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A note on hybrid hyper Leonardo numbers

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Abstract: Recently, many studies have been devoted to extending particular sets of integers to other special sets, such as complex, dual, hyperbolic, and hybrid numbers. In this study, we define a new generalization of the Leonardo sequence consisting of the hybrid hyper-Leonardo numbers. This sequence of numbers is an extension of the hyper Leonardo numbers in the hybrid set. We investigate some algebraic properties of this sequence, the generating function, and the Binet formula related to this type of sequence.

Key words: Leonardo sequence, hybrid Leonardo sequence, hybrid hyper Leonardo sequence, hybrid hyper Fibonacci sequence, the Binet formula, generating function

1. Introduction

In the last century, many researchers were interested in geometric and physical applications of sets of numbers with specific algebraic structures, such as complex, dual, hyperbolic, and hybrid numbers. These sets of numbers characterize a generalization of the entire set, thus allowing us to look at the extent of a subset of integers. Considering sets of special integers given by recurrence relations, research has shown that the relationship of recurrence, properties, and identities can be extended when we look at defining sequences in sets of complex, dual, hyperbolic numbers.

Recently, many studies have been devoted to sequences of numbers, and their generalizations and applications (see [1, 7, 10, 16]). Particularly, many studies have been devoted to sequences of hybrid numbers whose components are taken from special integer sequences, such as Fibonacci, Lucas, Pell, and Jacobsthal, (see [5, 9, 12, 13] and references therein).

Hybrid numbers have an algebraic and geometric structure. Still, especially when they are elements of recurring sequences, such as Horadam's and Leonardo's numbers, they define complex generalizations of generalized Fibonacci numbers. This new noncommutative numbering system, the hybrid numbers, was introduced by Özdemir [9] as a generalization of complex numbers, dual numbers, and hyperbolic numbers that are defined as the set

$$\mathbb{H} = \{a + bi + c\epsilon + dh | i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = -hi = \epsilon + i\}.$$

In particular, Szynal-Liana [12] introduced the hybrid numbers of Horadam, and several properties of special types of hybrid numbers were explored, [13–15]. Furthermore, Morales [8] worked on a generalization of the

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2010 *AMS Mathematics Subject Classification:* 11B39, 11B83, 11B37 hybrid numbers of $(p,q)-$ Fibonacci and $(p,q)-$ Lucas obtaining new identities between them. Another class explored was the $k-$ Pell hybrid numbers, investigated by Catarino [5]. In addition, recently Alp [2] introduced the hybrid-hyper-Fibonacci number defined by the recursive relation

$$\mathbb{H}F_n^{(r)} = F_n^{(r)} + F_{n+1}^{(r)} i + F_{n+2}^{(r)} \epsilon + F_{n+3}^{(r)} h, \tag{1.1}$$

where i, ϵ and h are hybrid units, and $F_n^{(r)}$ is the $n-$th hyper-Fibonacci number, introduced in [6] and defined by

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, F_n^{(0)} = F_n, F_0^{(r)} = 0, F_1^{(r)} = 1. \tag{1.2}$$

Similarly, several studies have been devoted to new sequences of integers and generalizations of sequences of integers. In special we are interested in the hyper-Leonardo numbers, $\{Le_n^{(r)}\}_{n\geq 0}$, $r \geq 1$, defined by the recurrence relation

$$Le_n^{(r)} = \sum_{k=0}^{n} Le_k^{(r-1)}, \tag{1.3}$$

with initial conditions given by $Le_n^{(0)} = Le_n$, $Le_0^{(r)} = 1$, $Le_1^{(r)} = r+1$, where Le_n is the $n-th$ Leonardo number recursively defined by

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \ (n \geq 2), \tag{1.4}$$

with initial conditions $Le_0 = Le_1 = 1$, and introduced by Catarino and Borges in [4] . This new generalization of the Leonardo sequence was introduced in [11]. Moreover, the authors established some algebraic properties of this sequence and also the generating function.

Motivated by the generalization given in [2], in this article, we introduce the hybrid hyper Leonardo sequence and provide some properties of this hybrid hyper Fibonacci related sequence.

The next section will be dedicated to the introduction of this new generalization of the Leonardo sequence that we shall call the hybrid hyper Leonardo sequence. Several identities, as well as the generating function and the Binet formula, will be established in the other sections of this work.

