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SELİM ÇETİN
seliimcetin@gmail.com

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Generalizations of Zassenhaus lemma and Jordan-Hölder theorem for $2-$crossed modules

Selim ÇETİN[*]
Department of Mathematics, Faculty of Arts and Humanities,
Burdur Mehmet Akif Ersoy University, Burdur, Turkiye

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Abstract: We present a comprehensive generalization of the Zassenhaus Lemma, the Scherier Refinement Theorem, and the Jordan-Hölder Theorem, extending their applicability to both crossed modules and $2-$crossed modules. It is discovered that the previously established normality conditions are insufficient for forming quotient objects of $2-$subcrossed modules, as demonstrated through an illustrative example, and these conditions are rigorously revised to allow these generalizations. These new conditions now yield results that are categorically accurate. Additionally, the study has led to the derivation of several supplementary results, including isomorphism theorems for $2-$crossed modules.

Key words: Crossed module, normality, Zassenhaus lemma, isomorphism theorems

1. Introduction and preliminaries

Crossed modules, in conjunction with their topological background, constitute structures that are algebraically widespread and significant. One of the most notable reasons for their algebraic importance is the fact that any group G can be analyzed through the crossed modules (T, G, ∂) that can be established on it. This situation has some similarities with the intimate connection between modules and rings, as it was stated in [22] that to study rings without some reference to modules is inconceivable.

First introduced by Whitehead [23, 24], crossed modules are structures that model the homotopy 2-type, just as homotopy 1-types can be classified by groups. While crossed modules can be studied within the framework of homotopy theory, which is the origin of the concept, they can also be studied with categorical or pure algebraic approaches. In fact, Lue and Norrie have addressed crossed modules from the axiomatic algebraic perspective and have shown in their studies that many concepts in group theory surprisingly correspond to similar concepts in crossed modules, which have sufficiently well-behaved properties [13, 19]. This made it possible to develop an interesting and comprehensive algebraic theory of crossed modules, which directly generalizes the theory of groups in many aspects.

Crossed modules, which are one of the best-known structures of higher-dimensional group theory, among others [1, 2, 4–10, 14–17] and which have many naturally equivalent structures as Cat-1 groups, simplicial groups, strict $2-$groups, have many equivalent definitions using different mathematical tools. For the purpose of this study, we prefer to give the axiomatic definition, which is the most common in the literature.

*Correspondence: seliimcetin@gmail.com
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A crossed module is a triple (T,G,∂), where T is a $G-$group (i.e. the group G acts on the group T by automorphisms) and $\partial : T \longrightarrow G$ is a $G-$equivariant group homomorphism satisfying the Peiffer rule. More explicitly ∂ fulfills the properties

CM1) $\partial(g \triangleright t) = g \partial t g^{-1}$

CM2) $\partial(s) \triangleright t = sts^{-1}$

for all $s,t \in T$ and $g \in G$, where $g \triangleright t$ stands for the action of the element g on t.

Crossed modules generalize both the notion of a group and the notion of a normal subgroup of a group. Indeed any group G can naturally be identified with both the crossed modules $(1,G,inc)$ and (G,G,id), where in the latter the crossed module G acts on itself by conjugation, and these identifications specify two fully faithful functors from the category of groups $\mathfrak{Gr}$ to the category of crossed modules $\mathfrak{XMod}$. On the other hand, any normal subgroup N of a group G naturally gives rise to the crossed module (N,G,inc), where G acts on N by conjugation, and as a crossed module (T,G,∂) is sometimes called a $G-$crossed module, a normal subgroup of G defines a $G-$crossed module, making it possible to interpret $G-$crossed modules as generalizations of normal subgroups of G.

In internal viewpoint, it is also possible to define subobjects and subobjects normal subobjects in $\mathfrak{XMod}$. Consider two crossed modules (T,G,∂) and (S,H,∂'). We call (S,H,∂') a subcrossed module of (T,G,∂) if S and H are subgroups of T and G, respectively, ∂' is the restriction of d and also the action of H on S is inherited from the action of G on T. For practical purposes, we often denote also the restriction of ∂ by ∂ again. The situation, where (S,H,∂) is a subcrossed module of (T,G,∂) is denoted by $(S,H,\partial) \leq (T,G,\partial)$. In this case, the normality of (S,H,∂) in (T,G,∂) is defined by the conditions

i) H is a normal subgroup of G,

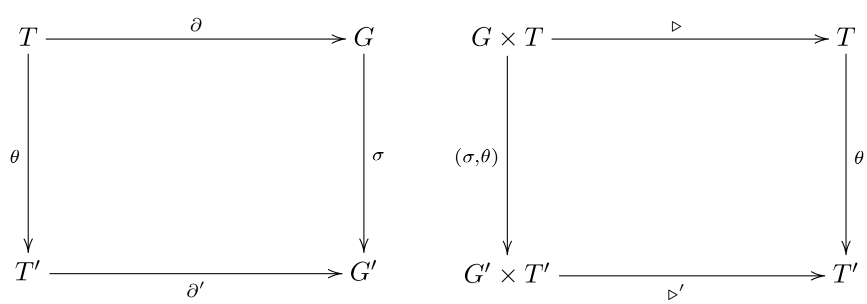
ii) $g \triangleright s \in S$ for all $g \in g$ and $s \in S$,

iii) $(h \triangleright t)t^{-1} \in S$ for all $h \in H$ and $t \in T$,

and denoted by $(S,H,\partial) \trianglelefteq (T,G,\partial)$.

If $(S,H,\partial) \trianglelefteq (T,G,\partial)$, then by definition we have $H \trianglelefteq G$. Additionally, it is easly seen that also $S \trianglelefteq T$. Thus, it is possible to form a crossed module $\left(T/S, G/H, \overline{\partial}\right)$, where $\overline{\partial}(tS) = \partial(t)H$ and G/H acts on T/S by the formula $(gH) \triangleright (tS) = (g \triangleright t)S$. This is called the quotient crossed module of (T,G,∂) by (S,H,∂) and is denoted by $\frac{(T,G,\partial)}{(S,H,\partial)}$.

A crossed module homomorphism from a crossed module (T,G,∂) to a crossed module (T',G',∂') is a pair (θ,σ) of group homomorphisms $\theta : T \longrightarrow T'$, $\sigma : G \longrightarrow G'$, such that $\sigma\partial = \partial'\theta$ and $\theta(g \triangleright t) = \sigma(g) \triangleright \theta(t)$ for all $g \in G$ and $t \in T$. Therefore, a crossed module homomorphism can be summarized by the following commutative diagrams.



Such a crossed module homomorphism defines a normal subcrossed module $(\ker\theta, \ker\sigma, \partial)$ of (T, G, ∂) and a subcrossed module $(\operatorname{im}\theta, \operatorname{im}\sigma, \partial')$ of (T', G', ∂'). While the former is called the kernel of the crossed module homomorphism (θ, σ), the latter is called the image of (θ, σ).

It is worth noting that there is a close relationship between the concepts of a normal subcrossed module and the kernel of a crossed module homomorphism. Namely, as we stated in the previous paragraph, the kernel of a crossed module homomorphism $(\theta, \sigma) : (T, G, \partial) \longrightarrow (T', G', \partial')$ is a normal subcrossed module of (T, G, ∂) and conversely, any normal subcrossed module (S, H, ∂) of (T, G, ∂) gives rise to a crossed module homomorphism $(\theta_q, \sigma_q) : (T, G, \partial) \longrightarrow \left(T/S, G/H, \overline{\partial}\right)$ defined as $\theta_q(t) = tS$, $\sigma_q(g) = gH$, whose kernel is equal to (S, H, ∂).

As a significant result of the discussion above, it can be stated that normal subcrossed modules of a crossed module (T, G, ∂) are exactly the kernels of crossed module homomorphisms originating from (T, G, ∂). This provides us an important criterion for defining normal objects. Extending this approach from the category of crossed modules to the category of $2-$crossed modules is the reason why we need to revise the definition of the normal $2-$subcrossed module in the literature. Indeed, a counterexample is also presented in this article to illustrate that the present definition may not always be sufficient to construct a well-defined quotient $2-$crossed module. For more details, see Example 2.

In the context of high-dimensional group theory $2-$crossed modules, which are first introduced by Conduché in [3], model $3-$groups, just as crossed modules model $2-$groups [18]. Some fundamental definitions regarding $2-$crossed modules are given in Section 3. Despite their seemingly complex structure, it is quite interesting to encounter examples of $2-$crossed modules in natural sciences, for instance in axion electrodynamics in the framework of particle physics [11].

Isomorphism theorems are well-known major results of group theory. As crossed modules generalize the group theory, the question arises whether if one can generalize isomorphism theorems to crossed modules. The answer is positive, and in fact the isomorphism theorems for crossed modules have really been well established and found their places in the literature. Relevant results can be found collectively in [19]. One way to further generalize these theorems is to investigate the validity of the results for $2-$crossed modules. A part of the current work involves identifying the structures that correctly describe the quotient objects of the $2-$crossed modules, and then to prove the generalizations of the isomorphism theorems.

Some of the important mathematical results that are somewhat related to isomorphism theorems are the Zassenhaus lemma, the Schreier refinement theorem, and the Jordan–Hölder theorem. The Zassenhaus lemma is also known as the butterfly lemma in the literature, and this nomenclature is based on the shape of the Hasse diagram for the lattice of subgroups involved in the lemma, given by Lang [12]. The Schreier refinement theorem is an important result on refinements of subnormal series of subgroups of a group, whose proof depends on the Zassenhaus lemma. The Jordan-Hölder theorem, on the other hand, is an important result that can be qualified as a fundamental theorem for the composition series of groups. Some of the results of this article are generalizations of the above lemmas and theorems, which play an important role in group theory, both in the theory of crossed modules and in the theory of $2-$crossed modules.

