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Alternative solution to the fractional differential equation with recurrence relationship

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Abstract: A different solution from the one already known for sequential fractional differential equations with recurrence relation is proposed. This solution involves a Mittag-Leffler type function, which satisfies a recurrence property compatible with the behavior of sequential fractional differential equations with recurrence relation.

Key words: Mittag-Leffler function, linear differential equation, trigonometric function

1. Introduction

The role of the exponential function in the solution of linear differential equations with constant coefficients has an analogy with the role of the Mittag-Leffler function and its generalizations in the solution of noninteger order differential equations. The exponential function has the important property of being invariant, except for constant, by the operations of differentiation and integration. In fractional calculus, the function that has this property is called α-exponential and it is defined in terms of the two-parameter Mittag-Leffler function. It is not possible to generalize the α-exponential function through generalizations of the Mittag-Leffler function with three or more parameters and to preserve its invariance under the operations of differentiation and fractional integration, for example, see [2, 3, 14]. This prompted us to introduce a Mittag-Leffler type function, the γ-α-n-exponential function, which has a similar property to the $\alpha-$exponential function but it involves recurrence relations when applying Miller-Ross sequential differentiation operators (see [9, 11]). The particular behavior of sequential derivatives makes a sequential differential equation an intuitive generalization of ordinary differential equations. In [10, 12] we introduced the bases for a new theory of fractional differential equations with a recurrence relation. In this article, we give an alternative solution to the linear differential equations with recurrence relationship homogeneous using both the γ-α-n-exponential function and the generalized fractional trigonometric functions defined in [9].

2. Preliminaries

The Mittag-Leffler function $E_{\alpha,\beta}(x)$ is defined by the following series:

$$E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha j + \beta)} \quad (x, \alpha, \beta \in \mathbb{C}; \mathfrak{Re}(\alpha) > 0), \tag{2.1}$$

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where $\Gamma(x)$ is the classical Gamma function; and $E_{1,1} = e^{\lambda x}$ $(\lambda \in \mathbb{C})$ (cf.[4]). Based on $E_{\alpha,\beta}(x)$, the α-Exponential Function is defined as follows:

$$e_{\alpha}^{\lambda x} = x^{\alpha-1} E_{\alpha,\alpha}\left(\lambda x^{\alpha}\right), \tag{2.2}$$

with $x \in \mathbb{C} \setminus \{0\}$, $\mathfrak{Re}(\alpha) > 0$, $\lambda \in \mathbb{C}$. The α-exponential function is a generalization of the exponential function, and $e_1^{\lambda x} = e^{\lambda x}$. If $x > a$ and $\lambda = b + ic$ $(b, c \in \mathbb{R})$, then the real and imaginary parts of $e_{\alpha}^{\lambda x}$ are defined as the α-trigonometrics functions:

$$\cos_{\alpha}(\lambda(x-a)) = \mathfrak{Re}\left[e_{\alpha}^{i\lambda(x-a)}\right] \qquad \text{and} \qquad \sin_{\alpha}(\lambda(x-a)) = \mathfrak{Im}\left[e_{\alpha}^{i\lambda(x-a)}\right]. \tag{2.3}$$

Prabhakar introduces in [13] the Mittag-Leffler type function

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{j=0}^{\infty} \frac{(\gamma)_j x^j}{\Gamma(\alpha j + \beta) j!}, \tag{2.4}$$

with α, β, $\gamma \in \mathbb{C}$; $\mathfrak{Re}(\alpha) > 0$, and $x \in \mathbb{C}$; where $(\gamma)_j$ is the Pochhammer symbol (cf.[4]), with $x \in \mathbb{C}$. And it verified $E_{\alpha,\beta}^{1} = E_{\alpha,\beta}$.

The Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$ of a integrable function $f(x)$ defined in $[a, b]$ is defined by (see [4])

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad n = [\Re(\alpha)] + 1. \tag{2.5}$$

In [4], it is proved that if $\alpha, \beta \in \mathbb{C}$ and $\mathfrak{Re}(\alpha), \mathfrak{Re}(\beta) > 0$, then

$$\left(D_{a+}^{\alpha} (t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x-a)^{\beta-\alpha-1} \; (\mathfrak{Re}(\alpha) \geq 0). \tag{2.6}$$

From (2.5) and (2.6), the following relationship is obtained:

$$\left(D_{a+}^{\alpha} e_{\alpha}^{\lambda(t-a)}\right)(x) = \lambda e_{\alpha}^{\lambda(x-a)}, \tag{2.7}$$

when $\mathfrak{Re}(\alpha) > 0$, and $\lambda \in \mathbb{C}$ (see [2]). In [9], the L-Mittag-Leffler function is introduced:

$$L_{\alpha,\beta}^{\gamma,n}(x) = \sum_{j=0}^{\infty} \frac{(\gamma)_{j+n} x^j}{\Gamma(\alpha j + \beta)(j+n)!}, \quad (x \in \mathbb{C}), \tag{2.8}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$; $\mathfrak{Re}(\alpha) > 0$, $\mathfrak{Re}(\beta) > 0$, $\mathfrak{Re}(\gamma) > 0$, $n \in \mathbb{N}_0$. The particular case $L_{\alpha,\beta}^{\gamma,0}(x) = E_{\alpha,\beta}^{\gamma}(x)$ is verified. Then the γ-α-n-Exponential is defined as follows:

$$e_{\alpha,\gamma,n}^{\lambda(x-a)} = (x-a)^{\alpha-1} L_{\alpha,\alpha}^{\gamma,n}(\lambda(x-a)^{\alpha}) \quad (x > a), \tag{2.9}$$

with λ, $\gamma \in \mathbb{C}$, $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The special cases $e_{\alpha,1,n}^{\lambda(x-a)} = e_{\alpha}^{\lambda(x-a)}$ are verified. The function (2.9) also exhibits the following properties:

$$\lim_{n\to\infty} \Gamma(\gamma) e_{\alpha,\gamma,n}^{\lambda(x-a)} = e_{\alpha}^{\lambda(x-a)}. \tag{2.10}$$

When $x \in A \subset (a,\infty)$, where A is a compact set; and

$$\left(\mathcal{D}_{a+}^{N\alpha} e_{\alpha,\gamma,n}^{\lambda(t-a)}\right)(x) = \lambda^N e_{\alpha,\gamma,n+N}^{\lambda(x-a)}, \tag{2.11}$$

where $\lambda \in \mathbb{C},\ N \in \mathbb{N},\ 0 < \alpha \le 1$, and $y^{(k\alpha} = \left(\mathcal{D}_{a+}^{k\alpha} y(x)\right)(x)\ (k = 1,2,...,N)$ represent a sequential fractional derivative, introduced by Miller and Ross in [11]:

$$\begin{aligned} \mathcal{D}_{a+}^{\alpha} &= \mathbf{D}_{a+}^{\alpha}\ \ (0<\alpha\le 1) \\ \mathcal{D}_{a+}^{k\alpha} &= \mathcal{D}_{a+}^{\alpha}\mathcal{D}_{a+}^{(k-1)\alpha}, \end{aligned} \tag{2.12}$$

where $\mathbf{D}_{a+}^{\alpha}$ is the Riemann-Liouville fractional derivative: $\mathbf{D}_{a+}^{\alpha} = D_{a+}^{\alpha}$.