2. The hybrid hyper Leonardo numbers

In this section, we define the hybrid hyper Leonardo sequence and provide some properties of this new sequence of numbers. We start considering the following definition,

Definition 2.1 *For integers $n \geq 0$ and $r \geq 1$, the hybrid hyper Leonardo sequence is defined recursively by*

$$\mathbb{H}Le_n^{(r)} = Le_n^{(r)} + Le_{n+1}^{(r)} i + Le_{n+2}^{(r)} \epsilon + Le_{n+3}^{(r)} h, \tag{2.1}$$

where $Le_n^{(r)}$ is the $n-th$ hyper-Leonardo number, i, ϵ and h are hybrid units.

In [11] was established a recurrence formula for the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$, namely,

Proposition 2.2 (Proposition 2.2, [11]) *For $n \geq 0$ and $r \geq 0$ the hyper-Leonardo sequence $\{Le_n^{(r)}\}_{n\geq 0}$ satisfies the recurrence relation*

$$Le_n^{(r)} = r + Le_{n-1}^{(r)} + Le_{n-2}^{(r)} + \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j}, \ \textit{for } n \geq 2, \tag{2.2}$$

with $Le_0^{(r)} = 1$ and $Le_1^{(r)} = r+1$.

Consider the expression $A_n^{(r)} = \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j} + r$. By replacing the relation (2.2) in (2.1) we obtain the following recurrence relation

$$\mathbb{H}Le_n^{(r)} = \mathbb{H}Le_{n-1}^{(r)} + \mathbb{H}Le_{n-2}^{(r)} + (A_n^{(r)} + A_{n+1}^{(r)}i + A_{n+2}^{(r)}\epsilon + A_{n+3}^{(r)}h),$$

with the initial relations derived from (1.3) given by, $\mathbb{H}Le_0^{(r)} = Le_0^{(r)} + Le_1^{(r)}i + Le_2^{(r)}\epsilon + Le_3^{(r)}h$ and $\mathbb{H}Le_1^{(r)} = Le_1^{(r)} + Le_2^{(r)}i + Le_3^{(r)}\epsilon + Le_4^{(r)}h$.

Under the previous discussion, we obtain the following proposition.

Proposition 2.3 *For $n \geq 0$ and $r \geq 0$ the hybrid hyper Leonardo sequence $\{\mathbb{H}Le_n^{(r)}\}_{n\geq 0}$ satisfies the recurrence relation*

$$\mathbb{H}Le_n^{(r)} = \mathbb{H}Le_{n-1}^{(r)} + \mathbb{H}Le_{n-2}^{(r)} + (A_n^{(r)} + A_{n+1}^{(r)}i + A_{n+2}^{(r)}\epsilon + A_{n+3}^{(r)}h), \tag{2.3}$$

where $A_n^{(r)} = \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j} + r$, with the initial conditions are given by $\mathbb{H}Le_0^{(r)} = Le_0^{(r)} + Le_1^{(r)}i + Le_2^{(r)}\epsilon + Le_3^{(r)}h$ and $\mathbb{H}Le_1^{(r)} = Le_1^{(r)} + Le_2^{(r)}i + Le_3^{(r)}\epsilon + Le_4^{(r)}h$.

As a consequence, we obtain the recurrence relation for the hybrid Leonardo sequence $\{\mathbb{H}Le_n^{(0)}\}_{n\geq 0}$ in the next corollary.

Corollary 2.4 *For $n \geq 0$, the hybrid Leonardo sequence $\{\mathbb{H}Le_n\}_{n\geq 0} = \{\mathbb{H}Le_n^{(0)}\}_{n\geq 0}$ satisfies the recurrence relation*

$$\mathbb{H}Le_n = \mathbb{H}Le_{n-1} + \mathbb{H}Le_{n-2} + (1 + i + \epsilon + h), \tag{2.4}$$

with the initial conditions are given by $\mathbb{H}Le_0 = 1 + i + 3\epsilon + 5h$ and $\mathbb{H}Le_1 = 1 + 3i + 5\epsilon + 9h$.

