)x_1 + f\left(t, x_1, \ldots, x_n\right)$, then the problem (1.5) reduces to the iterative differential problem (1.4) which has been studied by A. Bouakkaz, A. Ardjouni and A. Djoudi in [5] under Lipschitz conditions on the functions $g$ and $h$.

The paper is organized as follows. Section 2 is devoted to introducing necessary preliminary results. In Section 3 a new approach is constructed to ensure the existence and uniqueness of continuous solutions for hybrid iterative differential equations of type (1.5), and illustrative examples are provided to reinforce our findings.

## 2. Analytical preliminaries

Let $X$ be a Banach space with norm $\|\cdot\|$. We denote by $C(J,X)$ the Banach algebra of all continuous functions defined from $J$ into $X$ endowed with the supremum norm $\|x\| = \sup_{t\in J}\|x(t)\|$ and with the pointwise multiplication $(xy)(t) = x(t)y(t)$.

**Definition 2.1** [6, 9, 10]*Let $T : X \to X$. We say that $T$ is $\mathcal{D}$-Lipschitzian if there exists a continuous nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ such that*

$$\|T(x) - T(y)\| \leq f(\|x-y\|) \text{ for all } x, y \in X.$$

*If $f$ is not necessarily nondecreasing and $f(r) < r, r > 0$, then we say that $T$ is nonlinear contraction. Here, $f$ is called the $\mathcal{D}$-function associated to $T$.*

Notice that every Lipschitz mapping with Lipschitz constant $\alpha > 0$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\psi(t) = \alpha t, t \geq 0$, but the inverse is not necessarily true. It suffices to consider for example the operator $T : J \to J$ given by $T(x) = x - x^2$.

In the sequel, we need the following technical lemma. Consider the closed, convex subset

$$C(L,M) := \{x \in C(J,\mathbb{R}); \|x\| \leq L, |x(t) - x(s)| \leq M|t-s|\}.$$

**Lemma 2.2** *[21] For any $\xi, \zeta \in C(L,M)$,*

$$\left\|\xi^{[n]} - \zeta^{[n]}\right\| \leq \sum_{j=0}^{n-1} M^j \|\xi - \zeta\|.$$

A generalisation of the celebrated Banach fixed point theorem has been proved by Boyd and Wong in [6].

**Theorem 2.3** *Let $T : X \to X$ be a nonlinear contraction, then there exists a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$. In addition, for any $x \in X$, the sequence $T^n(x)$ converges to $x^*$.*

## 3. Existence results

This section is devoted to discussing the existence and uniqueness of a continuous solution for (1.5) under $\mathcal{D}$-Lischitzian conditions. For this purpose, we need to prove the following results.

**Definition 3.1** *A function $x \in C(J,\mathbb{R})$ is a solution of the iterative differential equation (1.5) if*

*(i) the function* $t \mapsto \dfrac{x(t) - h\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right)}{f\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right)}$ *is differentiable, and*

*(ii) $x$ satisfies the equations in (1.5).*

**Lemma 3.2** *Let $x \in C(J,\mathbb{R})$. Then for any function $g \in C(J,\mathbb{R})$, the function $x$ is a solution of the hybrid functional iterative problem*

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)-h\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}{f\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}\right)=g\left(t\right), t\in J, \\ x(0)=x_0, \end{cases} \tag{3.1}$$

*if and only if*

$$x(t) \quad = f\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)\left(\int_0^t g\left(s\right)ds+\frac{x_0-h\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)}{f\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)}\right)+h\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right). \tag{3.2}$$

**Proof** Let $g \in C(J,\mathbb{R})$. Assume first that $x$ is a solution of the iterative differential equations (3.1) defined on $J$. By definition, the function $t \to \dfrac{x(t)-h\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}{f\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}$ is continuous on the interval $J$, and is differentiable there, so $\dfrac{d}{dt}\left(\dfrac{x(t)-h\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}{f\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}\right)$ is integrable on $J$. Integrating the first equation in (3.1) from 0 to $t$, we obtain the hybrid functional integral equation (3.2) on $J$.
Conversely, assume that $x$ satisfies the hybrid functional integral equation (3.2). Then, we have

$$\frac{x(t)-h\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}{f\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}=\int_0^t g(s)ds+\frac{x_0-h\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)}{f\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)}.$$