The above generalized exponential function can be used to extend the ordinary trigonometric functions to the called γ-α-n-Cosine and γ-α-n-Sine functions, denoted as

$$\cos_n^{\alpha,\gamma}(\lambda(x-a)) = \mathfrak{Re}\left[e_{\alpha,\gamma,n}^{i\lambda(x-a)}\right] \quad \text{and} \quad \sin_n^{\alpha,\gamma}(\lambda(x-a)) = \mathfrak{Im}\left[e_{\alpha,\gamma,n}^{i\lambda(x-a)}\right] \quad (x>a), \tag{2.13}$$

respectively, where $\lambda \in \mathbb{C}$. In addition, since the relationship $e_{\alpha,1,n}^{i\lambda(x-a)} = e_{\alpha}^{i\lambda(x-a)}$ is verified, we obtain:

$$\cos_n^{\alpha,1}(\lambda(x-a)) = \cos_\alpha(\lambda(x-a)), \tag{2.14}$$

$$\sin_n^{\alpha,1}(\lambda(x-a)) = \sin_\alpha(\lambda(x-a)), \tag{2.15}$$

where $\sin_\alpha$ and $\cos_\alpha$ are given in (2.3), respectively. They also have the following properties:

$$\left(\mathcal{D}_{a+}^{N\alpha}\cos_n^{\alpha,\gamma}[\lambda(x-a)]\right)(x) = \begin{cases} \lambda^N \cos_{n+N}^{\alpha,\gamma}(\lambda(x-a)) & \text{if r=0,} \\ -\lambda^N \sin_{n+N}^{\alpha,\gamma}(\lambda(x-a)) & \text{if r=1,} \\ -\lambda^N \cos_{n+N}^{\alpha,\gamma}(\lambda(x-a)) & \text{if r=2,} \\ \lambda^N \sin_{n+N}^{\alpha,\gamma}(\lambda(x-a)) & \text{if r=3,} \end{cases} \tag{2.16}$$

and

$$\left(\mathcal{D}_{a+}^{N\alpha}\sin_n^{\alpha,\gamma}[\lambda(x-a)]\right)(x) = \begin{cases} \lambda^N \sin_{n+N}^{\alpha,\gamma}(\lambda(x-a)) & \text{if r=0,} \\ \lambda^N \cos_{n+N}^{\alpha,\gamma}(\lambda(x-a)) & \text{if r=1,} \\ -\lambda^N \sin_{n+N}^{\alpha,\gamma}(\lambda(x-a)) & \text{if r=2,} \\ -\lambda^N \cos_{n+N}^{\alpha,\gamma}(\lambda(x-a)) & \text{if r=3,} \end{cases} \tag{2.17}$$

where $N = 4q + r$, with $q \in \mathbb{N}_0$ and $0 \le r < 4$.

In [8] the basic general theory for the Linear Sequential Fractional Differential Equation which includes a recurrence relationship is introduced.

Definition 2.1 *Let $N \in \mathbb{N}$ and $0 < \alpha \le 1$. It is called Linear Sequential Fractional Differential Equations with Recurrence Relationship (LFDERR) of order $N\alpha$ to an equation of the type:*

$$\left[\mathbf{R}_{N\alpha}(y_n(t))_{n=0}^{\infty}\right](x) = \left(\mathcal{D}_{a+}^{N\alpha} y_n\right)(x) + \sum_{j=1}^{N} a_{N-j}(x)\left(\mathcal{D}_{a+}^{(N-j)\alpha} y_{n+j}\right)(x) = f_n(x), \tag{2.18}$$

($n \in \mathbb{N}_0$, $x > a$) where $\mathcal{D}_{a+}^{k\alpha}$ is defined by (2.12), $\{a_j(x)\}_{j=0}^{N-1}$ are real functions defined in $(a,b] \subset \mathbb{R}$, $a_0 \neq 0$, and $f_n(x) \in C((a,b])$, for each $n \in \mathbb{N}_0$. When $f_n \equiv 0$, the equation (2.18) is called homogeneous LFDERR

(LFDERRH) associated with (2.18). If $a_0, a_1, ..., a_{N-1}$ are constants, the equation (2.18) will be called an equation with constant coefficients

$$\left(\mathcal{D}_{a+}^{N\alpha} y_n\right)(x) + \sum_{j=1}^{N} a_{N-j}\left(\mathcal{D}_{a+}^{(N-j)\alpha} y_{n+j}\right)(x) = f_n(x); \tag{2.19}$$

and its corresponding homogeneous equation will be:

$$\left(\mathcal{D}_{a+}^{N\alpha} y_n\right)(x) + \sum_{j=1}^{N} a_{N-j}\left(\mathcal{D}_{a+}^{(N-j)\alpha} y_{n+j}\right)(x) = 0. \tag{2.20}$$

In [8], the set $\Delta^{N\alpha}(a,b)$ is defined as the set of functions that have sequential derivatives $\mathcal{D}_{a+}^{K\alpha}$, with $1 \leq K \leq N$, in (a,b); where it can be $N = \infty$, meaning that $\Delta^{\infty\alpha}(a,b)$ is the set of functions that have sequential derivatives of all orders. The set $\left[\Delta^{N\alpha}(a,b)\right]^{\mathbb{N}}$ is defined as the set of sequences of functions, such that each term belongs $\Delta^{N\alpha}(a,b)$: That is:

$$(y_n(x))_{n=0}^{\infty} \in \left[\Delta^{N\alpha}(a,b)\right]^{\mathbb{N}} \Leftrightarrow \forall n \in \mathbb{N}_0 \ : \ y_n(x) \in \Delta^{N\alpha}(a,b). \tag{2.21}$$