According to Proposition 2.2 in [4], the next relation between Leonardo and Fibonacci numbers is verified,

$$Le_n = 2F_{n+1} - 1. \tag{2.5}$$

This result was extended to the hyper-Leonardo numbers by considering the hyper Fibonacci sequence $\{F_n^{(r)}\}_{n\geq 0}$,

Proposition 2.5 (Proposition 2.6, [11]) *The hyper-Leonardo numbers $\{Le_n^{(r)}\}_{n\geq 0}$ satisfy the following property, for any integers $n \geq 0$ and $r \geq 0$,*

$$Le_n^{(r)} = 2F_{n+1}^{(r)} - \binom{n+r}{r}. \tag{2.6}$$

Consider the expression $B_n^{(r)} = \binom{n+r}{r}$. The next result gives us the relationship between the hybrid hyper Leonardo and hybrid hyper Fibonacci numbers.

Proposition 2.6 *The hybrid hyper Leonardo numbers $\{\mathbb{H}Le_n^{(r)}\}_{n\geq 0}$ satisfy the following property, for any integers $n \geq 0$ and $r \geq 0$,*

$$\mathbb{H}Le_n^{(r)} = 2\mathbb{H}F_{n+1}^{(r)} - (B_n^{(r)} + B_{n+1}^{(r)}i + B_{n+2}^{(r)}\epsilon + B_{n+3}^{(r)}h), \tag{2.7}$$

where $B_n^{(r)} = \binom{n+r}{r}$.

Proof By replacing the relation (2.6) in (2.1) we obtain

$$\begin{aligned}\mathbb{H}Le_n^{(r)} &= (2F_{n+1}^{(r)} - \binom{n+r}{r}) + (2F_{n+2}^{(r)} - \binom{n+1+r}{r})i \\ &+ (2F_{n+3}^{(r)} - \binom{n+2+r}{r})\epsilon + (2F_{n+4}^{(r)} - \binom{n+3+r}{r})h \\ &= 2\mathbb{H}F_{n+1}^{(r)} - (B_n^{(r)} + B_{n+1}^{(r)}i + B_{n+2}^{(r)}\epsilon + B_{n+3}^{(r)})h,\end{aligned}$$

where $B_n^{(r)} = \binom{n+r}{r}$.

□

As a consequence, we provide a relation between the hybrid Leonardo number $\mathbb{H}Le_n$ and the hybrid Fibonacci number $\mathbb{H}F_n$, in the next corollary.

Corollary 2.7 *The hybrid Leonardo numbers $\mathbb{H}Le_n^{(0)} = \mathbb{H}Le_n$ satisfy the following property, for any integers $n \geq 0$*

$$\mathbb{H}Le_n = 2\mathbb{H}F_{n+1} - (1 + i + \epsilon + h), \tag{2.8}$$

where $\mathbb{H}F_n$ is the n-th hybrid Fibonacci number.

3. The generating function for the hybrid hyper Leonardo sequence

In this section, we establish the generating function for the hybrid hyper Leonardo sequence by considering the generating function for the hybrid hyper-Fibonacci sequence established in [Theorem 2.1, [2]] and given by,

$$\sum_{n=0}^{\infty} \mathbb{H}F_n^{(r)} t^n = \frac{t^3 + it + \epsilon t + h}{t^2(1-t-t^2)(1-t)^r} - \frac{\epsilon t + h(1 + rt + t)}{t^2}.$$

Hence, we have the next result.

Theorem 3.1 *For $n \geq 0$, the generating function for hybrid hyper Leonardo number is given by,*

$$\sum_{n=0}^{\infty} \mathbb{H}Le_n^{(r)} t^n = \frac{2(t^3+it+\epsilon t+h)}{t(1-t-t^2)(1-t)^r} - \frac{2\epsilon t+2h(1+rt+t)-2(i+F_2^{(r)}\epsilon+F_3^{(r)}h)}{t} - \sum_{n=0}^{\infty}(B_n^{(r)}+B_{n+1}^{(r)}i+B_{n+2}^{(r)}\epsilon+B_{n+3}^{(r)}h)t^n. \tag{3.1}$$

where $B_n^{(r)} = \binom{n+r}{r}$.

