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Elastic strips along isotropic curves in complex 3-spaces

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Abstract: In this paper, we define the ruled surface in terms of the Darboux vector field of an isotropic curve in a complex 3-space, and we study the ruled surface as an elastic strip along the isotropic curve. First of all, we show that elastic strips along isotropic curves that serve critical points of the modified Sadowsky functional are characterized by three Euler-Lagrange equations. As a result, we give two conservation laws to characterize elastic strips of isotropic curves in complex 3-space and explain an isotropic helix in terms of the force vector. Finally, we give some examples to illustrate elastic strips with isotropic curves in a complex 3-space.

Key words: Elastic strip, isotropic curve, complex 3-space, conservation law

1. Introduction

An elastic curve is a solution to the variation problem of minimizing the bending energy $\int \kappa(s)^2 ds$ along a curve of a given length, where κ is the curvature of the curve. The problem of describing all planar elastic curves was proposed by Bernoulli and Euler in 1744. Euler studied the problem and showed that there are exactly two closed elastica: the circle and the Euler figure eight. Elastic curves play an important role in areas such as geometry, mathematical physics, differential equations, and complex analysis: for instance, the study of slender biological systems, like DNA, knotted or unknotted proteins [3, 4] or the construction of engineering structures, like cables or pipelines [9].

On the other hand, Sadowsky [8] studied the equilibria of a developable Möbius strip by minimizing the bending energy. He showed that if the developable Möbius strip shrinks to its centerline, the bending energy is reduced to a functional

$$S(\gamma) = \int \kappa^2 \left(1 + \frac{\tau^2}{\kappa^2}\right)^2 ds,$$

where κ and τ are the curvature and the torsion of the centerline γ, which is called the Sadowsky energy functional. It is well-known that a strip defined by a curve with constant curvature and torsion is evidently elastic. Hangan [5] discussed elastic strips generated by curves with nonconstant curvature and torsion. Chubelaschwili and Pinkall [2] gave two conservation laws to describe the equilibrium equations of elastic strips, and they funded new classes of integrable elastic strips which correspond to spherical elastic curves. In the Lorentz version, Tükel and Yücesan [10–12] studied elastic strips with spacelike, timelike, null, and

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pseudo-null curves in Minkowski 3-space, and Yüzbasi and Anco [15] studied elastic strips with null curves in terms of gauge transformation that belongs to the group contained in the Lorentz special orthogonal group in Minkowski 3-space. Wunderlich [13] argued that the Willmore functional $\int H^2 dA$ of an infinitely narrow strip is proportional to the Sadowsky functional.

On the other hand, Langer and Singer [6] considered an energy functional which penalized both the curvature and the torsion of the centerline of an elastic rod. In particular, they showed that the centerline of the Kirchhoff elastic rod is an equilibrium for a linear combination of the conserved Hamiltonians in the localized induction equation hierarchy. As a generalization of [6], Bevilacqua et. al [1] studied an energy density functional depending both on the curvature κ and the torsion τ of the curve $\gamma(s)$, that is, they deal with the following type of an elastic energy functional $\int_\gamma f(s,\kappa,\tau)dl$.

In this paper, we consider elastic strips with an isotropic curve in a complex 3-space. In particular, we study elastic strips as critical points of the modified Sadowsky functional in terms of Euler-Lagrange equations. Also, we provide the first and second conservation laws to characterize elastic strips and give some examples of elastic strips along isotropic helices as isotropic curves in a complex 3-space.

2. Preliminaries

It is well known that the simplest model of a holomorphic Riemannian manifold is a complex space. Let $\mathbb{C}^3$ be a 3-dimensional complex space with the standard holomorphic metric

$$\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 + u_2v_2 + u_3v_3 \tag{2.1}$$

for $\mathbf{u} = (u_1,u_2,u_3)$, $\mathbf{v} = (v_1,v_2,v_3) \in \mathbb{C}^3$. The norm of a vector $\mathbf{u} \in \mathbb{C}^3$ is defined by $||\mathbf{u}|| = \sqrt{|\langle \mathbf{u},\mathbf{u}\rangle|}$.