Furthemore, in [8], it is proved that the set $\mathbf{H} = \mathbf{E}_N^0(a,b) \cap [\Delta^{\infty\alpha}(a,b)]^{\mathbb{N}}$ is a vector space of N dimensions, which denotes $\mathbf{E}_N^0(a,b)$ the set of solutions to equation (2.20), with $x \in (a,b)$ and the operations $+$ and $\cdot$, defined as follows:

$$(y_n^1(x))_{n=0}^{\infty} + (y_n^2(x))_{n=0}^{\infty} = ((y_n^1 + y_n^2)(x))_{n=0}^{\infty} \tag{2.22}$$

$$d(y_n^1(x))_{n=0}^{\infty} = ((dy_n^1)(x))_{n=0}^{\infty}, \tag{2.23}$$

whenever $(y_n^1(x))_{n=0}^{\infty}$, $(y_n^2(x))_{n=0}^{\infty} \in \mathbf{E}_N^0(a,b)$, and d is a scalar.

3. Main results

In this section, we will use the α-γ-n-exponential function to find a fundamental set of solutions for equation (2.20). The following result will be required:

Lemma 3.1 *Let $k \in \mathbb{R}$ and $t \in \mathbb{N}$ be, there exist B_0, B_1, ..., $B_t \in \mathbb{R}$ such that we can write:*

$$k^t = \sum_{q=0}^{t} B_q \prod_{p=0}^{q} (k-p). \tag{3.1}$$

Proof It will be done by induction. $t = 1$ is taken in (3.1):

$$k = \sum_{q=0}^{1} B_q \prod_{p=0}^{q} (k-p) = (B_0 - B_1)k + B_1 k^2, \tag{3.2}$$

it is enough to take $B_0 = 1$ and $B_1 = 0$. For the remainder of the proof, it will be assumed that there exist B_1, B_2,..., $B_t \in \mathbb{R}$ such that (3.1) holds; and it will be proved that we can always find B_1', B_2',..., B_t', $B_{t+1}' \in \mathbb{R}$ such that

$$k^{t+1} = \sum_{q=0}^{t+1} B_q' \prod_{p=0}^{q} (k-p). \tag{3.3}$$

The expression $\prod_{p=0}^{t+1}(k-p)$ represents a polynomial of degree $t+1$ in k, which vanishes when $k \in \{0, 1, 2, ..., t, t+1\}$. Then, there exist a_0, a_1, ..., $a_t \in \mathbb{R}$, such that

$$\sum_{j=0}^{t+1} a_j k^{j+1} = \prod_{p=0}^{t+1}(k-p), \tag{3.4}$$

The following decomposition can be considered:

$$\sum_{q=0}^{t+1} B_q \prod_{p=0}^{q}(k-p) = \sum_{q=0}^{t} B_q \prod_{p=0}^{q}(k-p) + B_{t+1} \prod_{p=0}^{t+1}(k-p). \tag{3.5}$$

If we assume that $k \in \{0, 1, 2, ..., t, t+1\}$, then we can deduce that (3.5) has the shape

$$\sum_{q=0}^{t+1} B_q \prod_{p=0}^{q}(k-p) = \sum_{q=0}^{t} B_q \prod_{p=0}^{q}(k-p) + (B_{t+1})(0). \tag{3.6}$$

Therefore, B_{t+1} can be any real number. Furthermore, assuming (3.1) holds, (3.6) can be rewritten as:

$$\sum_{q=0}^{t+1} B_q \prod_{p=0}^{q}(k-p) = k^t. \tag{3.7}$$

Then, multiplying both sides of (3.7) by k we obtain:

$$\sum_{q=0}^{t+1} kB_q \prod_{p=0}^{q}(k-p) = k^{t+1}, \tag{3.8}$$

i.e. there exist $B'_0 = kB_0$, $B'_1 = kB_1$, ..., $B'_t = kB_t$, $B'_{t+1} = kB_{t+1} \in \mathbb{R}$, such that

$$\sum_{q=0}^{t+1} B'_q \prod_{p=0}^{q}(k-p) = k^{t+1}, \quad \text{since} \quad k \in \{0, 1, 2, ..., t, t+1\}. \tag{3.9}$$

On the other hand, if $k \notin \{0, 1, 2, ..., t, t+1\}$ is taken by (3.4), we can rewrite (3.5) in the following form

$$\sum_{q=0}^{t+1} B_q \prod_{p=0}^{q}(k-p) = \sum_{q=0}^{t} B_q \prod_{p=0}^{q}(k-p) + B_{t+1} \sum_{j=0}^{t+1} a_j k^{j+1}. \tag{3.10}$$

In addition, as (3.1) is supposed to hold, (3.10) can be rewritten as follows:

$$\sum_{q=0}^{t+1} B_q \prod_{p=0}^{q}(k-p) = k^t + B_{t+1} \sum_{j=0}^{t+1} a_j k^{j+1} = k^t \left(1 + B_{t+1} \sum_{j=0}^{t+1} a_j k^{j+1-t}\right). \tag{3.11}$$

Therefore, in order for (3.3) to be verified, it will suffice to write that:

$$1 + B_{t+1} \sum_{j=0}^{t+1} a_j k^{j+1-t} = k, \tag{3.12}$$

i.e.

$$B_{t+1} = \frac{k-1}{\sum_{j=0}^{t+1} a_j k^{j+1-t}} = \frac{(k-1)k^t}{\sum_{j=0}^{t+1} a_j k^{j+1}} = \frac{(k-1)k^t}{\prod_{p=0}^{t+1}(k-p)}. \tag{3.13}$$

Then there exist

$$B'_0 = B_0, \quad B'_1 = B_1, \; \ldots, \quad B'_t = B_t, \quad B'_{t+1} = \frac{(k-1)k^t}{\prod_{p=0}^{t+1}(k-p)} \in \mathbb{R}, \tag{3.14}$$

such that

$$\sum_{q=0}^{t+1} B'_q \prod_{p=0}^{q}(k-p) = k^{t+1}, \quad \text{since} \quad k \notin \{0,1,2,...,t,t+1\}. \tag{3.15}$$