This implies in particular that $t \mapsto \dfrac{x(t)-h\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}{f\left(t,x^{[1]}(t),\ldots,x^{[n]}(t)\right)}$ is differentiable, since $g \in C(J,\mathbb{R})$. Hence by direct differentiation, we get the first equation in (3.1). On the other hand, substituting $t=0$ in (3.2) in order to get $x(0)=x_0$. □

Our results will be considered under the following hypothesis: There exists $L>0$ such that:

$(H_1)$ The function $f: J\times\mathbb{R}^n \to \mathbb{R}\setminus\{0\}$ is such that:

(i) There exist $C_f, \delta > 0$ such that

$$|f(t,x_1,\ldots,x_n)-f(s,x_1,\ldots,x_n)| \le C_f|t-s| \text{ for all } t,s\in J \text{ and } x_i\in[-L,L],\ i=1,\ldots,n$$

and

$$\sup_{x_1,\ldots,x_n\in[-L,L]} |f(0,x_1,\ldots,x_n)|^{-1} \le \delta.$$

(ii) There are $\mathcal{D}$-functions $\varphi_i$ with $\varphi_i(r)<r$ for $r>0$, $i=1,\ldots,n$ and $\alpha\in C(J,\mathbb{R})$ such that

$$|f(t,x_1,\ldots,x_n)-f(t,y_1,\ldots,y_n)| \le |\alpha(t)|\sum_{i=1}^n \varphi_i(|x_i-y_i|) \text{ for } t\in J \text{ and } x_i,y_i\in[-L,L],\ i=1,\ldots,n.$$

($H_2$) The function $g : J \times \mathbb{R}^n \to \mathbb{R}$ is such that:

(i) The partial mapping $t \mapsto g(t, x_1, \ldots, x_n)$ is continuous on $J$.

(ii) There are $\mathcal{D}$-functions $\psi_i, i = 1, \ldots, n$ and $\eta \in C(J, \mathbb{R})$ such that

$$|g(t, x_1, \ldots, x_n) - g(t, y_1, \ldots, y_n)| \leq |\eta(t)| \sum_{i=1}^{n} \psi_i(|x_i - y_i|) \text{ for } t \in J \text{ and } x_i, y_i \in [-L, L],\ i = 1, \ldots, n.$$

($H_3$) The function $h : J \times \mathbb{R}^n \to \mathbb{R}$ is such that:

(i) There exists $C_h > 0$ such that

$$|h(t, x_1, \ldots, x_n) - h(s, x_1, \ldots, x_n)| \leq C_h |t - s| \text{ for } t, s \in J \text{ and } x_i \in [-L, L],\ i = 1, \ldots, n.$$

(ii) There are $\mathcal{D}$-functions $\theta_i$ with $\theta_i(r) < r$ for $r > 0$, $i = 1, \ldots, n$ and $\gamma \in C(J, \mathbb{R})$ such that

$$|h(t, x_1, \ldots, x_n) - h(t, y_1, \ldots, y_n)| \leq |\gamma(t)| \sum_{i=1}^{n} \theta_i(|x_i - y_i|) \text{ for } t \in J \text{ and } x_i, y_i \in [-L, L],\ i = 1, \ldots, n.$$

By an application of Lemma 3.2, we can see that the iterative differential equation (1.5) is equivalent to the hybrid fixed point problem

$$x = Ax \cdot Bx + Cx,$$

where the operators $A, B,$ and $C$ are defined by

$$(Ax)(t) = f\left(t, x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right),$$

$$(Bx)(t) = \int_0^t g\left(s, x^{[1]}(s), x^{[2]}(s), \ldots, x^{[n]}(s)\right) ds + \frac{x_0 - h\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right)}{f\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right)}$$

and

$$(Cx)(t) = h\left(t, x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right).$$