Now for complex vectors $\mathbf{u} = (u_1,u_2,u_3) = a + \mathrm{i}b$ and $\mathbf{v} = (v_1,v_2,v_3) = c + \mathrm{i}d$, where $a,b,c,d \in \mathbb{R}^3$, we define the cross product of $\mathbf{u}$ and $\mathbf{v}$ by

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= (a\times c - b\times d) + \mathrm{i}(a\times d + b\times c) \\ &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1). \end{aligned} \tag{2.2}$$

The cross product admits the following properties for $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{C}^3$ as:

$$\begin{aligned} &\langle \mathbf{u}\times\mathbf{v},\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{v}\times\mathbf{w}\rangle, \quad \mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}, \\ &\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \langle \mathbf{u},\mathbf{w}\rangle\mathbf{v} - \langle \mathbf{u},\mathbf{v}\rangle\mathbf{w}. \end{aligned} \tag{2.3}$$

A nonzero vector $\mathbf{u} \in \mathbb{C}^3$ is called an isotropic vector if $\langle \mathbf{u},\mathbf{u}\rangle = 0$. If the tangent vector $\gamma'(s)$ of a regular curve $\gamma(s)$ in $\mathbb{C}^3$ is an isotropic vector at every point along the curve $\gamma(s)$, then $\gamma(s)$ is called an isotropic curve. Because it is impossible to normalize the tangent vector of an isotropic curve, we can adopt the so-called pseudo arc length parameter normalizing the acceleration vector $\gamma''(s)$ such that $\langle \gamma''(s),\gamma''(s)\rangle = -1$. Throughout this paper, we assume that isotropic curves are parametrized by the pseudo arc length unless otherwise mentioned. Let $\gamma : \mathbb{C} \to \mathbb{C}^3$ be an isotropic curve with an unique Cartan Frenet frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ such that the Frenet formula is as follows (cf. [7, 14]):

$$\begin{aligned} \mathbf{e}_1'(s) &= -\mathrm{i}\mathbf{e}_2(s), \\ \mathbf{e}_2'(s) &= \mathrm{i}\kappa(s)\mathbf{e}_1(s) + \mathrm{i}\mathbf{e}_3(s), \\ \mathbf{e}_3'(s) &= -\mathrm{i}\kappa(s)\mathbf{e}_2(s), \end{aligned} \tag{2.4}$$

where

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle = 0, \quad \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = \langle \mathbf{e}_1, \mathbf{e}_3 \rangle = 1,$$
$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{i}\mathbf{e}_1, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{i}\mathbf{e}_3, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{i}\mathbf{e}_2.$$

Here the function

$$\kappa(s) = \frac{1}{2}\langle \gamma'''(s), \gamma'''(s) \rangle \tag{2.5}$$

is called the pseudo curvature of the isotropic curve $\gamma(s)$.

For an isotropic curve $\gamma(s)$ in $\mathbb{C}^3$ with the Frenet frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, there is a vector field D satisfying the conditions:

$$\mathbf{e}_1'(s) = D \times \mathbf{e}_1(s),$$
$$\mathbf{e}_2'(s) = D \times \mathbf{e}_2(s),$$
$$\mathbf{e}_3'(s) = D \times \mathbf{e}_3(s).$$

By using (2.4), the vector field D is given by

$$D = D(s) = \kappa(s)\mathbf{e}_1(s) - \mathbf{e}_3(s)$$

which is called the Darboux vector field of an isotropic curve $\gamma(s)$.