By (3.9) and (3.15), (3.3) it is proved; which concludes the proof. □

Example 3.2 *If $k \in \mathbb{R}$; by Lemma 3.1, with $t = 3$, we can always find B_0, B_1, B_2, $B_3 \in \mathbb{R}$ such that the following decomposition holds:*

$$\begin{aligned} k^3 &= \sum_{q=0}^{3} B_q \prod_{p=0}^{q}(k-p) = B_0 \prod_{p=0}^{0}(k-p) + B_1 \prod_{p=0}^{1}(k-p) + B_2 \prod_{p=0}^{2}(k-p) + B_3 \prod_{p=0}^{3}(k-p) && (3.16)\\ &= B_0 k + B_1 k(k-1) + B_2 k(k-1)(k-2) + B_3 k(k-1)(k-2)(k-3) && (3.17)\\ &= k(B_0 - B_1 + 2B_2 + 6B_3) + k^2(B_1 - 3B_2 - 7B_3) + k^3(B_2 - 6B_3) + k^4 B_3. && (3.18)\end{aligned}$$

Taking $B_3 = 0$ in (3.18), we have

$$k^3 = k(B_0 - B_1 + 2B_2) + K^2(B_1 - 3B_2) + k^3 B_2. \tag{3.19}$$

Letting be $B_2 = 1$ in (3.18):

$$0 = k(B_0 - B_1 + 2) + k^2(B_1 - 3). \tag{3.20}$$

Then, if $B_1 = 3$ in (3.20) it must be $B_0 = 1$. Therefore, a decomposition is

$$k^3 = k + 3k(k-1) + k(k-1)(k-2). \tag{3.21}$$

If, for example, $k = 5$ is taken in (3.21):

$$5^3 = 5 + (3)(5)(4) + (5)(4)(3) = 125. \tag{3.22}$$

From now on, we will study a fundamental set of solutions of (2.20); this set is an alternative to the one found in [8]. The set of solutions will be directly related to the γ-α-exponential function introduce in [9]. Taking $y_n(x) = e^{\lambda(x-a)}_{\alpha,\gamma,n}$, on the left side of (2.20), and applying (2.11) yields that

$$\left[\mathbf{R}_{N\alpha}\left(e^{\lambda(t-a)}_{\alpha,\gamma,n}\right)_{n=0}^{\infty}\right](x) = \lambda^N e^{\lambda(x-a)}_{\alpha,\gamma,n+N} + \sum_{j=1}^{N} a_{N-j}\lambda^{N-j} e^{\lambda(x-a)}_{\alpha,\gamma,n+N} = \left(\lambda^N + \sum_{j=1}^{N} a_{N-j}\lambda^{N-j}\right) e^{\lambda(x-a)}_{\alpha,\gamma,n+N}. \tag{3.23}$$

Then, the expression (3.23) suggests the following definition.

Definition 3.3 *The expression between parentheses in (3.23) will be called the characteristic polynomial associated with the equation (2.20), and will be denoted by*

$$P_N(\lambda) = \lambda^N + \sum_{j=1}^{N} a_{N-j}\lambda^{N-j}. \tag{3.24}$$

3.1. Alternative solution of the homogeneous LFDERR using the α-γ-n-exponential function

In [9], Corollary 2.5, a recurrence relation was established between two consecutive terms of the sequence of functions $(y_n(x))_{n=0}^{\infty}$, with

$$y_n(x) = (x-a)^{\alpha-1} L_{\alpha,\alpha}^{\gamma,n}\left(\lambda(x-a)^{\alpha}\right), \tag{3.25}$$

$n \in \mathbb{N}_0$, i.e. for all $x > a$: $y_n(x)$ is the solution of the recurrence equation:

$$\left(D_{a+}^{\alpha} y_n\right)(x) - \lambda y_{n+1}(x) = 0. \tag{3.26}$$

Theorem 3.4 *Let λ be a root of (3.24). Then, the sequence of the general term*

$$y_n(x) = e_{\alpha,\gamma,n}^{\lambda(x-a)} \tag{3.27}$$

is a solution of (2.20).

Proof It is evident from what has been stated in (3.23). □

Theorem 3.5 *Let λ be a root of (3.24) with the multiplicity ℓ. Then, for $0 < m \leq \ell - 1$, the sequence of general term*

$$y_n(x) = (n-N)^m e_{\alpha,\gamma,n}^{\lambda(x-a)} \tag{3.28}$$

is a solution of (2.20).

Proof Since λ has the multiplicity ℓ, it follows that:

$$P_N(\lambda) = \frac{dP_N(\lambda)}{d\lambda} = ... = \frac{d^{\ell-1}P_N(\lambda)}{d\lambda^{\ell-1}} = 0 \text{ and } \frac{d^{\ell}P_N(\lambda)}{d\lambda^{\ell}} \neq 0. \tag{3.29}$$

Whereas, proceeding as in (3.23), we derive that

$$\left[\mathbf{R}_{N\alpha}\left((n-N)^m e_{\alpha,\gamma,n}^{\lambda(t-a)}\right)_{n=0}^{\infty}\right] = e_{\alpha,\gamma,n+N}^{\lambda(x-a)}\left((n-N)^m\lambda^N + \sum_{j=1}^{N} a_{N-j}(n+j-N)^m\lambda^{N-j}\right). \tag{3.30}$$

If $a_N = 1$ is defined, we can write (3.30) as follows

$$\left[\mathbf{R}_{N\alpha}\left((n-N)^m e_{\alpha,\gamma,n}^{\lambda(t-a)}\right)_{n=0}^{\infty}\right] = e_{\alpha,\gamma,n+N}^{\lambda(x-a)}\left(\sum_{j=0}^{N} \lambda^j a_j (n-j)^m\right). \tag{3.31}$$

Now we will consider the expression between parentheses in (3.31). For this, we expand the binomial $(n-j)^m$ and apply Lemma 3.1 to j^t:

$$\sum_{j=0}^{N}\lambda^j a_j(n-j)^m=\sum_{j=0}^{N}\lambda^j a_j\left[\sum_{t=0}^{m}(-1)^t\binom{m}{t}n^{m-t}j^t\right]=$$
$$=\sum_{j=0}^{N}\lambda^j a_j\left\{\sum_{t=0}^{m}(-1)^t\binom{m}{t}n^{m-t}\left[\sum_{q=1}^{t}B_q\prod_{p=0}^{q}(j-p)\right]\right\}=$$
$$=\sum_{t=0}^{m}(-1)^t\binom{m}{t}n^{m-t}\left\{\sum_{q=0}^{t}B_q\left[\sum_{j=0}^{N}\lambda^j a_j\prod_{p=0}^{q}(j-p)\right]\right\}. \quad (3.32)$$