2. The case of crossed modules

Lemma 1 *Consider two subcrossed modules* (S,H,∂) *and* (R,K,∂) *of a crossed module* (T,G,∂) *and* (P,N,∂)*, and let* (Q,M,∂) *be normal subcrossed modules of* (S,H,∂) *and* (R,K,∂)*, respectively. Then*

$$(P,N,\partial)\big((S,H,\partial)\cap(Q,M,\partial)\big)$$

is a normal subcrossed module of

$$(P,N,\partial)\big((S,H,\partial)\cap(R,K,\partial)\big)$$

and

$$(Q,M,\partial)\big((P,N,\partial)\cap(R,K,\partial)\big)$$

is a normal subcrossed module of

$$(Q,M,\partial)\big((S,H,\partial)\cap(R,K,\partial)\big).$$

Moreover,

$$\frac{(P,N,\partial)\big((S,H,\partial)\cap(R,K,\partial)\big)}{(P,N,\partial)\big((S,H,\partial)\cap(Q,M,\partial)\big)}\cong\frac{(Q,M,\partial)\big((S,H,\partial)\cap(R,K,\partial)\big)}{(Q,M,\partial)\big((P,N,\partial)\cap(R,K,\partial)\big)}.$$

Proof From the definitions of the intersection of two subcrossed modules, the product of a subcrossed module and a normal subcrossed module, and the quotient crossed module, we must show that

$$\left(\frac{P(S\cap R)}{P(S\cap Q)},\frac{N(H\cap K)}{N(H\cap M)},\overline{\partial_1}\right)\cong\left(\frac{Q(S\cap R)}{Q(P\cap R)},\frac{M(H\cap K)}{M(N\cap K)},\overline{\partial_2}\right).$$

For this purpose, we consider the crossed module

$$\frac{(S,H,\partial)\cap(R,K,\partial)}{\big((P,N,\partial)\cap(R,K,\partial)\big)\big((S,H,\partial)\cap(Q,M,\partial)\big)}=\left(\frac{S\cap R}{(P\cap R)(S\cap Q)},\frac{H\cap K}{(N\cap K)(H\cap M)},\overline{\partial}\right).$$

It is known from the group theory that the functions

$$\theta:\frac{P(S\cap R)}{P(S\cap Q)}\longrightarrow\frac{S\cap R}{(P\cap R)(S\cap Q)}$$

$$\sigma:\frac{N(H\cap K)}{N(H\cap M)}\longrightarrow\frac{H\cap K}{(N\cap K)(H\cap M)}$$

given by $\theta\big(ptP(S\cap Q)\big)=t(P\cap R)(S\cap Q)$ and $\sigma\big(ngN(H\cap M)\big)=g(N\cap K)(H\cap M)$ are group isomorphisms [21].

Then for each $ptP(S\cap Q)\in\frac{P(S\cap R)}{P(S\cap Q)}$,

$$\begin{aligned}\sigma\overline{\partial_1}\big(ptP(S\cap Q)\big)&=\sigma\big(\partial(pt)N(H\cap M)\big)=\sigma\big(\partial p\partial tN(H\cap M)\big)\\&=\partial t(N\cap K)(H\cap M)=\overline{\partial}\big(t(P\cap R)(S\cap Q)\big)=\overline{\partial}\theta\big(ptP(S\cap Q)\big)\end{aligned}$$

and for each $ngN(H\cap M)\in\frac{N(H\cap K)}{N(H\cap M)}$,

$$\theta\big(ngN(H\cap M)\triangleright ptP(S\cap Q)\big)=\theta\big((ng\triangleright pt)P(S\cap Q)\big)$$

$$\begin{aligned}
&= \theta\big((ng \triangleright pt)(g \triangleright t)^{-1}(g \triangleright t)P(S\cap Q)\big)\\
&= \theta\big((ng \triangleright p)\big(n \triangleright (g \triangleright t)\big)(g \triangleright t)^{-1}(g \triangleright t)P(S\cap Q)\big)\\
&= \theta\big((p'(g \triangleright t)P(S\cap Q)\big) = (g \triangleright t)(P\cap R)(S\cap Q)\\
&= g(N\cap K)(H\cap M) \triangleright t(P\cap R)(S\cap Q)\\
&= \sigma\big(ngN(H\cap M)\big) \triangleright \theta\big((ptP(S\cap Q)\big)
\end{aligned}$$

which proves that (θ,σ) is a crossed module isomorphism.

The isomorphism $\big(\frac{Q(S\cap R)}{Q(P\cap R)}, \frac{M(H\cap K)}{M(N\cap K)}, \overline{\partial_2}\big) \cong \big(\frac{S\cap R}{(P\cap R)(S\cap Q)}, \frac{H\cap K}{(N\cap K)(H\cap M)}, \overline{\partial}\big)$ is similarly seen from the symmetry. □

Definition 1 *[19] A normal series (series of length n) of a crossed module (T,G,∂) consists of subcrossed modules (T_i,G_i,∂), $i=0,...,n$, such that, $(T_i,G_i,\partial) \trianglelefteq (T_{i-1},G_{i-1},\partial)$ for $i=1,...,n$ and*

$$\begin{array}{ccccccccc}
1=T_n & \longrightarrow & T_{n-1} & \longrightarrow & \cdots & \longrightarrow & T_1 & \longrightarrow & T_0=T\\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \partial}\\
1=G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0=G
\end{array}$$

In this case, the quotient crossed modules

$$\frac{(T_0,G_0,\partial_0)}{(T_1,G_1,\partial_1)}, \frac{(T_1,G_1,\partial_1)}{(T_2,G_2,\partial_2)}, \cdots, \frac{(T_{n-1},G_{n-1},\partial_{n-1})}{(T_n,G_n,\partial_n)}$$

are called factor crossed modules of (T,G,∂).

Definition 2 *Given two normal series*

$$\begin{array}{ccccccccc}
1=T_n & \longrightarrow & T_{n-1} & \longrightarrow & \cdots & \longrightarrow & T_1 & \longrightarrow & T_0=T\\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \partial}\\
1=G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0=G
\end{array}$$

and

$$\begin{array}{ccccccccc}
1=T_{i_m} & \longrightarrow & T_{i_{m-1}} & \longrightarrow & \cdots & \longrightarrow & T_{i_1} & \longrightarrow & T_{i_0}=T \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \partial \\
1=G_{i_m} & \longrightarrow & G_{i_{m-1}} & \longrightarrow & \cdots & \longrightarrow & G_{i_1} & \longrightarrow & G_{i_0}=G
\end{array}$$

of a crossed module (T,G,∂). *If* $i_0=0$, $i_m=n$ *and* $i_{k-1}<i_k$ *for each* $k=1,\cdots,m$, *then the former series is said to be a refinement of the latter.*

Definition 3 *A crossed module* (T,G,∂) *is called simple if its only normal subcrossed modules are* $(1,1,1)$ *and* (T,G,∂).

Definition 4 *A normal series of crossed modules, of which all nontrivial factor crossed modules are simple, is called composition series.*

Definition 5 *Two normal series of a crossed module* (T,G,∂) *are said to be equivalent if for each factor the crossed module corresponding to one of the normal series is isomorphic to some factor crossed modules in the other normal series.*

Theorem 1 (Scherier refinement theorem for crossed modules) *Consider two normal series*

$$\begin{array}{ccccccccc}
1=S_n & \longrightarrow & S_{n-1} & \longrightarrow & \cdots & \longrightarrow & S_1 & \longrightarrow & S_0=T \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \partial \\
1=H_n & \longrightarrow & H_{n-1} & \longrightarrow & \cdots & \longrightarrow & H_1 & \longrightarrow & H_0=G
\end{array}$$

and

$$\begin{array}{ccccccccc}
1=R_m & \longrightarrow & R_{m-1} & \longrightarrow & \cdots & \longrightarrow & R_1 & \longrightarrow & R_0=T \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \partial \\
1=K_m & \longrightarrow & K_{m-1} & \longrightarrow & \cdots & \longrightarrow & K_1 & \longrightarrow & K_0=G
\end{array}$$

of a crossed module (T,G,∂). *Then these normal series have equivalent refinements.*

Proof We define $T_{ij} = S_{i+1}(S_i \cap R_j)$, $G_{ij} = H_{i+1}(H_i \cap K_j)$ for all $0 \leq i \leq n-1$ and $0 \leq j \leq m$. Then $(T_{ij}, G_{ij}, \partial_{ij})$ are subcrossed modules of (T, G, ∂), since $(S_{i+1}, H_{i+1}, \partial_{i+1})$ is normal in (S_i, H_i, ∂_i) where ∂_{ij} is the restriction of ∂. Note that

$$T_{i0} = S_{i+1}(S_i \cap R_0) = S_{i+1}(S_i \cap T) = S_{i+1}S_i = S_i$$

$$T_{im} = S_{i+1}(S_i \cap R_m) = S_{i+1}(S_i \cap 1) = S_{i+1}1 = S_{i+1}$$

and $G_{i0} = H_i$, $G_{im} = H_{i+1}$, similarly. Thus, we obtain a refinement

$$\begin{array}{ccccccc}
1 = S_n = T_{(n-1)m} \rightarrowtail \cdots \rightarrowtail & T_{(n-1)0} = S_{n-1} = T_{(n-2)m} & \rightarrowtail \cdots \rightarrowtail & T_{10} = S_1 = T_{0m} & \rightarrowtail \cdots \rightarrowtail & T_{00} = S_0 = T \\
\downarrow & \downarrow & & \downarrow & & \downarrow{\scriptstyle \partial} \\
1 = H_n = G_{(n-1)m} \rightarrowtail \cdots \rightarrowtail & G_{(n-1)0} = H_{n-1} = G_{(n-2)m} & \rightarrowtail \cdots \rightarrowtail & G_{10} = H_1 = G_{0m} & \rightarrowtail \cdots \rightarrowtail & G_{00} = H_0 = G.
\end{array}$$

of the first normal series by the second. Similarly, we obtain a refinement of the second normal series by constructing the crossed module $T'_{ji} = R_{j+1}(R_j \cap S_i)$, $G'_{ji} = K_{j+1}(K_j \cap H_i)$. By Lemma 1, we have isomorphisms

$$\frac{(S_{i+1},H_{i+1},\partial_{i+1})\big((S_i,H_i,\partial_i)\cap(R_j,K_j,\partial_j)\big)}{(S_{i+1},H_{i+1},\partial_{i+1})\big((S_i,H_i,\partial_i)\cap(R_{j+1},K_{j+1},\partial_{j+1})\big)} \cong \frac{(R_{j+1},K_{j+1},\partial_{j+1})\big((R_j,K_j,\partial_j)\cap(S_i,H_i,\partial_i)\big)}{(R_{j+1},K_{j+1},\partial_{j+1})\big((R_j,K_j,\partial_j)\cap(S_{i+1},H_{i+1},\partial_{i+1})\big)}$$

as desired. □

Theorem 2 (Jordan-Hölder theorem for crossed modules) *All composition series of a crossed module are equivalent and have the same minimal length.*

Proof This is a direct consequence of the Scherier Refinement Theorem for crossed modules and the definition of a composition series. □

Example 1 *Let G be an abelian group given by a composition series of minimal length n, that is*

$$1 = G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 = G.$$

Then it is routine to check that the crossed module $(1, G, inc)$ also has a composition series of minimal length n:

$$\begin{array}{ccccccccc}
1 = 1 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 1 = 1 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle inc} \\
1 = G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 = G
\end{array}$$

However, the minimal length of a composite series of the abelian crossed module (G, G, id) *is* $2n$, *since*

$$\begin{array}{ccccccccccccc}
1 = G_n & \longrightarrow & G_{n-1} & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & G_0 = G \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\
1 = G_n & \longrightarrow & G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_1 & \longrightarrow & G_0 = G
\end{array}$$

is a composition series of (G, G, id). *Note that this is not always true for a nonabelian group* G, *since* (N, G, inc) *need not be a normal subcrossed module of* (G, G, id) *where* $N \trianglelefteq G$, *if* $[G, G]$ *is not contained in* N.