**Lemma 3.3** *The operators $A, B$ and $C$ map $C(L, M)$ into $C(J, \mathbb{R})$.*

**Proof** Let $x \in C(L, M)$ and let $t, t' \in J$. We have

$$\begin{aligned}
|(Ax)(t) - (Ax)(t')| &\leq \left| f\left(t, x^{[1]}(t), x^{[2]}(t), \ldots, x^{[n]}(t)\right) - f\left(t', x^{[1]}(t'), x^{[2]}(t'), \ldots, x^{[n]}(t')\right) \right| \\
&\leq \left| f\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right) - f\left(t', x^{[1]}(t), \ldots, x^{[n]}(t)\right) \right| \\
&\quad + \left| f\left(t', x^{[1]}(t), \ldots, x^{[n]}(t)\right) - f\left(t', x^{[1]}(t'), \ldots, x^{[n]}(t')\right) \right|.
\end{aligned}$$

Using assumption ($H_1$), we infer that

$$|(Ax)(t) - (Ax)(t')| \leq C_f |t - t'| + \|\alpha(\cdot)\| \sum_{i=1}^{n} \varphi_i\left(|x^{[i]}(t) - x^{[i]}(t')|\right). \tag{3.3}$$

Now, taking into account that

$$|x^{[i]}(t) - x^{[i]}(s)| \leq M^i|t-s| \text{ for all } t,s \in J \text{ and } i = 1,\dots,n,$$

on the base of (3.3) we derive that

$$|(Ax)(t) - (Ax)(t')| \quad \leq \quad C_f|t-t'| + \|\alpha(\cdot)\| \sum_1^n \varphi_i\left(M^i|t-t'|\right). \tag{3.4}$$

Using the continuity of $\varphi_i, i = 1,\dots,n,$ we get $A(x) \in C(J,\mathbb{R})$. By assumption $(H_2)$,

$$\begin{aligned} |(Bx)(t) - (Bx)(t')| &\leq \int_{t'}^{t} \left| g\left(s, x^{[1]}(s),\dots,x^{[n]}(s)\right)\right| ds \\ &\leq \int_{t'}^{t} \left| g\left(s, x^{[1]}(s),\dots,x^{[n]}(s)\right) - g\left(s,0,\dots,0\right)\right| + |g(s,0,\dots,0)|ds \\ &\leq \int_{t'}^{t} \left( |\eta(s)| \sum_1^n \psi_i\left(|x^{[i]}(s)|\right) + |g(s,0,\dots,0)|\right) ds, \end{aligned}$$

and therefore

$$|(Bx)(t) - (Bx)(t')| \quad \leq \quad \int_{t'}^{t} \left( |\eta(s)| \sum_1^n \psi_i\left(L\right) + |g(s,0,\dots,0)|\right) ds.$$

Taking into account assumption $(H_2) - (i)$ and using the dominated convergence theorem we infer that $B(x) \in C(J,\mathbb{R})$, moreover we get

$$|(Bx)(t) - (Bx)(t')| \quad \leq \quad \left( \|\eta(\cdot)\| \sum_1^n \psi_i\left(L\right) + \|g(\cdot,0,\dots,0)\|\right) |t-t'|. \tag{3.5}$$

Similarly, by using assumption $(H_3)$ we can obtain

$$|(Cx)(t) - (Cx)(t')| \quad \leq \quad C_h|t-t'| + \|\gamma(\cdot)\| \sum_1^n \theta_i\left(M^i|t-t'|\right), \tag{3.6}$$

which implies $C(x) \in C(J,\mathbb{R})$. □

**Lemma 3.4** *The operators $A, B$ and $C$ are $\mathcal{D}$-Lipschitzian on $C(L,M)$.*

**Proof** Let $x,y \in C(L,M)$ and let $t \in J$. The use of assumption $(H_1) - (ii)$ leads to

$$\begin{aligned} |(Ax)(t) - (Ay)(t)| &\leq \left| f\left(t, x^{[1]}(t), x^{[2]}(t),\dots,x^{[n]}(t)\right) - f\left(t, y^{[1]}(t), y^{[2]}(t),\dots,y^{[n]}(t)\right)\right| \\ &\leq |\alpha(t)| \sum_1^n \varphi_i\left(|x^{[i]}(t) - y^{[i]}(t)|\right). \end{aligned}$$