Now we will evaluate the sum between brackets in (3.32):

$$\left[\sum_{j=0}^{N}\eta^j a_j\prod_{p=0}^{q}(j-p)\right]_{\eta=\lambda}=\left[\eta^{q+1}\sum_{j=0}^{N}\eta^{j-(q+1)}a_j\prod_{p=0}^{q}(j-p)\right]_{\eta=\lambda}=\left[\eta^{q+1}\sum_{j=0}^{N}a_j\frac{d^q\eta^j}{d\eta^q}\right]_{\eta=\lambda}=$$
$$=\left[\eta^{q+1}\frac{d^q}{d\eta^q}\left(\sum_{j=0}^{N}a_j\eta^j\right)\right]_{\eta=\lambda}=\left[\eta^{q+1}\frac{d^qP_N(\eta)}{d\eta^q}\right]_{\eta=\lambda}. \quad (3.33)$$

Then, substituting (3.33) and (3.32) into (3.31) we obtain:

$$\mathbf{R}_{N\alpha}\left[(n-N)^m e_{\alpha,\gamma,n}^{\lambda(x-a)}\right]=e_{\alpha,\gamma,n+N}^{\lambda(x-a)}\left[\sum_{t=0}^{m}(-1)^t\binom{m}{t}n^{m-t}\left\{\sum_{q=0}^{t}B_q\left[\eta^{q+1}\frac{d^qP_N(\lambda)}{d\eta^q}\right]\right\}\right]=0, \quad (3.34)$$

for all $0<m\le\ell-1$. □

Theorem 3.6 *Let λ_1, $\lambda_2\in\mathbb{C}\backslash\{0\}$, $\lambda_1\neq\lambda_2$, then $\left(e_{\alpha,\gamma,n}^{\lambda_1(x-a)}\right)_{n=0}^{\infty}$ and $\left(e_{\alpha,\gamma,n}^{\lambda_2(x-a)}\right)_{n=0}^{\infty}$ are linearly independent.*

Proof Let $n\in\mathbb{N}$, c_1, $c_2\in\mathbb{R}$. Let the linear combination be:

$$0=c_1e_{\alpha,\gamma,n}^{\lambda_1(x-a)}+c_2e_{\alpha,\gamma,n}^{\lambda_2(x-a)}=\sum_{j=0}^{\infty}\frac{(\gamma)_{j+n}\left(c_1\lambda_1^j+c_2\lambda_2^j\right)}{\Gamma(\alpha(j+1))(j+n)!}(x-a)^{\alpha(j+1)-1}. \quad (3.35)$$

In order for (3.35) to vanish, for every $n\in\mathbb{N}_0$, it implies that $c_1\lambda_1^j+c_2\lambda_2^j=0$ for every $j\in\mathbb{N}_0$. Since $\lambda_1\neq\lambda_2$, there exists $d\in\mathbb{C}$ such that $d\lambda_1=\lambda_2$, hence: $0=c_1\lambda_1^j+c_2\lambda_2^j=\lambda_1^j\left(c_1+c_2d^j\right)$ for every $j\in\mathbb{N}_0$. Therefore $c_1=-d^jc_2$ for each $j\in\mathbb{N}_0$, i.e. $c_1=c_2=0$; namely, the only possible linear combination is a trivial one. Hence there must be $\left(e_{\alpha,\gamma,n}^{\lambda_1(x-a)}\right)_{n=0}^{\infty}$ and $\left(e_{\alpha,\gamma,n}^{\lambda_2(x-a)}\right)_{n=0}^{\infty}$ linearly independent. □

Theorem 3.7 *If $\lambda\in\mathbb{C}\setminus\{0\}$, $0<m_1<m_2$, then $\left((n-N)^{m_1}e_{\alpha,\gamma,n}^{\lambda(x-a)}\right)_{n=0}^{\infty}$ and $\left((n-N)^{m_2}e_{\alpha,\gamma,n}^{\lambda(x-a)}\right)_{n=0}^{\infty}$ are linearly independent.*

Proof Let $n \in \mathbb{N}$ and c_1, $c_2 \in \mathbb{R}$. If we propose the following null linear combination, then

$$0 = c_1(n-N)^{m_1} e_{\alpha,\gamma,n}^{\lambda(x-a)} + c_2(n-N)^{m_2} e_{\alpha,\gamma,n}^{\lambda(x-a)} = (n-N)^{m_1} e_{\alpha,\gamma,n}^{\lambda(x-a)} \left[c_1 + c_2(n-N)^{m_2-m_1}\right]. \tag{3.36}$$

Therefore, it should imply that $c_1 + c_2(n-N)^{m_2-m_1} = 0$, for every $n \in \mathbb{N}_0$; hence $c_1 = c_2 = 0$. □

Corollary 3.8 *If* λ_1, $\lambda_2 \in \mathbb{C} \setminus \{0\}$, $\lambda_1 \neq \lambda_2$, *and* $0 < m_1 < m_2$; *then* $\left((n-N)^{m_1} e_{\alpha,\gamma,n}^{\lambda_1(x-a)}\right)_{n=0}^{\infty}$ *and* $\left((n-N)^{m_2} e_{\alpha,\gamma,n}^{\lambda_2(x-a)}\right)_{n=0}^{\infty}$ *are linearly independent.*

Proof Let $n_0 \in \mathbb{N}$, c_1, $c_2 \in \mathbb{R}$. If we propose the following null linear combination, then

$$c_1(n-N)^{m_1} e_{\alpha,\gamma,n}^{\lambda_1(x-a)} + c_2(n-N)^{m_2} e_{\alpha,\gamma,n}^{\lambda_2(x-a)} = 0. \tag{3.37}$$

Therefore, it should imply that $c_1(n-N)^{m_1}\lambda_1^j + c_2(n-N)^{m_2}\lambda_2^j = 0$, for each $j,\ n \in \mathbb{N}_0$. Since $\lambda_1 \neq \lambda_2$, hence there exists $a \in \mathbb{C}$ such that $a\lambda_1 = \lambda_2$. Then

$$0 = c_1(n-N)^{m_1}\lambda_1^j + c_2(n-N)^{m_2} a^j \lambda_1^j = \lambda_1^j (n-N)^{m_1}\left[c_1 + c_2(n-N)^{m_2-m_1} a_j\right], \tag{3.38}$$

for every $j,\ n \in \mathbb{N}_0$; hence $c_1 = c_2 = 0$. □

Corollary 3.9 *Let* $\lambda \in \mathbb{C} \setminus \{0\}$, $m > 0$, *then* $\left(e_{\alpha,\gamma,n}^{\lambda(x-a)}\right)_{n=0}^{\infty}$ *and* $\left((n-N)^{m} e_{\alpha,\gamma,n}^{\lambda(x-a)}\right)_{n=0}^{\infty}$ *are linearly independent.*