3. The case of $2-$crossed modules

Definition 6 *A* $2-$*chain complex of groups consists of groups* S, L, M *and group homomorphisms* $\partial_1 : L \longrightarrow M$, $\partial_2 : S \longrightarrow L$

$$S \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M$$

such that $im\partial_2 \subseteq ker\partial_1$. *If in addition* $im\partial_2 \trianglelefteq L$ *and* $im\partial_1 \trianglelefteq M$, *then it is called a normal* $2-$*complex of groups.*

Definition 7 *A normal* $2-$*complex of groups*

$$S \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M$$

with a mapping

$$\{\ ,\ \} : L \times L \longrightarrow S$$

and the action of M *on* S, L, *and* M, *where the action of* M *on itself is given by conjugation, is called a* $2-$*crossed module if the following conditions hold, where* $m \triangleright l$ *and* $m \triangleright s$ *denote action of* $m \in M$ *on* $l \in L$ *and* $s \in S$, *respectively, and* $l \triangleright s$ *is defined to be equal to* $\{\partial_2 s,\, l\} s$.

1. ∂_1 *and* ∂_2 *are* $M-$*equivariant, that is*
   - *i)* $\partial_1(m \triangleright l) = m \triangleright \partial_1(l)$
   - *ii)* $\partial_2(m \triangleright s) = m \triangleright \partial_2(s)$
2. $\partial_2\{l_1,\, l_2\} = \big(\partial_1 l_1 \triangleright l_2\big)\big(l_1 {l_2}^{-1} {l_1}^{-1}\big)$
3. $\{\partial_2 s_1,\, \partial_2 s_2\} = [s_2,\, s_1]$
4. *i)* $\{l_1 l_2,\, l\} = \big(\partial_1 l_1 \triangleright \{l_2,\, l\}\big)\{l_1,\, l_2 l {l_2}^{-1}\}$

   ii) $\{l,\, l_1 l_2\} = \{l,\, l_1\}\big((l l_1 l^{-1}) \triangleright \{l,\, l_2\}\big)$

5. $\{l,\, \partial_2 s\} = (\partial_1 l \triangleright s)(l \triangleright s^{-1})$

6. $m \triangleright \{l_1,\, l_2\} = \{m \triangleright l_1,\, m \triangleright l_2\}$

for all $s, s_1, s_2 \in S$, $l, l_1, l_2 \in L$ *and* $m \in M$.

¶A $2-$crossed module

$$S \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M$$

is sometimes denoted by $(S, L, M, \partial_2, \partial_1)$ or (S, L, M) shortly, when no risk of confusion is present.

Here the curly bracket $\{\ ,\ \}$ is named Peiffer lifting.

As shown in [3], the notation $l \triangleright s$ introduced in the definition gives an action of L on S, keeping in mind that $\{\partial_2 s,\, l\}^{-1} = \partial_2 s \triangleright \{\partial_2 s^{-1},\, l\}$. On the other hand, L also acts on S via ∂_1. Note that the general

$$l \triangleright s \neq \partial_1 l \triangleright s$$

where the action on the left hand side is the action given by $\{\partial_2 s,\, l\}s$ and the one on the right hand side is derived from the action of M on S via ∂_1. Although we use the same notation for both actions, this will not cause confusion since the context clarifies it.

It is easy to verify that the condition (5) can be given equivalently

$$(5')\ \{l,\, \partial_2 s\}\{\partial_2 s,\, l\} = (\partial_1 l \triangleright s)s^{-1}$$

using the definition of the action of L on S.

Given a $2-$crossed module

$$S \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M$$

we have a crossed module

$$S \xrightarrow{\partial_2} L$$

with the action $l \triangleright s = \{\partial_2 s,\, l\}s$ and two precrossed modules

$$L \xrightarrow{\partial_1} M$$

and

$$S \xrightarrow{\partial_1 \partial_2} M$$

where the latter has trivial boundary map $\partial_1 \partial_2 = 1_M$.

Definition 8 *Consider two* $2-$ *crossed modules*

$$S \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M$$

and

$$T \xrightarrow{\partial_2'} K \xrightarrow{\partial_1'} N.$$

A homomorphism

$$\Phi : (S, L, M, \partial_2, \partial_1) \longrightarrow (T, K, N, \partial_2', \partial_1')$$

is a triple $\Phi = (\alpha, \beta, \gamma)$ *of group homomorphisms*

$$\alpha : S \longrightarrow T$$

$$\beta : L \longrightarrow K$$

$$\gamma : M \longrightarrow N$$

making the diagram

$$\begin{array}{ccccc} S & \xrightarrow{\partial_2} & L & \xrightarrow{\partial_1} & M \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ T & \xrightarrow[\partial_2']{} & K & \xrightarrow[\partial_1']{} & N \end{array}$$

commuting $(\beta\partial_2 = \partial_2'\alpha,\ \gamma\partial_1 = \partial_1'\beta)$ *and having the properties*

$$\alpha(m \triangleright s) = \gamma(m) \triangleright \alpha(s)$$

$$\beta(m \triangleright l) = \gamma(m) \triangleright \beta(l)$$

$$\alpha\{l, l'\} = \{\beta(l), \beta(l')\}.$$

If each of α *,* β *, and* γ *is an isomorphism of groups, then* Φ *is said to be a* $2-$ *crossed module isomorphism.*

¶A homomorphism

$$\Phi : (S, L, M, \partial_2, \partial_1) \longrightarrow (T, K, N, \partial_2', \partial_1')$$

also preserves both actions of L on S since

$$\alpha(l \triangleright s) = \alpha\big(\{\partial_2 s, l\}s\big) = \{\beta\partial_2 s, \beta l\}\alpha s = \{\partial_2'\alpha s, \beta l\}\alpha s = \beta(l) \triangleright \alpha(s)$$

and

$$\alpha(\partial_1 l \triangleright s) = \gamma\partial_1 l \triangleright \alpha s = \partial_1'\beta(l) \triangleright \alpha(s).$$

Also, if $\Phi = (\alpha, \beta, \gamma)$ is a $2-$crossed module homomorphism, then

$$(\alpha, \beta) : (S, L, \partial_2) \longrightarrow (T, K, \partial_2')$$

is a crossed module homomorphism.

Definition 9 *Let*

$$S \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M$$

be a $2-$*crossed module. If also*

$$T \xrightarrow{\partial_2'} K \xrightarrow{\partial_1'} N$$

is a $2-$*crossed module with* $T \subseteq S$, $K \subseteq L$, $N \subseteq M$ *with* $\partial_1' = \partial_1|_K$, $\partial_2' = \partial_2|_T$ *and restricted actions and Peiffer lifting, then* $(T, K, N, \partial_2', \partial_1')$ *is said to be a* $2-$*subcrossed module of* $(S, L, M, \partial_2, \partial_1)$*. This case is denoted by* $(T, K, N) \leq (S, L, M)$*.*

Definition 10 *[20] Consider a* $2-$*subcrossed module* (T, K, N) *of a* $2-$*crossed module*

$$S \xrightarrow{\partial_2} L \xrightarrow{\partial_1} M.$$

If

1. $K \trianglelefteq L$ *and* $N \trianglelefteq M$
2. $m \triangleright k \in K$, $m \triangleright t \in T$
3. $(n \triangleright l)l^{-1} \in K$, $(n \triangleright s)s^{-1} \in T$
4. $\{k, l\}, \{l, k\} \in T$

for all $s \in S$, $l \in L$, $m \in M$, $t \in T$, $k \in K$, *and* $n \in N$, *then* (T, K, N) *is said to be a normal* $2-$*subcrossed module of* (S, L, M) *and this is denoted by* $(T, K, N) \trianglelefteq (S, L, M)$*.*

Note that, from the condition (4), we also have $l \triangleright t \in T$ and $(k \triangleright s)s^{-1} \in T$ for $l \in L$, $t \in T$, $k \in K$ and $s \in S$. Indeed, if (T, K, N) is a normal $2-$subcrossed module of $(S, L, M, \partial_2, \partial_1)$ then for all $l \in L$, $t \in T$, $k \in K$ and $s \in S$, we have

$$l \triangleright t = \{\partial t, l\}t \in T,$$

$$(k \triangleright s)s^{-1} = \{\partial s, k\}ss^{-1} = \{\partial s, k\} \in T$$

and also

$$\partial_1 l \triangleright t \in T, \ (\partial_1 k \triangleright s)s^{-1} \in T.$$

In addition, $T \trianglelefteq S$ since

$$sts^{-1} = \partial_2(s) \triangleright t \in T.$$

As a consequence of these arguments, (T, K) is a normal subcrossed module of (S, L), if (T, K, N) is a normal $2-$subcrossed module of $(S, L, M, \partial_2, \partial_1)$.