Since $\varphi_i, i = 1, \ldots, n,$ are nondecreasing functions, passing to the supremum over $t \in J,$ we get

$$\|Ax - Ay\| \quad \leq \quad \|\alpha(\cdot)\| \sum_1^n \varphi_i \left( \|x^{[i]} - y^{[i]}\| \right).$$

Accordingly, by Lemma 2.2 we deduce that

$$\|Ax - Ay\| \quad \leq \quad \|\alpha(\cdot)\| \sum_1^n \varphi_i \left( \sum_1^{i-1} M^j \|x - y\| \right),$$

which means that $A$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function

$$\Phi(t) = \|\alpha(\cdot)\| \sum_1^n \varphi_i \left( \sum_1^{i-1} M^j t \right).$$

Now from assumption $(H_2) - (ii),$ it follows that

$$\begin{aligned}
|(Bx)(t) - (By)(t)| \quad &\leq \quad \int_0^t \left| g\left(s, x^{[1]}(s), \ldots, x^{[n]}(s)\right) - g\left(s, y^{[1]}(s), \ldots, y^{[n]}(s)\right) \right| ds \\
&+ \left| \frac{x_0}{f\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right)} - \frac{x_0}{f\left(0, y^{[1]}(0), \ldots, y^{[n]}(0)\right)} \right| \\
&+ \left| \frac{h\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right)}{f\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right)} - \frac{h\left(0, y^{[1]}(0), \ldots, y^{[n]}(0)\right)}{f\left(0, y^{[1]}(0), \ldots, y^{[n]}(0)\right)} \right| \\
&\leq \quad \int_0^t \left[ |\eta(s)| \sum_1^n \psi_i \left( |x^{[i]}(s) - y^{[i]}(s)| \right) \right] ds + A_1 + A_2,
\end{aligned}$$

where

$$A_1 = \left| \frac{x_0}{f\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right)} - \frac{x_0}{f\left(0, y^{[1]}(0), \ldots, y^{[n]}(0)\right)} \right|.$$

and

$$A_2 = \left| \frac{h\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right)}{f\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right)} - \frac{h\left(0, y^{[1]}(0), \ldots, y^{[n]}(0)\right)}{f\left(0, y^{[1]}(0), \ldots, y^{[n]}(0)\right)} \right|.$$

From assumption $(H_1),$ it follows that

$$\begin{aligned}
A_1 \quad &\leq \quad |x_0|\delta^2 \left| f\left(0, y^{[1]}(0), \ldots, y^{[n]}(0)\right) - f\left(0, x^{[1]}(0), \ldots, x^{[n]}(0)\right) \right| \\
&\leq |x_0|\delta^2 |\alpha(0)| \sum_1^n \varphi_i \left( |x^{[i]}(0) - y^{[i]}(0)| \right).
\end{aligned}$$

Similarly, by assumptions $(H_1)$ and $(H_3)$, we obtain

$$\begin{aligned} A_2 &\leq \frac{\left|h\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)-h\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)\right|}{\left|f\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)\right|} \\ &\quad +\delta^2\left|h\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)\right|\left|f\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)-f\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)\right| \\ &\leq \delta\left|h\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)-h\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)\right| \\ &\quad +\delta^2\left|h\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)\right|\left|f\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)-f\left(0,x^{[1]}(0),\ldots,x^{[n]}(0)\right)\right| \\ &\leq |\gamma(0)|\delta\sum_1^n\theta_i\left(|x^{[i]}(0)-y^{[i]}(0)|\right)+|\alpha(0)|\delta^2\left|h\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)\right|\sum_1^n\varphi_i\left(|x^{[i]}(0)-y^{[i]}(0)|\right). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} |(Bx)(t)-(By)(t)| &\leq \int_0^t\left[|\eta(s)|\sum_1^n\psi_i\left(|x^{[i]}(s)-y^{[i]}(s)|\right)\right]ds+|\gamma(0)|\delta\sum_1^n\theta_i\left(|x^{[i]}(0)-y^{[i]}(0)|\right) \\ &\quad +|\alpha(0)|\delta^2\left(|x_0|+\left|h\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)\right|\right)\sum_1^n\varphi_i\left(|x^{[i]}(0)-y^{[i]}(0)|\right). \end{aligned}$$