Proof This proof is similar to that of the Theorem 3.7. □

Corollary 3.10 *Let* λ_1, $\lambda_2 \in \mathbb{C} \setminus \{0\}$, $\lambda_1 \neq \lambda_2$, $m > 0$, *then* $\left(e_{\alpha,\gamma,n}^{\lambda_1(x-a)}\right)_{n=0}^{\infty}$ *and* $\left((n-N)^{m} e_{\alpha,\gamma,n}^{\lambda_2(x-a)}\right)_{n=0}^{\infty}$ *are linearly independent.*

Proof The proof of this Corollary is similar to that of the Corollary 3.8. □

Theorem 3.11 *If* $P_N(\lambda) = (\lambda-\lambda_1)^{\ell_1}(\lambda-\lambda_2)^{\ell_2}...(\lambda-\lambda_M)^{\ell_M}$, *i.e.,* λ_1, λ_2, $...,\lambda_M$ *are the different roots of* $P_N(\lambda)$, *of multiplicity* ℓ_1, ℓ_2, ..., ℓ_M, *respectively; where* $\ell_j \geq 1$ $(j = 1, 2, ..., M)$, *and* $\ell_1 + \ell_2 + ... + \ell_M = N$. *Then, an expression for the general solution of (2.20), is given by* $(y_n(x))_{n=0}^{\infty}$, $x \in (a,b)$, *where*

$$y_n(x) = \sum_{q=1}^{M}\left[\sum_{j=1}^{\ell_q-1} c_{j,q}(n-N)^j e_{\alpha,\gamma,n}^{\lambda_q(x-a)} + c_{0,q} e_{\alpha,\gamma,n}^{\lambda_q(x-a)}\right], \tag{3.39}$$

and $c_{j,q}$ *'s are arbitrary constants.*

Proof For each q, such that $1 \leq q \leq M$, by the Theorems 3.4 and 3.5 we can obtain ℓ_q solutions of (2.20):

$$\bigcup_{j=1}^{\ell_q-1}\left\{\left((n-N)^j e_{\alpha,\gamma,n}^{\lambda_q(x-a)}\right)_{n=0}^{\infty}\right\} \ ; \ \left(e_{\alpha,\gamma,n}^{\lambda_q(x-a)}\right)_{n=0}^{\infty} \tag{3.40}$$

which also, by the Theorem 3.7 and Corollary 3.9 are linearly independent. If we also take into account Theorem 3.6 and Corollaries 3.8 and 3.10, hence

$$\mathbf{A} = \left\{ \bigcup_{j=1}^{\ell_q - 1} \left\{ \left((n-N)^j e_{\alpha,\gamma,n}^{\lambda_q(x-a)} \right)_{n=0}^{\infty} \right\} \; ; \; \left(e_{\alpha,\gamma,n}^{\lambda_q(x-a)} \right)_{n=0}^{\infty} \right\}_{q=1}^{M} \tag{3.41}$$

is a fundamental set of solutions of (2.20). Then, by [10], Lemma 3, every solution $(y_n(x))_{n=0}^{\infty}$ of the equation (2.20), in (a,b), can be written as

$$y_n(x) = \sum_{q=1}^{M} \left[\sum_{j=1}^{\ell_q - 1} c_{j,q} (n-N)^j e_{\alpha,\gamma,n}^{\lambda_q(x-a)} + c_{0,q} e_{\alpha,\gamma,n}^{\lambda_q(x-a)} \right] \tag{3.42}$$

where $c_{j,q}$'s are arbitrary constants. □

Corollary 3.12 *If $P_N(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)...(\lambda-\lambda_N)$, with λ_1, λ_2, ...,$\lambda_N \in \mathbb{C}$. Then, an expression for the general solution of (2.20), is given by $(y(x))_{n=0}^{\infty}$, $x \in (a,b)$, with*

$$y_n(z) = \sum_{j=1}^{N} c_j e_{\alpha,\gamma,n}^{\lambda_j(x-a)} \tag{3.43}$$

where $c_j \in \mathbb{C}$, $i = 1,2,...,N$.

Proof It follows as a particular case of the Theorem 3.11 taking $\ell_i = 1$ for $i = 1,2,...,M = N$. □

Example 3.13 *Consider the following LFDERRH of order 2α:*

$$\left(\mathcal{D}_{0+}^{2\alpha} y_n\right)(x) - 4y_{n+2}(x) = 0, \tag{3.44}$$

with $x \in (0,+\infty)$, and $n \in \mathbb{N}_0$. The characteristic polynomial associated with this equation is $P_2(\lambda) = (\lambda-2)(\lambda+2)$. By applying Theorems 3.4 and 3.6 we can certify that $\left(e_{\alpha,\gamma,n}^{2x}\right)_{n=0}^{\infty}$ and $\left(e_{\alpha,\gamma,n}^{-2x}\right)_{n=0}^{\infty}$ represent two linearly independent solutions of (3.44). Therefore, according to Corollary 3.12, the general solution of (3.44) is given by $(y_n(x))_{n=0}^{\infty}$ with

$$y_n(x) = Ae_{\alpha,\gamma,n}^{2x} + Be_{\alpha,\gamma,n}^{-2x}, \tag{3.45}$$

where A and B are arbitrary constants. Furthermore, by (2.10) it is known that, if $n \to \infty$:

$$Ae_{\alpha,\gamma,n}^{2x} + Be_{\alpha,\gamma,n}^{-2x} \;\to\; \left(\frac{A}{\Gamma(\gamma)}\right) e_{\alpha}^{2x} + \left(\frac{B}{\Gamma(\gamma)}\right) e_{\alpha}^{-2x}, \tag{3.46}$$

uniformly in any compact set contained in $(0,+\infty)$. Finally, by [4], Chapter 7-7.2, the function

$$y(x) = \left(\frac{A}{\Gamma(\gamma)}\right) e_{\alpha}^{2x} + \left(\frac{B}{\Gamma(\gamma)}\right) e_{\alpha}^{-2x}, \tag{3.47}$$

is an expression of the general solution of the following LFDE:

$$\left(\mathcal{D}_{0+}^{2\alpha}y\right)(x) - 4y(x) = 0. \tag{3.48}$$

In conclusion, we were able to find an expression for the general solution of (3.44) that converges uniformly to the solution to the equation (3.48) on compact sets.