Note that the definition of normality given here is somewhat different from the one given in [20]. In addition to the definition given there, we add conditions on Peiffer lifting of a quotient $2-$crossed module. See Lemma 2 and Example 2.

Definition 11 *[20] Consider a* $2-$ *crossed module homomorphism*

$$\Phi = (\alpha, \beta, \gamma) : (S, L, M, \partial_2, \partial_1) \longrightarrow (T, K, N, \partial_2', \partial_1').$$

The kernel of Φ *is defined by* $ker\Phi = (ker\alpha, ker\beta, ker\gamma)$ *and the image of* Φ *is defined by* $im\Phi = (im\alpha, im\beta, im\gamma)$*.*

It is known that $\ker\Phi$ is a normal $2-$subcrossed module of (S, L, M) in the sense of the definition given in [20], and also $\operatorname{im}\Phi$ is a $2-$subcrossed module of (T, K, N). The normality of $\ker\Phi$ in the sense of the definition given here is also seen by showing that

$$\{k, l\}, \{l, k\} \in \ker\alpha$$

for all $k \in \ker\beta$ and $l \in L$. Indeed,

$$\begin{aligned} \alpha\{k, l\} &= \{\beta(k), \beta(l)\} = \{1, \beta(l)\} \\ &= \{\partial_2(1), \beta(l)\}1 = l \triangleright 1 = 1, \end{aligned}$$

$$\begin{aligned} \alpha\{l, k\} &= \{\beta(l), \beta(k)\} = \{\beta(l), 1\}\{\beta(l), \partial_2(1)\} \\ &= \left(\partial_1'\beta(l) \triangleright 1\right)\left(\beta(l) \triangleright 1^{-1}\right) = 1 \cdot 1 = 1 \end{aligned}$$

gives $\{k, l\}$, $\{l, k\} \in \ker\alpha$.

Definition 12 *[20] Let* $(S, L, M, \partial_2, \partial_1)$ *be a* $2-$ *crossed module and* (T, K, N) *be a normal* $2-$ *subcrossed module of* (S, L, M)*. Then the quotient* $2-$ *crossed module* $\frac{(S,L,M)}{(T,K,N)}$ *is given by the quintuble* $\left(\frac{S}{T}, \frac{L}{K}, \frac{M}{N}, \overline{\partial_2}, \overline{\partial_1}\right)$ *where* $\overline{\partial_2}$*,* $\overline{\partial_1}$ *and Peiffer lifting are given by*

$$\begin{aligned} \overline{\partial_2}(sT) &= \partial_2(s)K \\ \overline{\partial_1}(lK) &= \partial_1(l)N \\ \{l_1K, l_2K\} &= \{l_1, l_2\}T, \end{aligned}$$

and the actions of $\frac{M}{N}$ *on* $\frac{S}{T}$ *and* $\frac{L}{K}$ *are given by*

$$\begin{aligned} mN \triangleright sT &= (m \triangleright s)T \\ mN \triangleright lK &= (m \triangleright l)K. \end{aligned}$$

Proposition 1 *Let* $(S, L, M, \partial_2, \partial_1)$ *be a* $2-$ *crossed module and* (T, K, N) *be a normal* $2-$ *subcrossed module of* (S, L, M)*. Then, the triple* $\boldsymbol{Q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ *given by* $\mathbf{q}_1(s) = sT$*,* $\mathbf{q}_2(l) = lK$*,* $\mathbf{q}_3(m) = mN$ *for all* $s \in S$*,* $l \in L$ *and* $m \in M$ *gives a* $2-$ *crossed module homomorphism from* $(S, L, M, \partial_2, \partial_1)$ *to* $\left(\frac{S}{T}, \frac{L}{K}, \frac{M}{N}, \overline{\partial_2}, \overline{\partial_1}\right)$ *and kernel of* $\boldsymbol{Q}$ *is equal to* (T, K, N)*.*

Proof It is known that $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ are group homomorphisms, namely canonical mappings. The other conditions are easily seen from the following equations, for all $s \in S$, $l \in L$, $m \in M$.

$$\overline{\partial_2}\mathbf{q}_1(s) = \overline{\partial_2}(sT) = \partial_2(s)K = \mathbf{q}_2\partial_2(s)$$

$$\overline{\partial_1}\mathbf{q}_2(l) = \overline{\partial_1}(lK) = \partial_1(l)N = \mathbf{q}_3\partial_1(l)$$

$$\{\mathbf{q}_2(l_1), \mathbf{q}_2(l_2)\} = \{l_1K, l_2K\} = \{l_1, l_2\}T = \mathbf{q}_1\big(\{l_1, l_2\}\big)$$

$$\mathbf{q}_3(m) \triangleright \mathbf{q}_1(s) = mN \triangleright sT = (m \triangleright s)T = \mathbf{q}_1(m \triangleright s)$$

$$\mathbf{q}_3(m) \triangleright \mathbf{q}_2(l) = mN \triangleright lK = (m \triangleright l)K = \mathbf{q}_2(m \triangleright l).$$

As a result of group theoretical identities, we also have

$$\ker \mathbf{Q} = (\ker \mathbf{q}_1, \ker \mathbf{q}_2, \ker \mathbf{q}_3) = (T, K, N).$$

□

The additional requirements on normality that $\{k, l\}$, $\{l, k\} \in T$ for $k \in K$, $l \in L$ are needed to ensure the well-definedness of quotient Peiffer lifting. In fact, these are necessary and sufficient conditions for the Peiffer lifting to be well-defined as seen in the following Lemma.

Lemma 2 *Consider a $2-$crossed module $(S, L, M, \partial_2, \partial_1)$ and a $2-$subcrossed module (T, K, N) of (S, L, M), which satisfies all conditions of normality definition in [20]. Then the Peiffer lifting of the quotient is well-defined if and only if $\{k, l\}$, $\{l, k\} \in T$ for all $k \in K$, $l \in L$.*

Proof Let $\{k, l\}$, $\{l, k\} \in T$ for all $k \in K$ and $l \in L$. Assume that $l_1K = l_2K$ in L/K. Then $l_1{l_2}^{-1} \in K$. Observing that

$$\begin{aligned}
\{l_2, l\}\{l_1, l\}^{-1} &= \{l_2, l\}\{l_1l_2^{-1}l_2, l\}^{-1} \\
&= \{l_2, l\}\Big[\big(\partial_1(l_1l_2^{-1}) \triangleright \{l_2, l\}\big)\{l_1l_2^{-1}, l_2ll_2^{-1}\}\Big]^{-1} \\
&= \{l_2, l\}\{l_1l_2^{-1}, l_2ll_2^{-1}\}^{-1}\big(\partial_1(l_1l_2^{-1}) \triangleright \{l_2, l\}^{-1}\big) \\
&= \{l_2, l\}\{l_1l_2^{-1}, l_2ll_2^{-1}\}^{-1}\Big(\big(\partial_1(l_1l_2^{-1}) \triangleright \{l_2, l\}^{-1}\big)\{l_2, l\}\Big)\{l_2, l\}
\end{aligned}$$

we see $\{l_2, l\}\{l_1, l\}^{-1} \in T$, hence $\{l_1K, lK\} = \{l_2K, lK\}$ and from

$$\begin{aligned}
\{l, l_1\}^{-1}\{l, l_2\} &= \big((ll_1l^{-1}) \triangleright \{l, l_1^{-1}\}\big)\{l, l_2\} \\
&= (ll_1l^{-1}) \triangleright \Big(\{l, l_1^{-1}\}\big((ll_1^{-1}l^{-1}) \triangleright \{l, l_2\}\big)\Big) \\
&= (ll_1l^{-1}) \triangleright \{l, l_1^{-1}l_2\}
\end{aligned}$$

we have $\{l, l_1\}^{-1}\{l, l_2\} \in T$ so that $\{lK, l_1K\} = \{lK, l_2K\}$. For the converse, we have $\{l_1, l\}\{l_2, l\}^{-1}, \{l, l_1\}\{l, l_2\}^{-1} \in T$ if $l_1l_2^{-1} \in K$. Then for all $k \in K$ and $l \in L$, since $k = k1^{-1} \in K$,

$$\{k, l\} = \{k, l\}1 = \{k, l\}\{1, l\}^{-1} \in T,$$

$$\{l, k\} = \{l, k\}1 = \{l, k\}\{l, 1\}^{-1} \in T.$$

□

As a result of Proposition 1, Lemma 2, and the discussion following Definition 11, the following corollary is immediately seen, which further emphasizes that the given definition of normality is categorically correct.

Corollary 1 *Let $(S,L,M,\partial_2,\partial_1)$ be a $2-$crossed module and (T,K,N) be a $2-$subcrossed module of (S,L,M). The necessary and sufficient condition for (T,K,N) to be normal is that there exists a $2-$crossed module homomorphism $\Phi:(S,L,M,\partial_2,\partial_1)\longrightarrow(S',L',M',\partial_2',\partial_1')$, where $(S',L',M',\partial_2',\partial_1')$ is an arbitrary $2-$crossed module, such that $ker\Phi=(T,K,N)$.*

The following example illustrates that there are $2-$subcrossed modules that are not normal but satisfy all normality conditions except the Peiffer lifting condition. Therefore, the the Peiffer lifting condition is essential for an accurate definition of normality for $2-$subcrossed modules.

Example 2 *Consider the multiplicative groups $T=M=N=\{1\}$, $S=\{-1,1\}$, $K=\{-1,1,i,-i\}$ and $L=\{1,i,j,k,-1,-i,-j,-k\}$, as subgroups of multiplicative group of nonzero quaternions. Then $S\hookrightarrow L\rightarrow M$ and $T\hookrightarrow K\rightarrow N$ are $2-$crossed modules with the homomorphisms given as inclusions and trivial homomorphisms, all actions of $M=N$ are trivial, and both Peiffer liftings are inverse commutator, that is*

$$\{l_1,l_2\}=l_2l_1l_2^{-1}l_1^{-1}$$

$$\{k_1,k_2\}=k_2k_1k_2^{-1}k_1^{-1}.$$

Thus $l\triangleright s=\{\partial s,l\}s=l\partial sl^{-1}\partial s^{-1}s=lsl^{-1}s^{-1}s=lsl^{-1}$ and similarly $k\triangleright t=ktk^{-1}$ are actions by conjugation. It is routine to check that (S,L,M) and (T,K,N) are $2-$crossed modules and (T,K,N) is a $2-$subcrossed module of (S,L,M).