Using the fact that

$$\begin{aligned} \left|h\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)\right| &\leq \left|h\left(0,y^{[1]}(0),\ldots,y^{[n]}(0)\right)-h(0,\ldots,0)\right|+|h(0,\ldots,0)| \\ &\leq |\gamma(0)|\sum_1^n\theta_i\left(|y^{[i]}(0)|\right)+|h(0,\ldots,0)| \\ &\leq |\gamma(0)|\sum_1^n\theta_i(L)+|h(0,\ldots,0)| \end{aligned}$$

and, taking into account the monotonicity of $\psi_i,\varphi_i,\theta_i, i=1,\ldots,n,$ we can obtain

$$\begin{aligned} \|(Bx)-(By)\| &\leq \|\eta(\cdot)\|\sum_1^n\psi_i\left(\|x^{[i]}-y^{[i]}\|\right)+|\gamma(0)|\delta\sum_1^n\theta_i\left(\|x^{[i]}-y^{[i]}\|\right) \\ &\quad +\kappa\sum_1^n\varphi_i\left(\|x^{[i]}-y^{[i]}\|\right), \end{aligned}$$

where $\kappa=|\alpha(0)|\delta^2\left(|x_0|+|\gamma(0)|\sum_1^n\theta_i(L)+|h(0,\ldots,0)|\right)$.

Thus in view of Lemma 2.2, we obtain that

$$\begin{aligned} \|(Bx)-(By)\| &\leq \|\eta(\cdot)\|\sum_1^n\psi_i\left(\sum_1^{i-1}M^j\|x-y\|\right)+|\gamma(0)|\delta\sum_1^n\theta_i\left(\sum_1^{i-1}M^j\|x-y\|\right) \\ &\quad +\kappa\sum_1^n\varphi_i\left(\sum_1^{i-1}M^j\|x-y\|\right). \end{aligned}$$

Then $B$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function

$$\Psi(t) = \|\eta(\cdot)\| \sum_1^n \psi_i \left( \sum_1^{i-1} M^j t \right) + |\gamma(0)| \delta \sum_1^n \theta_i \left( \sum_1^{i-1} M^j t \right) + \kappa \sum_1^n \varphi_i \left( \sum_1^{i-1} M^j t \right).$$

Proceeding as above, the operator $C$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function

$$\Xi(t) = \|\gamma(\cdot)\| \sum_1^n \theta_i \left( \sum_1^{i-1} M^j t \right).$$

□

**Lemma 3.5** *The sets* $A(C(L,M))$, $B(C(L,M))$ *and* $C(C(L,M))$ *are bounded respectively by* $\kappa_A, \kappa_B$, *and* $\kappa_C$, *where*

$$\kappa_A = \|\alpha(\cdot)\| \sum_1^n \varphi_i (L) + \|f(\cdot, 0, \ldots, 0)\|,$$

$$\kappa_B = \Psi(L) + \|g(\cdot, 0, \ldots, 0)\| + \delta \left( |x_0 - h(0, \ldots, 0)| \right)$$

*and*

$$\kappa_C = \|\gamma(\cdot)\| \sum_1^n \theta_i (L) + \|h(\cdot, 0, \ldots, 0)\|.$$

**Proof** Let $x \in C(L,M)$ and let $t \in J$. From assumption $(H_1) - (ii)$, it follows that

$$\begin{aligned} |(Ax)(t)| &\leq \left| f\left(t, x^{[1]}(t), \ldots, x^{[n]}(t)\right) - f(t, 0, \ldots, 0) \right| + |f(t, 0, \ldots, 0)| \\ &\leq |\alpha(t)| \sum_1^n \varphi_i \left( |x^{[i]}(t)| \right) + |f(t, 0, \ldots, 0)|. \end{aligned}$$

Passing to the supremum over $t \in J$, we get

$$\|Ax\| \leq \|\alpha(\cdot)\| \sum_1^n \varphi_i (L) + \|f(\cdot, 0, \ldots, 0)\|.$$

Similarly, from $(H_3) - (ii)$ we can result that

$$\|Cx\| \leq \|\gamma(\cdot)\| \sum_1^n \theta_i (L) + \|h(\cdot, 0, \ldots, 0)\|.$$

On the other hand, since $B$ is $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function $\Psi$, we have

$$\begin{aligned} \|Bx\| &\leq \|B(x) - B(0)\| + \|B(0)\| \\ &\leq \Psi(L) + \|g(\cdot, 0, \ldots, 0)\| + \delta \left( |x_0 - h(0, \ldots, 0)| \right). \end{aligned}$$

□

Therefore, we are in position to present the main result of this section.