Remark 3.14 *Regarding the parameter γ, it was only required that $\mathfrak{Re}(\gamma) > 0$; and in [9] it was observed that $e_{\alpha,1,n}^{\lambda(x-a)} = e_{\alpha}^{\lambda(x-a)}$, then the sequence of general term:*

$$y_n(x) = Ae_{\alpha}^{2x} + Be_{\alpha}^{-2x}, \tag{3.49}$$

with A and B arbitrary constants, is a solution of (3.44), that is: the sequence of functions whose terms are all equal to a solution (in this case the general solution) of the equation (3.48), is a solution of the recurrence equation (3.44).

3.2. Solution of the LFDERR using Generalized Fractional Trigonometric Functions

There are many generalizations of the trigonometric functions; some of thereferences for this section are found in [1, 5]. Recently, fractional trigonometry has attracted great interest, for example, in [5] they are used to model different phenomena that respond to the behavior of spirals. Generalized trigonometric functions are also used to solve fractional differential equations, as seen, for example, in [6, 7]. In this section, we will show that the trigonometric functions (2.13) introduced in [9] are useful for studying the solutions of the LFDERR.

Example 3.15 *We look for the possible solutions of the LFDERRH of order 2α, and $a = 0$:*

$$\left(\mathcal{D}_{0+}^{2\alpha}y_n\right)(x) + a_1\left(\mathcal{D}_{0+}^{\alpha}y_{n+1}\right)(x) + a_0 y_{n+2}(x) = 0, \quad (x > 0). \tag{3.50}$$

Taking $y_n(x) = e_{\alpha,\gamma,n}^{\lambda x}$ in (3.50), the following is verified:

$$0 = \left(\mathcal{D}_{0+}^{2\alpha}e_{\alpha,\gamma,n}^{\lambda t}\right)(x) + a_1\left(\mathcal{D}_{0+}^{\alpha}e_{\alpha,\gamma,n+1}^{\lambda t}\right)(x) + a_0 e_{\alpha,\gamma,n+2}^{\lambda x} = (\lambda^2 + a_1\lambda + a_0)e_{\alpha,\gamma,n+2}^{\lambda x} = P_2(\lambda)e_{\alpha,\gamma,n+2}^{\lambda x}. \tag{3.51}$$

Therefore, the roots of the characteristic polynomial $P_2(\lambda)$ in (3.51), determine the values of λ which $\left(e_{\alpha,\gamma,n}^{\lambda x}\right)_{n=0}^{\infty}$ is a solution of (3.50). There are three possible cases:

1) *If $\lambda_1, \lambda_2 \in \mathbb{R}\setminus\{0\}$, $\lambda_1 \neq \lambda_2$: $\left(e_{\alpha,\gamma,n}^{\lambda_1 x}\right)_{n=0}^{\infty}$ and $\left(e_{\alpha,\gamma,n}^{\lambda_2 x}\right)_{n=0}^{\infty}$ are solutions to (3.50), linearly independent (Theorem 3.6); then the general term of the solution to (3.50) can be written, by Corollary 3.12, as follows:*

$$y_n(x) = c_1 e_{\alpha,\gamma,n}^{\lambda_1 x} + c_2 e_{\alpha,\gamma,n}^{\lambda_2 x}, \tag{3.52}$$

   with c_1 and c_2 arbitrary constants.

2) *When λ_1 and $\lambda_2 \in \mathbb{C}\setminus\{0\}$, that is, $\overline{\lambda_1} = \lambda_2$; then $\left(e_{\alpha,\gamma,n}^{\lambda_1 x}\right)_{n=0}^{\infty}$ and $\left(e_{\alpha,\gamma,n}^{\overline{\lambda_1}x}\right)_{n=0}^{\infty}$ are linearly independent solutions to equation (3.50) (Theorem 3.6), which are sequences of complex functions of real variables, but*

it is possible to obtain real solutions from them. Therefore, since there exists $w \in \mathbb{C}$ *such that* $\lambda_1 = \mathrm{i}\omega$ ($\lambda_2 = -\mathrm{i}\omega$)*, solving the equation (3.50) is equivalent to finding the solution to*

$$\left(\mathcal{D}_{0+}^{2\alpha} y_n\right)(x) + \omega^2 y_{n+2}(x) = 0, \tag{3.53}$$

Furthermore, since $\left(e_{\alpha,\gamma,n}^{\lambda_1 x}\right)_{n=0}^{\infty}$ *and* $\left(e_{\alpha,\gamma,n}^{\overline{\lambda_1} x}\right)_{n=0}^{\infty}$ *are solutions of (3.50), it follows that*

$$\frac{1}{2} e_{\alpha,\gamma,n}^{\lambda_1 x} + \frac{1}{2} e_{\alpha,\gamma,n}^{\overline{\lambda_1} x} = \cos_n^{\alpha,\gamma}(\omega x) \quad ; \quad \frac{1}{2i} e_{\alpha,\gamma,n}^{\lambda_1 x} - \frac{1}{2i} e_{\alpha,\gamma,n}^{\overline{\lambda_1} x} = \sin_n^{\alpha,\gamma}(\omega x), \tag{3.54}$$

by Theorem 3.11, the sequences of functions $(\cos_n^{\alpha,\gamma}(\omega x))_{n=0}^{\infty}$ *and* $(\sin_n^{\alpha,\gamma}(\omega x))_{n=0}^{\infty}$ *are linearly independent solutions to equation (3.50). Then, by Corollary 3.12, it will be possible to write the solution to (3.50) as*

$$(y_n(x))_{n=0}^{\infty} = (c_1 \cos_n^{\alpha,\gamma}(\omega z) + c_2 \sin_n^{\alpha,\gamma}(\omega x))_{n=0}^{\infty}, \tag{3.55}$$

with c_1, c_2 *arbitrary constants.*

3) *Finally, if* $\lambda_1 = \lambda_2$*, then* $(y_n^1(x))_{n=0}^{\infty} = \left(e_{\alpha,\gamma,n}^{\lambda_1 x}\right)_{n=0}^{\infty}$ *is a solution to (3.50), and from this we can obtain another linearly independent with respect to it (Theorems 3.7 and 3.5) defining, for example:*