Proof By Expression (2.6), namely,

$$\mathbb{H}Le_n^{(r)} = 2\mathbb{H}F_{n+1}^{(r)} - (B_n^{(r)}+B_{n+1}^{(r)}i+B_{n+2}^{(r)}\epsilon+B_{n+3}^{(r)}h),$$

we have,

$$\begin{aligned}
t\sum_{n=0}^{\infty}\mathbb{H}Le_n^{(r)}t^n &= 2\sum_{n=0}^{\infty}\mathbb{H}F_{n+1}^{(r)}t^{n+1} - \sum_{n=0}^{\infty}(B_n^{(r)}+B_{n+1}^{(r)}i+B_{n+2}^{(r)}\epsilon+B_{n+3}^{(r)}h)t^{n+1}\\
t\sum_{n=0}^{\infty}\mathbb{H}Le_n^{(r)}t^n &= 2\left(\sum_{n=0}^{\infty}\mathbb{H}F_n^{(r)}t^n - \mathbb{H}F_0^{(r)}\right) - \sum_{n=0}^{\infty}(B_n^{(r)}+B_{n+1}^{(r)}i+B_{n+2}^{(r)}\epsilon+B_{n+3}^{(r)}h)t^{n+1}\\
t\sum_{n=0}^{\infty}\mathbb{H}Le_n^{(r)}t^n &= 2\left(\sum_{n=0}^{\infty}\mathbb{H}F_n^{(r)}t^n - (i+F_2^{(r)}\epsilon+F_3^{(r)}h)\right)\\
&\quad - \sum_{n=0}^{\infty}(B_n^{(r)}+B_{n+1}^{(r)}i+B_{n+2}^{(r)}\epsilon+B_{n+3}^{(r)}h)t^{n+1}\\
&= \frac{2(t^3+it+\epsilon t+h)}{t^2(1-t-t^2)(1-t)^r} - \frac{2\epsilon t+2h(1+rt+t)}{t^2} - 2(i+F_2^{(r)}\epsilon+F_3^{(r)}h)\\
&\quad - \sum_{n=0}^{\infty}(B_n^{(r)}+B_{n+1}^{(r)}i+B_{n+2}^{(r)}\epsilon+B_{n+3}^{(r)}h)t^{n+1}.
\end{aligned}$$

Then,

$$\sum_{n=0}^{\infty} \mathbb{H}Le_n^{(r)} t^n = \frac{2(t^3+it+\epsilon t+h)}{t(1-t-t^2)(1-t)^r} - \frac{2\epsilon t+2h(1+rt+t)-2(i+F_2^{(r)}\epsilon+F_3^{(r)}h)}{t} - \sum_{n=0}^{\infty}(B_n^{(r)}+B_{n+1}^{(r)}i+B_{n+2}^{(r)}\epsilon+B_{n+3}^{(r)}h)t^n.$$

where $B_n^{(r)} = \binom{n+r}{r}$. □

Remark 3.2 *Note that, for $r \geq 1$, the Expression $\sum_{n=0}^{\infty}\binom{n+r}{r}t^n$ can be written as*

$$\frac{1}{r!}\sum_{n=0}^{\infty}(n+r)(n+r-1)\cdots(n+1)t^n = \frac{1}{r!}\frac{d^r}{dt^r}\left(\frac{1}{1-t}\right).$$

Then, for $r \geq 1$, the generating function for hybrid hyper Leonardo number is given by

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{H}Le_n^{(r)} t^n &= \frac{2(t^3+it+\epsilon t+h)}{t(1-t-t^2)(1-t)^r} - \frac{2\epsilon t + 2h(1+rt+t) - 2(i+F_2^{(r)}\epsilon + F_3^{(r)} h)}{t} \\ &\quad - \frac{1}{r!}\frac{d^r}{dt^r}\left(\frac{1}{1-t}\right) - \frac{i}{r!}\frac{d^{r+1}}{dt^{r+1}}\left(\frac{1}{1-t}\right) - \frac{\epsilon}{r!}\frac{d^{r+2}}{dt^{r+2}}\left(\frac{1}{1-t}\right) \\ &\quad - \frac{h}{r!}\frac{d^{r+2}}{dt^{r+2}}\left(\frac{1}{1-t}\right).\end{aligned}$$

As a consequence of Theorem 3.1, for $r = 0$ we obtain the generating function of hybrid Leonardo numbers $\{\mathbb{H}Le_n^{(0)}\}_{n\geq 0}$.

Corollary 3.3 *For $n \geq 0$, the hybrid Leonardo number $\mathbb{H}Le_n^{(0)} = \mathbb{H}Le_n$ is the coefficient of t^n in the expansion of*

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{H}Le_n t^n &= \frac{t^3(2+2\epsilon - t^2(2i) + t(i-\epsilon-2h) + 2(i+\epsilon+2h))}{t(1-t-t^2)} \\ &\quad - \left(\frac{1+i+\epsilon+h}{1-t}\right)\end{aligned}$$

The generating function of hybrid Leonardo numbers $\{\mathbb{H}Le_n^{(0)}\}_{n\geq 0}$ was also established in [Proposition 2.2, [3]].