**Theorem 3.6** *Suppose that*

$$\kappa_A\,\Psi(r)+\kappa_B\,\Phi(r)+\Xi(r)<r, r>0, \tag{3.7}$$

$$\kappa_A\left(\|\eta(\cdot)\|\sum_{1}^{n}\psi_i\,(L)+\|g(\cdot,0,\ldots,0)\|\right)+\kappa_B\left(C_f+\|\alpha(\cdot)\|\sum_{1}^{n}M^i\right)+\left(C_h+\|\gamma(\cdot)\|\sum_{1}^{n}M^i\right)\leq M, \tag{3.8}$$

*and*

$$\kappa_A\kappa_B+\kappa_C\leq L. \tag{3.9}$$

*Then (1.5) has a unique solution in $C(L,M)$.*

**Proof** By using Lemma 3.2, $x\in C(J)$ is a solution for the problem (1.5) if it satisfies the operator equation

$$Q(x):=A(x)\cdot B(x)+C(x)=x.$$

Therefore, in order to apply the Boyd-Wong Theorem, we shall prove that $Q$ is a nonlinear contraction mapping on $C(L,M)$. This will be achieved in the following steps.

*Step* 1. $A\cdot B+C$ maps $C(L,M)$ into $C(L,M)$. Let $x\in C(L,M)$ and let $t,t'\in J$. Without loss of generality, we suppose that $x\neq 0$ and $t\neq t'$. We have

$$\begin{aligned}|Q(x)(t)-(Qx)(t')| &\leq |A(x)(t)B(x)(t)-A(x)(t')B(x)(t')|+|C(x)(t)-C(x)(t')|\\ &\leq \|A(x)\||B(x)(t)-B(x)(t')|+\|B(x)\||A(x)(t)-A(x)(t')|\\ &\quad +|C(x)(t)-C(x)(t')|.\end{aligned}$$

From assumption $(H_1)$-$(ii)$, it follows that $\varphi_i(r)<r, r>0$ for all $i=1,\ldots,n$. In view of inequality (3.4) we get

$$|A(x)(t)-A(x)(t')|\leq\left(C_f+\|\alpha(\cdot)\|\sum_{1}^{n}M^i\right)|t-t'|.$$

Similarly from (3.5)-(3.6), we can obtain

$$|B(x)(t)-B(x)(t')|\leq\left[\|\eta(\cdot)\|\sum_{1}^{n}\psi_i\,(L)+\|g(\cdot,0,\ldots,0)\|\right]|t-t'|.$$

and

$$|C(x)(t)-C(x)(t')|\leq\left(C_h+\|\gamma(\cdot)\|\sum_{1}^{n}M^i\right)|t-t'|.$$

Consequently, using Lemma 3.5, we get

$$\begin{aligned}|Q(x)(t)-(Qx)(t')| &\leq |A(x)(t)B(x)(t)-A(x)(t')B(x)(t')|+|C(x)(t)-C(x)(t')|\\ &\leq |A(x)(t)B(x)(t)-A(x)(t)B(x)(t')|\\ &\quad +|A(x)(t)B(x)(t')-A(x)(t')B(x)(t')|+|C(x)(t)-C(x)(t')|\\ &\leq \|A(x)\|\left[\|\eta(\cdot)\|\sum_{1}^{n}\psi_i\left(L\right)+\|g(\cdot,0,\dots,0)\|\right]|t-t'|\\ &\quad +\left[\|B(x)\|\left(C_f+\|\alpha(\cdot)\|\sum_{1}^{n}M^i\right)+\left(C_h+\|\gamma(\cdot)\|\sum_{1}^{n}M^i\right)\right]|t-t'|.\end{aligned}$$