Definition 2.1 *Let $\gamma = \gamma(s)$ be an isotropic curve in complex 3-space with Frenet frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. If there exists a nonzero constant vector $\mathbf{u}$ in complex 3-space such that $\langle \mathbf{e}_1, \mathbf{u} \rangle$ is a (complex) constant, it is called an isotropic helix, and $\mathbf{u}$ is called the axis of γ.*

3. Elastic strips

The classical curve known as the elastic strip is the solution to a variational problem that of minimizing the bending energy of a thin inextensible wire. For a detailed description of elastic strips in complex 3-space $\mathbb{C}^3$, we first introduce a developable ruled surface.

Let $\gamma : [0, L] \to \mathbb{C}^3$ be an isotropic curve with length L in $\mathbb{C}^3$. Then we consider the ruled surface parametrized as follows:

$$\begin{aligned} \Gamma_\gamma : [0, L] \times (-\varepsilon, \varepsilon) &\to \mathbb{C}^3 \\ (s, t) &\to \Gamma_\gamma(s, t) = \gamma(s) + tD(s), \end{aligned} \tag{3.1}$$

where D is the Darboux vector field of $\gamma(s)$.

On the other hand, we know that infinitely narrow strips are critical points of the Willmore functional $E(\Gamma_\gamma) = \int_M H^2 dA$ among all space curves with fixed endpoints in Euclidean 3-space, where H is the mean curvature of Γ_γ. In [13] Wunderlich showed that $\lim_{\epsilon -> 0} \int_M H^2 dA$ is proportional to the Sadowsky functional

$$S(\gamma) = \int_0^L \kappa^2 (1 + \lambda^2)^2 ds, \tag{3.2}$$

where $\lambda = \frac{\tau}{\kappa}$.

Now we discuss elastic strips along an isotropic curve corresponding to the Sadowsky functional (3.2) and the condition

$$\delta L := \frac{\partial}{\partial t}|_{t=0} L(\gamma_s) = 0. \tag{3.3}$$

Therefore we can define elastic strips with an isotropic curve as follows:

Definition 3.1 *A strip Γ_γ is elastic if an isotropic curve γ in complex 3-space is a critical point of the modified Sadowsky functional*

$$S_\mu(\gamma) = \int_0^L (\kappa^2(1+\rho^2)^2 - \mu)\kappa^{\frac{1}{2}} ds, \tag{3.4}$$

where μ is a Lagrange multiplier standing for the length constraint and $\rho = \frac{1}{\kappa}$.

Lemma 3.2 *Let γ_0 be an isotropic curve in complex 3-space $\mathbb{C}^3$ and $\gamma_t(s) = \gamma(s,t)$ be a variation of γ_0 with a variational vector field*

$$\frac{\partial}{\partial t}\gamma_t(s)|_{t=0} = p_1(s)\mathbf{e}_1(s) + p_2(s)\mathbf{e}_2(s) + p_3(s)\mathbf{e}_3(s). \tag{3.5}$$

Then we have

$$\begin{aligned} \delta\kappa &= \kappa' p_1 + (\mathrm{i}\kappa'' + 4\mathrm{i}\kappa^2)p_2 + 5\kappa\kappa' p_3 + 6\kappa p_1' + p_1''' + 3\mathrm{i}\kappa' p_2' + 6\mathrm{i}\kappa p_2'' + 6\kappa^2 p_3' + \kappa p_3''', \\ \delta\rho &= -\rho^2\kappa' p_1 - (\mathrm{i}\rho^2\kappa'' + 4\mathrm{i})p_2 - 5\rho\kappa' p_3 - 6\rho p_1' - \rho^2 p_1''' - 3\mathrm{i}\rho^2\kappa' p_2' - 6\mathrm{i}\rho p_2'' - 6p_3' - \rho p_3'''. \end{aligned} \tag{3.6}$$