4. The Binet formula and some identities

Given a recurrence sequence, the Binet formula is a closed analytic formula that describes the sequence of numbers defined by its associated recurrence relation. This section establishes the Binet formula for the hybrid hyper Leonardo numbers.

Consider the Binet formula for the hybrid hyper Fibonacci numbers established in [Proposition 2.2, [2]], namely,

$$\mathbb{H}F_n = \frac{\underline{\alpha}\alpha^{n+2r} - \underline{\beta}\beta^{n+2r}}{\alpha - \beta} - \mathbb{H}A^r, \tag{4.1}$$

where α and β are the roots of the characteristic equation of the Fibonacci sequence, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 h\alpha^3$, $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$, and

$$\mathbb{H}A^r = \sum_{k=0}^{r-1} \binom{n+r+k}{r-1-k} + i\binom{n+r+k+1}{r-1-k} + \epsilon\binom{n+r+k+2}{r-1-k} + h\binom{n+r+k+3}{r-1-k}.$$

Then, by Expression (2.7), we obtain the Binet formula for the hybrid hyper Leonardo, given in the next theorem.

Theorem 4.1 (Binet's formula) *For $n \geq 0$, the hybrid hyper Leonardo of order n, $\mathbb{H}Le_n^{(r)}$, is given by*

$$\mathbb{H}Le_n^{(r)} = \frac{2(\underline{\alpha}\alpha^{n+2r} - \underline{\beta}\beta^{n+2r})}{\alpha - \beta} - \mathbb{H}B^r \tag{4.2}$$

where α and β are the roots of the characteristic equation of the Fibonacci sequence, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 h\alpha^3$, $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$, and

$$\begin{aligned}\mathbb{H}B^r = &\left(\binom{n+r}{r} + 2\sum_{k=0}^{r-1}\binom{n+r+k}{r-1-k}\right) + i\left(\binom{n+r+1}{r} + 2\sum_{k=0}^{r-1}\binom{n+r+k+1}{r-1-k}\right)\\ &+ \epsilon\left(\binom{n+r+2}{r} + 2\sum_{k=0}^{r-1} n\binom{n+r+k+2}{r-1-k}\right)\\ &+ h\left(\binom{n+r+3}{r} + 2\sum_{k=0}^{r-1}\binom{n+r+k+3}{r-1-k}\right).\end{aligned}$$

Proof By replacing Expression (4.1) in Expression (2.7), we obtain

$$\begin{aligned}\mathbb{H}Le_n^{(r)} =&2\left(\frac{\underline{\alpha}\alpha^{n+2r} - \underline{\beta}\beta^{n+2r}}{\alpha - \beta} - \mathbb{H}A^r\right) - (B_n^{(r)} + B_{n+1}^{(r)}i + B_{n+2}^{(r)}\epsilon + B_{n+3}^{(r)}h),\\ =& \frac{2(\underline{\alpha}\alpha^{n+2r} - \underline{\beta}\beta^{n+2r})}{\alpha - \beta} - 2\mathbb{H}A^r - (B_n^{(r)} + B_{n+1}^{(r)}i + B_{n+2}^{(r)}\epsilon + B_{n+3}^{(r)}h),\\ =& \frac{2(\underline{\alpha}\alpha^{n+2r} - \underline{\beta}\beta^{n+2r})}{\alpha - \beta} - \mathbb{H}B^r,\end{aligned}$$

where

$$\begin{aligned}\mathbb{H}B^r = &\left(\binom{n+r}{r} + 2\sum_{k=0}^{r-1}\binom{n+r+k}{r-1-k}\right) + i\left(\binom{n+r+1}{r} + 2\sum_{k=0}^{r-1}\binom{n+r+k+1}{r-1-k}\right)\\ &+ \epsilon\left(\binom{n+r+2}{r} + 2\sum_{k=0}^{r-1} n\binom{n+r+k+2}{r-1-k}\right)\\ &+ h\left(\binom{n+r+3}{r} + 2\sum_{k=0}^{r-1}\binom{n+r+k+3}{r-1-k}\right).\end{aligned}$$