By Lemma 3.5, we deduce that

$$\begin{aligned}|Q(x)(t)-(Qx)(t')| &\leq \kappa_A\left[\|\eta(\cdot)\|\sum_{1}^{n}\psi_i\left(L\right)+\|g(\cdot,0,\dots,0)\|\right]|t-t'|\\ &\quad +\left[\kappa_B\left(C_f+\|\alpha(\cdot)\|\sum_{1}^{n}M^i\right)+\left(C_h+\|\gamma(\cdot)\|\sum_{1}^{n}M^i\right)\right]|t-t'|.\end{aligned}$$

So, the use of assumption (3.8) leads to

$$|Q(x)(t)-(Qx)(t')|\leq M|t-t'|.$$

Now, using Lemma 3.5 together with assumption (3.9), we get

$$\|Q(x)\|\leq\|A(x)\|\|B(x)\|+\|C(x)\|\leq\kappa_A\kappa_B+\kappa_C\leq L.$$

Therefore, $Q$ maps $C(L,M)$ into itself.
*Step* 2. $A\cdot B+C$ defines a nonlinear contraction.
Let $x,y\in C(L,M)$, since $A$, $B$ and $C$ are $D$-Lipschitzian it follows that

$$\begin{aligned}\|Q(x)-Q(y)\| &\leq \|A(x)\cdot(B(x)-B(y))\|+\|(A(x)-A(y))\cdot B(y)\|+\|C(x)-C(y)\|\\ &\leq \|A(x)\|\ \Psi(\|x-y\|)+\|B(x)\|\ \Phi(\|x-y\|)+\Xi(\|x-y\|).\end{aligned}$$

Consequently, from Lemma 3.5, we obtain

$$\|Q(x)-Q(y)\|\leq\kappa_A\Psi(\|x-y\|)+\kappa_B\Phi(\|x-y\|)+\Xi(\|x-y\|).$$

Hence, by virtue of assumption (3.7) we deduce that $A\cdot B+C$ defines a nonlinear contraction on $C(L,M)$ with $D$-function

$$\Theta(r)=\kappa_A\Psi(r)+\kappa_B\Phi(r)+\Xi(r),r>0.$$

As an application of Theorem 2.3, we conclude that $A\cdot B+C$ has a unique unique fixed point $x\in C(L,M)$, which is the solution of the functional iterative problem (1.5). □

**Remark 3.7** *Theorem 3.6 extends and generalizes several results in the literature of iterative differential equations. For example*

*(i) If* $f = 1$, $h = 0$ *and* $g(t, x_1, \ldots, x_2, x_n) = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n + f(t)$, *then Theorem 3.6 reduces to the existence results for the iterative differential problem (1.3), which has been studied in [21] via the principle of contraction mappings. Moreover, we get an extension of H.Y Zhao and J. Liu results [21] to the class of nonlinear contraction mappings.*

*(ii) In the special case when* $g(t, x_1, \ldots, x_n) = -a(t)x_1 + f(t, x_1, \ldots, x_n)$, *then Theorem 3.6 reduces to the existence results for the iterative differential problem (1.4), which has been studied by A. Bouakkaz, A. Ardjouni, and A. Djoudi in [5] under Lipschitz conditions on the functions* $g$ *and* $h$, *i.e.,* $g$ *and* $h$ *satisfy the following conditions: There exist some* $\kappa_i, c_i > 0$, $i = 1, \ldots, n$, *such that:*

$$|g(t, x_1, \ldots, x_n) - g(t, y_1, \ldots, y_n)| \leq \sum_{i=1}^{n} \kappa_i |x_i - y_i| \tag{3.10}$$

*and*

$$|h(t, x_1, \ldots, x_n) - h(t, y_1, \ldots, y_n)| \leq \sum_{i=1}^{n} c_i\, |x_i - y_i|. \tag{3.11}$$

*Theorem 3.6 generalizes and extends the results in [5], and shows that the conditions (3.10) and (3.11) can be relaxed by assuming that* $g$ *and* $h$ *satisfy, respectively,* $(H_2)$-$(ii)$ *and* $(H_3)$-$(ii)$.