(T,K,N) satisfies the first three conditions in Definition 10. In fact,

1. *$N\trianglelefteq M$ trivially, and observing that $jij^{-1}=-i$, $kik^{-1}=-i$ in particular, guides the normality of K in L.*
2. *$1\triangleright z\in K$, $1\triangleright 1\in T$ and $l\triangleright 1\in T$ for $1\in M$, $z\in K$, $1\in T$, $l\in L$.*
3. *$(1\triangleright l)l^{-1}=1\in K$, $(1\triangleright s)s^{-1}=1\in T$, $(z\triangleright 1)1^{-1}=z1z^{-1}1=1\in T$ and $\big(z\triangleright(-1)\big)(-1)^{-1}=z(-1)z^{-1}(-1)=1\in T$ for $1\in N$, $l\in L$, $s\in S$, $z\in K$ and $1,-1\in S$.*

However, note that the condition (4) is not satisfied since

$$\{i,j\}=jij^{-1}i^{-1}=ji(-j)(-i)=jiji=(-k)(-k)=k^2=-1\notin T.$$

In this circumstance, if we try to form the quotient $\big(S/T,L/K,M/N\big)$, then

$$jK=\{j1,ji,j(-1),j(-i)\}=\{j,-k,-j,k\}$$

$$kK=\{k1,ki,k(-1),k(-i)\}=\{k,j,-k,-j\}$$

hence $jK=kK$. However,

$$\{jK,jK\}=\{j,j\}T=jjj^{-1}j^{-1}T=1T=T=\{1\}$$

$$\begin{aligned}\{jK,kK\}&=\{j,k\}T=kjk^{-1}j^{-1}T=kj(-k)(-j)T\\&=kjkjT=(-i)(-i)T=(-1)T=\{-1\}\end{aligned}$$

so that $\{jK,jK\}\neq\{jK,kK\}$, which is a matter of being not well-defined.

Theorem 3 *[20] Consider a* $2-$ *crossed module homomorphism*

$$\Phi = (\alpha, \beta, \gamma) : (S, L, M, \partial_2, \partial_1) \longrightarrow (T, K, N, \partial_2', \partial_1').$$

Then

$$\frac{(S, L, M)}{ker\,\Phi} \cong im\,\Phi.$$

The above theorem is the First Isomorphism Theorem for $2-$crossed modules, which was easily stated and proven in [20] and its proof is also valid under the definitions given here. Now, we give some preliminary definitions and propositions as a preparation for the statements of the Second and Third Isomorphism Theorems for $2-$crossed modules and some of their consequences.

Definition 13 *[20] Let* (T_1, K_1, N_1) *and* (T_2, K_2, N_2) *be two* $2-$*subcrossed modules of* $(S, L, M, \partial_2, \partial_1)$. *Then the intersection of* (T_1, K_1, N_1) *and* (T_2, K_2, N_2) *is given by* $(T_1, K_1, N_1) \cap (T_2, K_2, N_2) = (T_1 \cap T_2, K_1 \cap K_2, N_1 \cap N_2)$.

Note that the additional condition on Peiffer lifting for normality follows immediately, which makes the following result valid also under the definitions given here.

Proposition 2 *[20] Intersection of* $2-$*subcrossed modules is also a* $2-$*subcrossed module. In addition, if each* $2-$*subcrossed module is normal, then the intersection is also normal.*

Definition 14 *Let* $(S, L, M, \partial_2, \partial_1)$ *be a* $2-$*crossed module and* (T, K, N), (V, J, P) *be* $2-$*subcrossed modules of* (S, L, M). *Then the multiplication of* (T, K, N) *and* (V, J, P) *is given by* $(T, K, N)(V, J, P) := (TV, KJ, NP)$.

Proposition 3 (T, K, N) *be a normal* $2-$*subcrossed module and* (V, J, P) *be a* $2-$*subcrossed module of* $(S, L, M, \partial_2, \partial_1)$.

(i) $(T, K, N)(V, J, P)$ *is a* $2-$*subcrossed module of* (S, L, M)

(ii) If in addition (V, J, P) *is normal, then also* $(T, K, N)(V, J, P)$ *is a normal* $2-$*subcrossed module of* (S, L, M).

Proof

(i) $\partial_2(TV) \subseteq KJ$ and $\partial_1(KJ) \subseteq NP$ is clear. For $t \in T$, $k \in K$, $n \in N$, $v \in V$, $j \in J$, and $p \in P$, we have

$$\begin{aligned}(np) \triangleright (tv) &= \big((np) \triangleright t\big)\big((np) \triangleright v\big)(p \triangleright v)^{-1}(p \triangleright v) \\ &= \big((np) \triangleright t\big)\big(n \triangleright (p \triangleright v)\big)(p \triangleright v)^{-1}(p \triangleright v) \in TV\end{aligned}$$

and

$$\begin{aligned}(np) \triangleright (kj) &= \big((np) \triangleright k\big)\big((np) \triangleright j\big)(p \triangleright j)^{-1}(p \triangleright j) \\ &= \big((np) \triangleright k\big)\big(n \triangleright (p \triangleright j)\big)(p \triangleright j)^{-1}(p \triangleright j) \in KJ.\end{aligned}$$

Also, for $k_1, k_2, \in K$, $j_1, j_2, \in J$,

$$\{k_1 j_1, k_2 j_2\} = \big(\partial_1 k_1 \triangleright \{j_1, k_2 j_2\}\big)\{k_1, j_1 k_2 j_2 j_1^{-1}\}$$

$$= \Big(n \triangleright \Big(\{j_1, k_2\}\big((j_1 k_2 j_1^{-1}) \triangleright \{j_1, j_2\}\big) \Big) \Big) t$$
$$= (n \triangleright t')\big(n \triangleright (k' \triangleright v)\big) t$$
$$= t''(n \triangleright l) l^{-1} (k' \triangleright v) v^{-1} v t$$
$$= t'' t''' t'''' v t$$
$$= t'' t''' t'''' t''''' v \in TV$$

where $n = \partial_1 k$, $t = \{k_1, j_1 k_2 j_2 j_1^{-1}\}$, $t' = \{j_1, k_2\}$, $k' = j_1 k_2 j_1^{-1}$, $t'' = n \triangleright t'$, $l = k' \triangleright v$, $t''' = (n \triangleright l) l^{-1}$, $t'''' = (k' \triangleright v) v^{-1}$, $t''''' = v t v^{-1}$.

(ii) (1) $KJ \trianglelefteq L$ and $NP \trianglelefteq M$ follows from $K, J \trianglelefteq L$ and $N, P \trianglelefteq M$.

(2) $m \triangleright (kj) = (m \triangleright k)(m \triangleright j) \in KJ$ and $m \triangleright (tv) = (m \triangleright t)(m \triangleright v) \in TV$.

(3) $\big((np) \triangleright l\big) l^{-1} = \big(n \triangleright (p \triangleright l)\big)(p \triangleright l)^{-1}(p \triangleright l) l^{-1} \in KJ$ and
$\big((np) \triangleright s\big) s^{-1} = \big(n \triangleright (p \triangleright s)\big)(p \triangleright s)^{-1}(p \triangleright s) s^{-1} \in TV$.

(4) $\{kj, l\} = \big(\partial_1 k \triangleright \{j, l\}\big)\{k, jlj^{-1}\} \in VT = TV$
$\{l, kj\} = \{l, k\}\big((lkl^{-1}) \triangleright \{l, j\}\big) \in TV$.

□

Definition 15 *[20] Direct product of two* $2-$ *crossed modules* $(S, L, M, \partial_2, \partial_1)$ *and* $(S', L', M', \partial_2', \partial_1')$ *is defined as the* $2-$ *crossed module*

$$(S, L, M, \partial_2, \partial_1) \times (S', L', M', \partial_2', \partial_1') = (S \times S', L \times L', M \times M', \partial_2 \times \partial_2', \partial_1 \times \partial_1')$$

where

$$(\partial_1 \times \partial_1')(l, l') = (\partial_1 l, \partial_1' l'),$$
$$(\partial_2 \times \partial_2')(s, s') = (\partial_2 s, \partial_2' s'),$$
$$(m, m') \triangleright (s, s') = (m \triangleright s, m' \triangleright s'),$$
$$(m, m') \triangleright (l, l') = (m \triangleright l, m' \triangleright l'),$$
$$\big\{(l_1, l_1'), (l_2, l_2')\big\} = \big(\{l_1, l_2\}, \{l_1', l_2'\}\big).$$

Proposition 4 *Let* $(S, L, M, \partial_2, \partial_1)$ *be a* $2-$ *crossed module and* (T, K, N), (V, J, P) *be normal* $2-$ *subcrossed modules of* (S, L, M) *such that*

i) $(T, K, N)(V, J, P) = (S, L, M)$,

ii) $(T, K, N) \cap (V, J, P) = (1, 1, 1)$.