□

By using the classical approach, consider the hybrid hyper Leonardo sequence $\{\mathbb{H}Le_n^{(r)}\}_{n\geq 0}$ satisfying the recurrence relation (2.4), namely,

$$\mathbb{H}Le_n^{(r)} = \mathbb{H}Le_{n-1}^{(r)} + \mathbb{H}Le_{n-2}^{(r)} + (A_n^{(r)} + A_{n+1}^{(r)}i + A_{n+2}^{(r)}\epsilon + A_{n+3}^{(r)}h),$$

where $A_n^{(r)} = \binom{n+r-2}{r} + \sum_{j=1}^{r-1} j\binom{n+r-2-j}{r-j} + r$.

Then, the solutions of $\{\mathbb{H}Le_n^{(r)}\}_{n\geq 0}$ are given as a sum of the solution of the homogeneous part of (**??**) and a particular solution of (2.2). By considering the homogeneous part of (2.4), its characteristic polynomial is given by,

$$P(z) = z^2 - z - 1,$$

with simple roots given by $\frac{1-\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2}$, roots of the characteristic equation associated to Fibonacci sequence. Then, the solution of the homogeneous part of (2.2) is given by $C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n$, where C_1 and C_2 are constants which depend on particular solution and initial conditions.

By fixing r, a particular solution is given by

$$\sum_{j=0}^{r}(B_j^0 + iB_j^1 + \epsilon B_j^2 + hB_j^3)n^j$$

where B_j^p, $0 \leq p \leq 3$, are constants for each $0 \leq j \leq r$, obtained by solving the equation

$$\begin{aligned}\sum_{j=0}^{r}(B_j^0 + iB_j^1 + \epsilon B_j^2 + hB_j^3)n^j &= \sum_{j=0}^{r}(B_j^0 + iB_j^1 + \epsilon B_j^2 + hB_j^3)(n-1)^j \\ &\quad + \sum_{j=0}^{r}(B_j^0 + iB_j^1 + \epsilon B_j^2 + hB_j^3)(n-2)^j \\ &\quad + A_n^{(r)}.\end{aligned} \tag{4.3}$$

Theorem 4.2 *For $n \geq 0$, the hybrid hyper Leonardo of order n, $Le_n^{(r)}$, is given by*

$$\mathbb{H}Le_n^{(r)} = C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n + \sum_{j=0}^{r}(B_j^0 + iB_j^1 + \epsilon B_j^2 + hB_j^3)n^j, \tag{4.4}$$

where C_1 and C_2 are constants that depend on particular solutions and initial conditions, and B_j^p, $0 \leq p \leq 3$, are constants for each $0 \leq j \leq r$, obtained by solving the system (4.3).

For $r = 0$, we obtain the function $A(n) = 1 + i + \epsilon + h$, then the particular solution is given by hybrid constant $-(1 + i + \epsilon + h)$. Then, the next result is verified.

Proposition 4.3 *For $n \geq 0$, the Leonardo number of order n is given as*

$$Le_n = C_1\left(\frac{1-\sqrt{5}}{2}\right)^n + C_2\left(\frac{1+\sqrt{5}}{2}\right)^n - (1 + i + \epsilon + h), \tag{4.5}$$

where where $C_1 = \frac{1}{\sqrt{5}}(-1 + \sqrt{5} + i(-3 + \sqrt{5}) + \epsilon(-4 + 2\sqrt{5})) + h(-7 + 3\sqrt{5}))$ and $C_2 = \frac{1}{\sqrt{5}}(-3 + \sqrt{5} + i(-5 + \sqrt{5}) + \epsilon(-8 + 2\sqrt{5})) + h(-13 + 3\sqrt{5}))$.

The Binet formula for the hybrid Leonardo numbers $\{\mathbb{H}Le_n^{(0)}\}_{n\geq 0}$ was also established in [Proposition 2.4, [3]], using the Binet formula of hybrid Fibonacci numbers.

5. Conclusion

In this paper, we introduced a new generalization of the Leonardo sequence, the hybrid hyper Leonardo numbers. Moreover, the algebraic properties of this sequence are studied and also the generating function and several identities are provided. It seems to us that all results given here are not current in the literature and this new sequence of numbers is a subject that can still be studied in several aspects such as combinatorial, analytical, and matrix perspectives.

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