*(iii) In [20], P. Zhang and X. Gong established existence results for the iterative differential problem (1.2) via Schauder's fixed point theorem.*
*If we take* $f = 1$ *and* $h = 0$ *in the iterative differential problem (1.5), it reduces to problem (1.2). Therefore, as a special case Theorem 3.6 ensures the existence and the uniqueness of a continuous solution for the iterative differential problem (1.2).*

Now, we illustrate the applicability of our Theorem 3.6 by considering the following examples of iterative differential equations.

**Example 3.8** *Consider the following iterative differential equation*

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t) - c\,x^{[2]}(t)}{f(t, x^{[1]}(t), x^{[2]}(t))}\right) = ae^{-x^{[2]}(t)}, \quad t \in J, \\ \\ x(0) = 0.1, \end{cases} \tag{3.12}$$

*where* $a, c > 0$, $f(t, x_1, x_2) = \dfrac{0.5}{1 + ae^{-0.5}t}$.

*Let us define the functions* $g, h : J \times \mathbb{R}^2 \to \mathbb{R}$ *by*

$$g(t, x_1, x_2) = ae^{-x_2} \textit{ for all } t \in J \textit{ and } x_1, x_2 \in \mathbb{R}^2$$

*and*

$$h(t,x_1,x_2)=cx_2 \text{ for all } t\in J \text{ and } x_1,x_2\in\mathbb{R}^2.$$

*By an elementary calculus we can show that the function* $f$ *satisfies the condition* $(H_1)$, *with* $C_f=\frac{1+ae^{-0.5}}{0.5}$, $\delta=2$, $\alpha(t)=\varphi_1(t)=\varphi_2(t)=0$, *the function* $g$ *satisfies the condition* $(H_2)$, *with* $\eta(t)=ae^L$, $\psi_1(t)=0$ *and* $\psi_2(t)=(1-e^{-t})$, *and* $h$ *satisfies the condition* $(H_3)$, *with* $C_h=0$, $\gamma(t)=1$, $\theta_1(t)=0$ *and* $\theta_2(t)=c\,t$.
*Applying Theorem 3.6, we obtain that (3.12) has a unique solution in* $C(L,M)$ *with* $L=M=1$, *when* $a$ *and* $c$ *are small enough.*

**Example 3.9** *Consider the following iterative differential equation*

$$\begin{cases} \frac{d}{dt}\Big(x(t)-c\,t\,\sin(x^{[2]}(t))\Big)=a\,t\,x^{[2]}(t)+a\,t\,\log(1+|x(t)|), \quad t\in J, \\ x(0)=0, \end{cases} \tag{3.13}$$

*where* $a,c>0$.

*Notice that the problem (3.13) can be reformulated into (1.5) with* $x_0=0$, $f=1$, *and the functions* $g,h:J\times\mathbb{R}^2\to\mathbb{R}$ *are defined by*

$$g(t,x_1,x_2)=a\,t\log(1+|x_1|)+a\,t\,x_2 \text{ for } t\in J \text{ and } x_1,x_2\in\mathbb{R}^2$$

*and*

$$h(t,x_1,x_2)=c\,t\,\sin(x_2) \text{ for } t\in J \text{ and } x_1,x_2\in\mathbb{R}^2.$$

*By an elementary calculus we can show that the function* $f=1$ *satisfies the condition* $(H_1)$, *with* $C_f=0$, $\delta=1$, $\alpha(t)=\varphi_1(t)=\varphi_2(t)=0$, *the function* $g$ *satisfies the condition* $(H_2)$, *with* $\eta(t)=at$, $\psi_1(t)=\log(1+t)$ *and* $\psi_2(t)=t$, *and* $h$ *satisfies the condition* $(H_3)$, *with* $C_h=0$, $\gamma(t)=t$, $\theta_1(t)=0$ *and* $\theta_2(t)=c\,t$.
*Applying Theorem 3.6, we obtain that (3.12) has a unique solution in* $C(L,M)$ *with* $L=M=1$, *when* $a$ *and* $c$ *are small enough.*

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