Then,

$$(S, L, M) \cong (T, K, N) \times (V, J, P).$$

Proof Noting that $T, V \trianglelefteq S$, $K, J \trianglelefteq L$, $N, P \trianglelefteq M$, $TV = S$, $KJ = L$, $NP = M$, $T \cap V = 1$, $K \cap J = 1$ and $N \cap P = 1$, we can conclude from the group theory that

$$\begin{aligned}
&\alpha : T \times V \longrightarrow S, \quad \alpha(t, v) = tv \\
&\beta : K \times J \longrightarrow L, \quad \beta(k, j) = kj \\
&\gamma : N \times P \longrightarrow M, \quad \gamma(n, p) = np
\end{aligned}$$

are group isomorphisms. We see that

$$\Phi = (\alpha, \beta, \gamma) : (T, K, N) \times (V, J, P) \longrightarrow (S, L, M)$$

satisfies the remaining conditions for 2−crossed module homomorphisms.

$$\begin{array}{ccccc}
T \times V & \xrightarrow{\partial_2 \times \partial_2} & K \times J & \xrightarrow{\partial_1 \times \partial_1} & N \times P \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
S & \xrightarrow{\partial_2} & L & \xrightarrow{\partial_1} & M
\end{array}$$

$$\begin{aligned}
&\beta(\partial_2 \times \partial_2)(t, v) = \beta(\partial_2 t, \partial_2 v) = \partial_2 t \partial_2 v = \partial_2(tv) = \partial_2 \alpha(t, v), \\
&\gamma(\partial_1 \times \partial_1)(k, j) = \gamma(\partial_1 k, \partial_1 j) = \partial_1 k \partial_1 j = \partial_1(kj) = \partial_1 \beta(k, j).
\end{aligned}$$

Note that from the normality, for all $p \in P$, $t \in T$ and $k \in K$, $(p \triangleright t)t^{-1} \in V$ and also $p \triangleright t \in T$ and $t^{-1} \in T$ gives $(p \triangleright t)t^{-1} \in T \cap V = \{1\}$ which implies $p \triangleright t = t$ and similarly $p \triangleright k = k$. So P acts on T and K trivially. On the other side, N acts on V and J trivially. Hence,

$$\begin{aligned}
\alpha\big((n, p) \triangleright (t, v)\big) &= \alpha(n \triangleright t, p \triangleright v) = (n \triangleright t)(p \triangleright v) = \big(n \triangleright (p \triangleright t)\big)\big(n \triangleright (p \triangleright v)\big) \\
&= (np \triangleright t)(np \triangleright v) = np \triangleright tv = \gamma(n, p) \triangleright \alpha(t, v).
\end{aligned}$$

$$\begin{aligned}
\beta\big((n, p) \triangleright (k, j)\big) &= \alpha(n \triangleright k, p \triangleright j) = (n \triangleright k)(p \triangleright j) = \big(n \triangleright (p \triangleright k)\big)\big(n \triangleright (p \triangleright j)\big) \\
&= (np \triangleright k)(np \triangleright j) = np \triangleright kj = \gamma(n, p) \triangleright \beta(k, j).
\end{aligned}$$

Let $k_1, k_2 \in K$ and $j_1, j_2 \in J$. Observe that

$$\{k_1 j_1, k_2 j_2\} = \{k_1 j_1, k_2\}\Big(\big(k_1 j_1 k_2 (k_1 j_1)^{-1}\big) \triangleright \{k_1 j_1, j_2\}\Big)$$

Now, K acts on V trivially since for $k \in K$ and $v \in V$, $k \triangleright v = \{\partial_2 v, k\}v$ and $k \in K$, $\partial_2 v \in J$ together implies that $\{\partial_2 v, k\} \in T \cap V = \{1\}$ so that $k \triangleright v = v$. Here we have $(k_1 j_1)k_2(k_1 j_1)^{-1} \in K$ and by $j_2 \in J$, $\{k_1 j_1, j_2\} \in V$. Thus

$$\begin{aligned}
\{k_1 j_1, k_2 j_2\} &= \{k_1 j_1, k_2\}\{k_1 j_1, j_2\} \\
&= \big(\partial_1 k_1 \triangleright \{j_1, k_2\}\big)\{k_1, j_1 k_2 j_1^{-1}\}\big(\partial_1 k_1 \triangleright \{j_1, j_2\}\big)\{k_1, j_1 j_2 j_1^{-1}\}.
\end{aligned}$$

Note that since $j_1 \in J$ and $k_2 \in K$, $\{j_1, k_2\} \in V \cap T = \{1\}$ and similarly $\{k_1, j_1 j_2 j_1^{-1}\} = 1$. Also, $\partial_1 k_1 \in N$ and $\{j_1, j_2\} \in V$ gives $\partial_1 k_1 \triangleright \{j_1, j_2\} = \{j_1, j_2\}$ from the triviality of the action. Lastly, $j_1 k_2 j_1^{-1} k_2^{-1} \in J \cap K = \{1\}$ so that $j_1 k_2 j_1^{-1} = k_2$. So we have

$$\{k_1 j_1, k_2 j_2\} = \{k_1, k_2\}\{j_1, j_2\}$$

which means that

$$\big\{\beta(k_1, j_1), \beta(k_2, j_2)\big\} = \alpha\big(\{k_1, k_2\}, \{j_1, j_2\}\big).$$

□

Definition 16 *If a $2-$crossed module $(S, L, M, \partial_2, \partial_1)$ and its given two normal $2-$subcrossed modules (T, K, N) and (V, J, P) satisfies the conditions*

i) $(T, K, N)(V, J, P) = (S, L, M)$,

ii) $(T, K, N) \cap (V, J, P) = (1, 1, 1)$,

then (S, L, M) is said to be internal direct product of (T, K, N) and (V, J, P).

Theorem 4 *Let $(S, L, M, \partial_2, \partial_1)$ be a $2-$crossed module, (T, K, N) be a normal $2-$subcrossed module of (S, L, M) and (V, J, P) be a $2-$subcrossed module of (S, L, M). Then,*

$$\frac{(V, J, P)}{(T, K, N) \cap (V, J, P)} \cong \frac{(T, K, N)(V, J, P)}{(T, K, N)}.$$

Proof We shall show that

$$\Big(\frac{V}{T \cap V}, \frac{J}{K \cap J}, \frac{P}{N \cap P}\Big) \cong \Big(\frac{TV}{T}, \frac{KJ}{K}, \frac{NP}{N}\Big).$$

Consider the functions

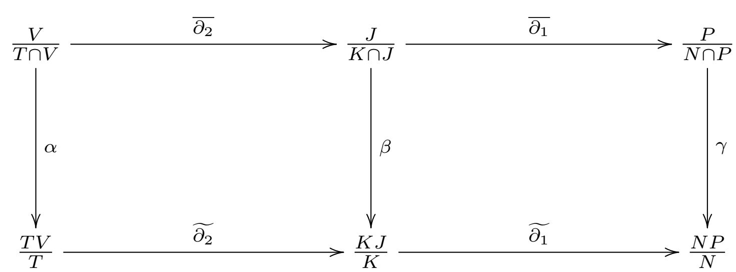
$$\alpha : \frac{V}{T \cap V} \longrightarrow \frac{TV}{T}, \quad \alpha\big(v(T \cap V)\big) = vT$$

$$\beta : \frac{J}{K \cap J} \longrightarrow \frac{KJ}{K}, \quad \beta\big(j(K \cap J)\big) = jK$$

$$\gamma : \frac{P}{N \cap P} \longrightarrow \frac{NP}{N}, \quad \gamma\big(p(N \cap P)\big) = pN.$$

It is well-known that α, β, and γ are group isomorphisms. We want to show that $\Phi = (\alpha, \beta, \gamma)$ is a $2-$crossed module homomorphism.

Consider first the diagram



For all $v \in V$, $k \in K$, and $p \in P$,

$$\beta\overline{\partial_2}\big(v(T\cap V)\big) = \beta\big(\partial_2 v(K\cap J)\big) = \partial_2 vK = \widetilde{\partial_2}(vT) = \widetilde{\partial_2}\alpha\big(v(T\cap V)\big).$$

and

$$\gamma\overline{\partial_1}\big(j(K\cap J)\big) = \gamma\big(\partial_1 j(N\cap P)\big) = \partial_1 jN = \widetilde{\partial_1}(jK) = \widetilde{\partial_1}\beta\big(j(K\cap J)\big).$$

In addition,

$$\begin{aligned}\alpha\big(p(N\cap P)\triangleright v(T\cap V)\big) &= \alpha\big((p\triangleright v)(T\cap V)\big) = (p\triangleright v)T\\ &= pN\triangleright vT = \gamma\big(p(N\cap P)\big)\triangleright\alpha\big(v(T\cap V)\big)\end{aligned}$$

and

$$\begin{aligned}\beta\big(p(N\cap P)\triangleright j(K\cap J)\big) &= \beta\big((p\triangleright j)(K\cap J)\big) = (p\triangleright j)K\\ &= pN\triangleright jK = \gamma\big(p(N\cap P)\big)\triangleright\beta\big(j(K\cap J)\big).\end{aligned}$$

Lastly, for all $j_1, j_2 \in J$, we have

$$\begin{aligned}\alpha\big\{j_1(K\cap J), j_2(K\cap J)\big\} &= \alpha\big(\{j_1,j_2\}(T\cap V)\big) = \{j_1,j_2\}T\\ &= \{j_1K, j_2K\} = \Big\{\beta\big(j_1(K\cap J)\big), \beta\big(j_2(K\cap J)\big)\Big\}.\end{aligned}$$

□

Proposition 5 *Consider a normal* $2-$ *subcrossed module* (T,K,N) *and a* $2-$ *subcrossed module* (V,J,P) *of* $(S,L,M,\partial_2,\partial_1)$ *such that* $T\subseteq V$, $K\subseteq J$ *and* $N\subseteq P$. *Then,* $\frac{(V,J,P)}{(T,K,N)}$ *is a normal* $2-$ *subcrossed module of* $\frac{(S,L,M)}{(T,K,N)}$ *if and only if* (V,J,P) *is a normal* $2-$ *subcrossed module of* (S,L,M).

Proof It is obvious that $\frac{(V,J,P)}{(T,K,N)}$ is a $2-$subcrossed module of $\frac{(S,L,M)}{(T,K,N)}$.

1. For groups, it is known that $J\trianglelefteq L \iff \frac{J}{K}\trianglelefteq\frac{L}{K}$ and $P\trianglelefteq M \iff \frac{P}{N}\trianglelefteq\frac{M}{N}$.

2. $m\triangleright j\in J \iff (m\triangleright j)K\in\frac{J}{K} \iff mN\triangleright jK\in\frac{J}{K}$
   $m\triangleright v\in V \iff (m\triangleright v)T\in\frac{V}{T} \iff mN\triangleright vT\in\frac{V}{T}$.

3. $(p\triangleright l)l^{-1}\in J \iff (p\triangleright l)l^{-1}K\in\frac{J}{K} \iff \big(pN\triangleright lK\big)(lK)^{-1}\in\frac{J}{K}$
   $(p\triangleright s)s^{-1}\in V \iff (p\triangleright s)s^{-1}T\in\frac{V}{T} \iff \big(pN\triangleright sT\big)(sT)^{-1}\in\frac{V}{T}$.