$$y_n^2(x) = (n-2) y_n^1(x). \tag{3.56}$$

Then, by Corallary 3.12, the following general solution is obtained

$$(y_n(x))_{n=0}^{\infty} = \left(c_1 e_{\alpha,\gamma,n}^{\lambda_1 x} + c_2 (n-2) e_{\alpha,\gamma,n}^{\lambda_1 x}\right)_{n=0}^{\infty}, \tag{3.57}$$

with c_1, c_2 *arbitrary constants.*

Example 3.16 *We analyze the following nonhomogeneous equation whose independent term is a linear combination of* α*-*γ*-*n*-trigonometric functions:*

$$\left[\mathbf{R}_{2\alpha}(y_n(t))_{n=0}^{\infty}\right](x) = A_0 \sin_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] + B_0 \cos_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)], \tag{3.58}$$

where A_0 *and* B_0 *are constants. To solve this equation, a solution* $(y_n^p(x))_{n=2}^{\infty}$ *is proposed, with*

$$y_n^p(x) = r\ \sin_n^{\gamma,\alpha}[\lambda_0(x-a)] + s \cos_n^{\gamma,\alpha}[\lambda_0(x-a)]. \tag{3.59}$$

Hence,

$$\begin{aligned} A_0 \sin_{n+2}^{\gamma;\alpha}[\lambda_0(x-a)] + B_0 \cos_{n+2}^{\gamma;\alpha}[\lambda_0(x-a)] &= \\ = \left[\mathbf{R}_{2\alpha}\left(r \sin_n^{\gamma,\alpha}[\lambda_0(t-a)] + s \cos_n^{\gamma,\alpha}[\lambda_0(t-a)]\right)_{n=2}^{\infty}\right](x) &= \\ = r\left[\mathbf{R}_{2\alpha}\left(\sin_n^{\gamma,\alpha}[\lambda_0(t-a)]\right)_{n=0}^{\infty}\right](x) + s\left[\mathbf{R}_{2\alpha}\left(\cos_n^{\gamma,\alpha}[\lambda_0(t-a)]\right)_{n=0}^{\infty}\right](x)&. \end{aligned} \tag{3.60}$$

Applying (2.16) and (2.17):

$$[\mathbf{R}_{2\alpha}\,(\sin_n^{\gamma,\alpha}[\lambda_0(t-a)])_{n=0}^{\infty}]\,(x) =$$

$$\begin{aligned}
&= \left(\mathcal{D}_{a+}^{2\alpha}\sin_n^{\gamma,\alpha}[\lambda_0(t-a)]\right)(x) + a_1\left(\mathcal{D}_{a+}^{\alpha}\sin_{n+1}^{\gamma,\alpha}[\lambda_0(t-a)]\right)(x) + a_0\sin_{n+2}^{\gamma,\alpha}[\lambda_0(t-a)](x)\\
&= -\lambda_0^2\sin_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] + a_1\lambda_0\cos_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] + a_0\sin_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)]\\
&= \left(a_0-\lambda_0^2\right)\sin_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] + a_1\lambda_0\cos_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)],
\end{aligned} \tag{3.61}$$

and

$$[\mathbf{R}_{2\alpha}\,(\cos_n^{\gamma,\alpha}[\lambda_0(t-a)])_{n=0}^{\infty}]\,(x) =$$

$$\begin{aligned}
&= \left(\mathcal{D}_{a+}^{2\alpha}\cos_n^{\gamma,\alpha}[\lambda_0(t-a)]\right)(x) + a_1\left(\mathcal{D}_{a+}^{\alpha}\cos_{n+1}^{\gamma,\alpha}[\lambda_0(t-a)]\right)(x) + a_0\cos_{n+2}^{\gamma,\alpha}[\lambda_0(t-a)](x)\\
&= -\lambda_0^2\cos_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] - a_1\lambda_0\sin_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] + a_0\cos_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)]\\
&= (a_0-\lambda_0^2)\cos_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] - a_1\lambda_0\sin_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)].
\end{aligned} \tag{3.62}$$

Substituting (3.61) and (3.62) in (3.60), and grouping the terms accordingly, the following is obtained:

$$A_0\sin_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] + B_0\cos_{n+2}^{\gamma,\alpha}[\lambda_0(x-a)] =$$
$$= \left[r(a_0-\lambda_0^2) + s(-a_1)\lambda\right]\sin_{n+2}^{\gamma,\alpha}[\lambda(x-a)] + \left[r(\lambda a_1) + s(a_0-\lambda^2)\right]\cos_{n+2}^{\gamma,\alpha}[\lambda(x-a)]. \tag{3.63}$$

Then, for $y_n^p(x)$ to be solution, it will suffice to take r and s such that

$$\begin{cases} r(a_0-\lambda_0^2) + s(-a_1)\lambda_0 &= A_0\\ r(\lambda_0 a_1) + s(a_0-\lambda_0^2) &= B_0. \end{cases} \tag{3.64}$$

In particular, this procedure can be applied to the equation below:

$$\left(\mathcal{D}_{0+}^{2\alpha}y_n\right)(x) + 3\left(\mathcal{D}_{0+}^{\alpha}y_{n+1}\right)(x) + 2y_{n+2}(x) = 17\cos_{n+2}^{\gamma,\alpha}(x) - 11\sin_{n+2}^{\gamma,\alpha}(x), \tag{3.65}$$

$n\in\mathbb{N}_0$, $0<\alpha\le 1$.

Equation (3.65) is a particular case of equation (3.58), where $a=0$, $\lambda_0=1$, $a_0=2$, $a_1=3$, $A_0=-11$ and $B_0=17$. Therefore, taking into account (3.59) and (3.64), the solution to equation (3.65), will be given by a sequence of general terms

$$y_n(x) = r\sin_n^{\gamma,\alpha}(x) + s\cos_n^{\gamma,\alpha}(x), \tag{3.66}$$

where r and s are such that

$$\begin{cases} r-3s &= -11\\ 3r+s &= 17, \end{cases} \tag{3.67}$$

that is, $r=4$ and $s=5$. Therefore

$$y_n(x) = 4\sin_n^{\gamma,\alpha}(x) + 5\cos_n^{\gamma,\alpha}(x), \quad (n\in\mathbb{N}_0). \tag{3.68}$$

4. Conclusion

We studied the solution of LFDERR in a different way from what is presented in [9], and we showed that we can solve these equations by means of the γ-α-n-Exponential Function. We also established relationships between LFDERR and LFDE through this solution, as shown in Example 3.13. This approach allows us to rethink the already known problems, and study them using LFDERR.

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