4. $\{j,l\}\in V \iff \{j,l\}T\in\frac{V}{T} \iff \{jK,lK\}\in\frac{V}{T}$
   $\{l,j\}\in V \iff \{l,j\}T\in\frac{V}{T} \iff \{lK,jK\}\in\frac{V}{T}$.

□

Theorem 5 *Let $(S, L, M, \partial_2, \partial_1)$ be a 2− crossed module, (T, K, N) and (V, J, P) be two normal 2− subcrossed modules of (S, L, M), such that $T \subseteq V$, $K \subseteq J$ and $N \subseteq P$. Then,*

$$\frac{\frac{(S,L,M)}{(T,K,N)}}{\frac{(V,J,P)}{(T,K,N)}} \cong \frac{(S, L, M)}{(V, J, P)}.$$

Proof Define α, β, and γ as

$$\alpha : \frac{S/T}{V/T} \longrightarrow \frac{S}{V}, \quad \alpha\big(sT(V/T)\big) = sV$$

$$\beta : \frac{L/K}{J/K} \longrightarrow \frac{L}{J}, \quad \beta\big(lK(J/K)\big) = lJ$$

$$\gamma : \frac{M/N}{P/N} \longrightarrow \frac{M}{P}, \quad \gamma\big(mN(P/N)\big) = mP.$$

which are known to be group isomorphisms.

Consider the diagram:

$$\begin{array}{ccccc}
\frac{S/T}{V/T} & \xrightarrow{\overline{\overline{\partial_2}}} & \frac{L/K}{J/K} & \xrightarrow{\overline{\overline{\partial_1}}} & \frac{M/N}{P/N} \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
\frac{S}{V} & \xrightarrow[\overline{\partial_2}]{} & \frac{L}{J} & \xrightarrow[\overline{\partial_1}]{} & \frac{M}{P}
\end{array}$$

For all $s \in S$, $l, l_1, l_2 \in L$ and $m \in M$,

$$\begin{aligned}
\beta\overline{\overline{\partial_2}}\big(sT(V/T)\big) &= \beta\big(\widetilde{\partial_2}(sT)J/K\big) = \beta\big(\partial_2 sK(J/K)\big) \\
&= \partial_2 sJ = \overline{\partial_2}(sV) = \overline{\partial_2}\alpha\big(sT(V/T)\big),
\end{aligned}$$

$$\begin{aligned}
\gamma\overline{\overline{\partial_1}}\big(lK(J/K)\big) &= \gamma\big(\widetilde{\partial_1}(lK)P/N\big) = \gamma\big(\partial_1 lN(P/N)\big) \\
&= \partial_1 lP = \overline{\partial_1}(lJ) = \overline{\partial_1}\beta\big(lK(J/K)\big),
\end{aligned}$$

and

$$\begin{aligned}
\alpha\big(mN(P/N) \triangleright sT(V/T)\big) &= \alpha\big((mN \triangleright sT)V/T\big) = \alpha\big((m \triangleright s)T(V/T)\big) \\
&= (m \triangleright s)V = mP \triangleright sV \\
&= \gamma\big(mN(P/N)\big) \triangleright \alpha\big(sT(V/T)\big),
\end{aligned}$$

$$\begin{aligned}
\beta\big(mN(P/N) \triangleright lK(J/K)\big) &= \beta\big((mN \triangleright lK)J/K\big) = \beta\big((m \triangleright l)K(J/K)\big) \\
&= (m \triangleright l)J = mP \triangleright lJ
\end{aligned}$$

$$= \gamma\big(mN(P/N)\big) \triangleright \beta\big(lK(J/K)\big).$$

In addition,

$$\begin{aligned}\alpha\big\{l_1K(J/K), l_2K(J/K)\big\} &= \alpha\big(\{l_1K, l_2K\}(V/T)\big) = \alpha\big(\{l_1, l_2\}T(V/T)\big)\\ &= \{l_1, l_2\}V = \{l_1J, l_2J\}\\ &= \Big\{\beta\big(l_1K(J/K)\big), \beta\big(l_2K(J/K)\big)\Big\}.\end{aligned}$$

□

Lemma 3 *Let* $(S, L, M, \partial_2, \partial_1)$ *be a* $2-$*crossed module,* (T, K, N) *and* (V, J, P) *be* $2-$*subcrossed modules of* (S, L, M)*,* (Y, H, Q) *be a normal* $2-$*subcrossed module of* (T, K, N)*, and* (Z, G, R) *be a normal* $2-$*subcrossed module of* (V, J, P)*. Then*

$$(Y, H, Q)\big((T, K, N) \cap (Z, G, R)\big)$$

is a normal $2-$*subcrossed module of*

$$(Y, H, Q)\big((T, K, N) \cap (V, J, P)\big)$$

and

$$(Z, G, R)\big((Y, H, Q) \cap (V, J, P)\big)$$

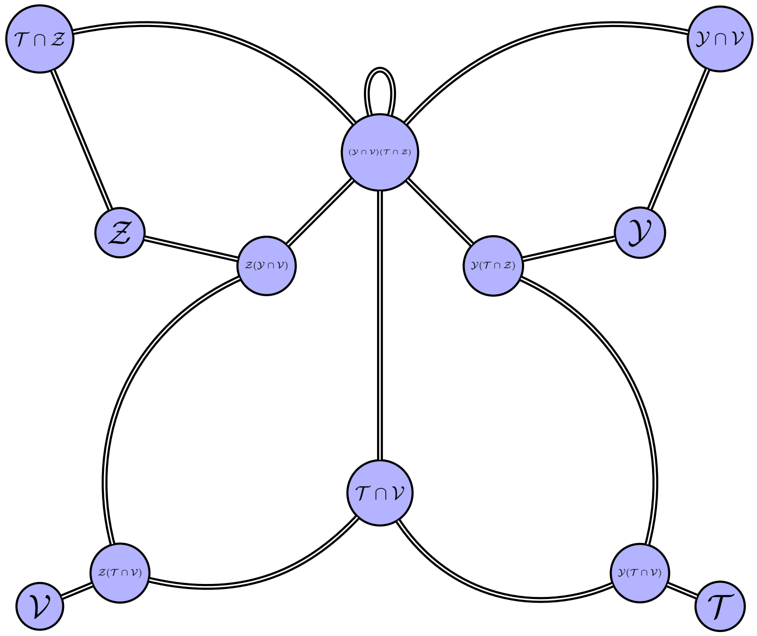
is a normal $2-$*subcrossed module of*

$$(Z, G, R)\big((T, K, N) \cap (V, J, P)\big)$$

and also,

$$\frac{(Y, H, Q)\big((T, K, N) \cap (V, J, P)\big)}{(Y, H, Q)\big((T, K, N) \cap (Z, G, R)\big)} \cong \frac{(Z, G, R)\big((T, K, N) \cap (V, J, P)\big)}{(Z, G, R)\big((Y, H, Q) \cap (V, J, P)\big)}.$$

Note that the elements of the lattice of $2-$subcrossed modules of (S,L,M), which also includes some $2-$subcrossed modules in this lemma, can be represented in the diagram below. This is why the group-theoretical analogue of this lemma is sometimes called the butterfly lemma. For a more economical use of space in the diagram, we represent each $2-$subcrossed module with a calligraphic version of the first letter involved.



Proof We consider the 2−crossed module

$$\frac{(T,K,N)\cap(V,J,P)}{\big((Y,H,Q)\cap(V,J,P)\big)\big((T,K,N)\cap(Z,G,R)\big)}\cong\Big(\frac{T\cap V}{(Y\cap V)(T\cap Z)},\frac{K\cap J}{(H\cap J)(K\cap G)},\frac{N\cap P}{(Q\cap P)(N\cap R)}\Big).$$

From the group theory, it is known that

$$\alpha:\frac{Y(T\cap V)}{Y(T\cap Z)}\longrightarrow\frac{T\cap V}{(Y\cap V)(T\cap Z)},\quad \alpha\big(ysY(T\cap Z)\big)=s(Y\cap V)(T\cap Z)$$

$$\beta:\frac{H(K\cap J)}{H(K\cap G)}\longrightarrow\frac{K\cap J}{(H\cap J)(K\cap G)},\quad \beta\big(hlH(K\cap G)\big)=l(H\cap J)(K\cap G)$$

$$\gamma:\frac{Q(N\cap P)}{Q(N\cap R)}\longrightarrow\frac{N\cap P}{(Q\cap P)(N\cap R)},\quad \gamma\big(qmQ(N\cap R)\big)=m(Q\cap P)(N\cap R).$$

are group isomorphisms [21]. Then for $ysY(T\cap Z)\in\frac{Y(T\cap V)}{Y(T\cap Z)}$

$$\begin{aligned}\beta\overline{\partial_2}\big(ysY(T\cap Z)\big)&=\beta\big(\partial_2(ys)H(K\cap G)\big)=\partial_2 s(H\cap J)(K\cap G)\\&=\widetilde{\partial}_2\big(s(Y\cap V)(T\cap Z)\big)=\widetilde{\partial}_2\alpha\big(ysY(T\cap Z)\big)\end{aligned}$$

and for $hlH(K\cap G)\in\frac{H(K\cap J)}{H(K\cap G)}$

$$\begin{aligned}\gamma\overline{\partial_1}\big(hlH(K\cap G)\big)&=\gamma\big(\partial_1(hl)Q(N\cap R)\big)=\partial_1 l(Q\cap P)(N\cap R)\\&=\widetilde{\partial}_1\big(l(H\cap J)(K\cap G)\big)=\widetilde{\partial}_1\beta\big(hlH(K\cap G)\big).\end{aligned}$$

For $ysY(T\cap Z)\in\frac{Y(T\cap V)}{Y(T\cap Z)}$, $qmQ(N\cap R)\in\frac{Q(N\cap P)}{Q(N\cap R)}$

$$\alpha\big(qmQ(N\cap R)\triangleright ysY(T\cap Z)\big)=\alpha\big((qm\triangleright ys)Y(T\cap Z)\big)$$

$$\begin{aligned}
&= \alpha\big((qm \triangleright ys)(m \triangleright s)^{-1}(m \triangleright s)Y(T\cap Z)\big)\\
&= \alpha\big((qm \triangleright y)\big(q \triangleright (m \triangleright s)\big)(m \triangleright s)^{-1}(m \triangleright s)Y(T\cap Z)\big)\\
&= \alpha\big(y'(m \triangleright s)Y(T\cap Z)\big) = (m \triangleright s)(Y\cap V)(T\cap Z)\\
&= m(Q\cap P)(N\cap R) \triangleright s(Y\cap V)(T\cap Z)\\
&= \gamma\big(qmQ(N\cap R)\big) \triangleright \alpha\big(ysY(T\cap Z)\big).
\end{aligned}$$

where $y' = (qm \triangleright y)\big(q \triangleright (m \triangleright s)\big)(m \triangleright s)^{-1}$. For $hlH(K\cap G) \in \frac{H(K\cap J)}{H(K\cap G)}$, $qmQ(N\cap R) \in \frac{Q(N\cap P)}{Q(N\cap R)}$

$$\begin{aligned}
\beta\big(qmQ(N\cap R) \triangleright hlH(K\cap G)\big) &= \beta\big((qm \triangleright hl)H(K\cap G)\big)\\
&= \beta\big((qm \triangleright hl)(m \triangleright l)^{-1}(m \triangleright l)H(K\cap G)\big)\\
&= \beta\big((qm \triangleright h)\big(q \triangleright (m \triangleright l)\big)(m \triangleright l)^{-1}(m \triangleright l)H(K\cap G)\big)\\
&= \beta\big(h'(m \triangleright l)H(K\cap G)\big) = (m \triangleright l)(H\cap J)(K\cap G)\\
&= m(Q\cap P)(N\cap R) \triangleright l(H\cap J)(K\cap G)\\
&= \gamma\big(qmQ(N\cap R)\big) \triangleright \beta\big(hlH(K\cap G)\big),
\end{aligned}$$

where $h' = (qm \triangleright h)\big(q \triangleright (m \triangleright l)\big)(m \triangleright l)^{-1}$.

For $h_1l_1H(K\cap G), h_2l_2H(K\cap G) \in \frac{H(K\cap J)}{H(K\cap G)}$ noting that

$$(h_il_i)(l_ih_i)^{-1} = h_il_ih_i^{-1}l_i^{-1} \in H \subseteq H(K\cap G)$$

we have

$$h_il_iH(K\cap G) = l_ih_iH(K\cap G) = l_iH(K\cap G)$$

and

$$\begin{aligned}
\alpha\big\{h_1l_1H(K\cap G), h_2l_2H(K\cap G)\big\} &= \alpha\big\{l_1H(K\cap G), l_2H(K\cap G)\big\}\\
&= \alpha\big(\{l_1, l_2\}Y(T\cap Z)\big) = \alpha\big(1\{l_1, l_2\}Y(T\cap Z)\big)\\
&= \{l_1, l_2\}(Y\cap V)(T\cap Z)\\
&= \big\{l_1(H\cap J)(K\cap G), l_2(H\cap J)(K\cap G)\big\}\\
&= \Big\{\beta\big(h_1l_1H(K\cap G)\big), \beta\big(h_2l_2H(K\cap G)\big)\Big\}.
\end{aligned}$$

□

Definition 17 *A normal series (of length n) of a $2-$crossed module $(S, L, M, \partial_2, \partial_1)$ consists of $2-$subcrossed modules (S_i, L_i, M_i), $i = 0, ..., n$, of (S, L, M) such that for each $i = 1, ..., n$, (S_i, L_i, M_i) is a normal*

$2-$ *subcrossed module of* $(S_{i-1}, L_{i-1}, M_{i-1})$, $(S_n, L_n, M_n) = (1,1,1)$ *and* $(S_0, L_0, M_0) = (S, L, M)$.

$$\begin{array}{ccccccccc}
1 = S_n & \longrightarrow & S_{n-1} & \longrightarrow & \cdots & \longrightarrow & S_1 & \longrightarrow & S_0 = S \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \partial_2 \\
1 = L_n & \longrightarrow & L_{n-1} & \longrightarrow & \cdots & \longrightarrow & L_1 & \longrightarrow & L_0 = L \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \partial_1 \\
1 = M_n & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0 = M
\end{array}$$

If (S_i, L_i, M_i) *is a normal series of* (S, L, M), *then the quotients*

$$\frac{(S_0, L_0, M_0)}{(S_1, L_1, M_1)}, \frac{(S_1, L_1, M_1)}{(S_2, L_2, M_2)}, \cdots, \frac{(S_{n-1}, L_{n-1}, M_{n-1})}{(S_n, L_n, M_n)}$$

are called factor $2-$ *crossed modules of* (S, L, M). *If a normal series* (S_i, L_i, M_i) *of* (S, L, M) *can be obtained by eleminating some* $2-$ *crossed modules in a normal series* (S'_i, L'_i, M'_i) *of* (S, L, M), *then* (S'_i, L'_i, M'_i) *is called a refinement of* (S_i, L_i, M_i).

Definition 18 *A* $2-$ *crossed module* $(S, L, M, \partial_2, \partial_1)$ *is called simple, if its only normal* $2-$ *subcrossed modules are* $(1,1,1)$ *and* (S, L, M).

Definition 19 *A normal series of a* $2-$ *crossed module is said to be a composition series, if all nontrivial factor* $2-$ *crossed modules are simple.*

Definition 20 *Two normal series of a* $2-$ *crossed module are said to be equivalent if for each factor* $2-$ *crossed module corresponding to one of the normal series is isomorphic to some factor* $2-$ *crossed modules in the other normal series.*

Theorem 6 (Scherier refinement theorem for $2-$crossed modules) *Let*

$$\begin{array}{ccccccccc}
1 = T_n & \longrightarrow & T_{n-1} & \longrightarrow & \cdots & \longrightarrow & T_1 & \longrightarrow & T_0 = S \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \partial_2 \\
1 = K_n & \longrightarrow & K_{n-1} & \longrightarrow & \cdots & \longrightarrow & K_1 & \longrightarrow & K_0 = L \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \partial_1 \\
1 = N_n & \longrightarrow & N_{n-1} & \longrightarrow & \cdots & \longrightarrow & N_1 & \longrightarrow & N_0 = M
\end{array}$$

and

$$\begin{array}{ccccccccc}
1=V_m & \longrightarrow & V_{m-1} & \longrightarrow & \cdots & \longrightarrow & V_1 & \longrightarrow & V_0=S \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \partial_2} \\
1=J_m & \longrightarrow & J_{m-1} & \longrightarrow & \cdots & \longrightarrow & J_1 & \longrightarrow & J_0=L \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \partial_1} \\
1=P_m & \longrightarrow & P_{m-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0=M
\end{array}$$

be two normal series of a $2-$ *crossed module* $(S,L,M,\partial_2,\partial_1)$. *Then* (T_i,K_i,N_i) *and* (V_i,J_i,M_i) *have equivalent refinements.*

Proof Define $S_{ij}=T_{i+1}(T_i\cap V_j)$, $L_{ij}=K_{i+1}(K_i\cap J_j)$ and $M_{ij}=N_{i+1}(N_i\cap P_j)$ for $0\leq i\leq n-1$ and $0\leq j\leq m$. Note that each (S_{ij},L_{ij},M_{ij}) is a $2-$subcrossed module of $(S,L,M,\partial_2,\partial_1)$ since $(T_{i+1},K_{i+1},N_{i+1})$ is normal in (T_i,K_i,N_i). Moreover,

$$S_{i0}=T_{i+1}(T_i\cap V_0)=T_{i+1}(T_i\cap S)=T_{i+1}T_i=T_i$$

and

$$S_{im}=T_{i+1}(T_i\cap V_m)=T_{i+1}(T_i\cap 1)=T_{i+1}1=T_{i+1}$$

and similarly $L_{i0}=K_i$, $M_{i0}=N_i$, $L_{im}=K_{i+1}$, $M_{im}=N_{i+1}$. Thus, we obtain a refinement of the normal series (T_i,K_i,N_i), namely

$$\begin{array}{ccccccccc}
1=T_n=S_{(n-1)m} & \to\cdots\to & S_{(n-1)0}=T_{n-1}=S_{(n-2)m} & \to\cdots\to & S_{10}=T_1=S_{0m} & \to\cdots\to & S_{00}=T_0=S \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \partial_2} \\
1=K_n=L_{(n-1)m} & \to\cdots\to & L_{(n-1)0}=K_{n-1}=L_{(n-2)m} & \to\cdots\to & L_{10}=K_1=L_{0m} & \to\cdots\to & L_{00}=K_0=L \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \partial_1} \\
1=N_n=M_{(n-1)m} & \to\cdots\to & M_{(n-1)0}=N_{n-1}=M_{(n-2)m} & \to\cdots\to & M_{10}=N_1=M_{0m} & \to\cdots\to & M_{00}=N_0=M.
\end{array}$$

In the same way, we can construct a refinement of the normal series (V_i,J_i,M_i) by defining $S'_{ij}=V_{j+1}(V_j\cap T_i)$, $L'_{ij}=J_{j+1}(J_j\cap K_i)$ and $M'_{ij}=P_{j+1}(P_j\cap N_i)$. Then by Lemmma 3, we have isomorphisms

$$\frac{(T_{i+1},K_{i+1},N_{i+1})\big((T_i,K_i,N_i)\cap(V_j,J_j,P_j)\big)}{(T_{i+1},K_{i+1},N_{i+1})\big((T_i,K_i,N_i)\cap(V_{j+1},J_{j+1},P_{j+1})\big)}\cong\frac{(V_{j+1},J_{j+1},P_{j+1})\big((V_j,J_j,P_j)\cap(T_i,K_i,N_i)\big)}{(V_{j+1},J_{j+1},P_{j+1})\big((V_j,J_j,P_j)\cap(T_{i+1},K_{i+1},N_{i+1})\big)}$$

which completes the proof. □

Considering Definition 19, the following theorem is a direct consequence of the Scherier refinement theorem for $2-$crossed modules.

Theorem 7 (Jordan-Hölder Theorem for $2-$crossed modules) *All composition series of a* $2-$ *crossed module are equivalent and have the same minimal length.*

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