Proof Let γ_0 be an isotropic curve with Frenet frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ satisfying (2.4) in complex 3-space $\mathbb{C}^3$. We put the vector field along γ_0 as

$$W(s) := \frac{\partial}{\partial t}\gamma_t(s)|_{t=0}.$$

Then one finds

$$\langle W'(s), \mathbf{e}_1(s)\rangle = 0.$$

On the other hand, the vector field $W(s)$ can be expressed by

$$W(s) = p_1(s)\mathbf{e}_1(s) + p_2(s)\mathbf{e}_2(s) + p_3(s)\mathbf{e}_3(s),$$

where $p_i(s)$ $(i = 1,2,3)$ are smooth functions. From this, we get the third derivatives of $W(s)$ as follows:

$$\begin{aligned} W'''(s) = &\big((p_1' + \mathrm{i}\kappa p_2)'' + (\kappa p_1 + \kappa^2 p_3 + \mathrm{i}\kappa p_2')' + \mathrm{i}\kappa(p_2' - \mathrm{i}p_1 - \mathrm{i}\kappa p_3)' \\ &+\mathrm{i}\kappa(2\kappa p_2 - \mathrm{i}p_1' - \mathrm{i}\kappa p_3')\big)\,\mathbf{e}_1 \\ &+ \big((p_2' - \mathrm{i}p_1 - \mathrm{i}\kappa p_3)'' + (2\kappa p_2 - \mathrm{i}p_1' - \mathrm{i}\kappa p_3')' - \mathrm{i}(p_1' + \mathrm{i}\kappa p_2)' \\ &+2(\kappa p_2' - \mathrm{i}\kappa p_1 - \mathrm{i}\kappa^2 p_3) - \mathrm{i}\kappa(p_3' + \mathrm{i}p_2)'\big)\,\mathbf{e}_2 \\ &+ \big((p_3' + \mathrm{i}p_2)'' + 2(p_1 + \kappa p_3 + \mathrm{i}p_2')' + p_1' + \kappa p_3' + 2\mathrm{i}\kappa p_2\big)\,\mathbf{e}_3. \end{aligned}$$

From (2.5) and the last equation, the first variations of κ and ρ become

$$\delta\kappa := \frac{\partial \kappa_t}{\partial t}|_{t=0} = \kappa\langle W'''(s), \mathbf{e}_1\rangle + \langle W'''(s), \mathbf{e}_3\rangle,$$
$$\delta\rho := \frac{\partial \rho_t}{\partial t}|_{t=0} = -\rho\langle W'''(s), \mathbf{e}_1\rangle - \rho^2\langle W'''(s), \mathbf{e}_3\rangle,$$

where κ_t denotes the pseudo curvature of the curve $\gamma_t(s)$ and $\rho_t = \frac{1}{\kappa_t}$, which imply that we can obtain (3.6).

□

Now we may compute the first variation of the modified Sadowsky functional by using the variation of γ having the variation vector field (3.5).

Theorem 3.3 *Let γ be an isotropic curve in complex 3-space and define an elastic strip. Consider a variation of γ with the variation vector field*

$$\frac{\partial}{\partial t}\gamma_t(s)|_{t=0} = p_1(s)\mathbf{e}_1(s) + p_2(s)\mathbf{e}_2(s) + p_3(s)\mathbf{e}_3(s),$$

then we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\kappa_t^2(1+\rho_t^2)^2 - \mu\right)\kappa_t^{\frac{1}{2}}|_{t=0} = \varphi_1 p_1 + \varphi_2 p_2 + \varphi_3 p_3 + \varphi', \tag{3.7}$$

where

$$\begin{aligned}
\varphi_1 = {} & \frac{1}{4}\kappa'(5(1+\rho^2)^2 - \mu) - 2\kappa'\rho^3(1+\rho^2) - \frac{3}{2}(5\kappa(1+\rho^2)^2 - \kappa\mu)' + 12(\rho^2(1+\rho^2))' \\
& - 5(\rho\rho'(1+\rho^2))'' + 2(\rho^3(1+\rho^2))''', \\
\varphi_2 = {} & (5(1+\rho^2)^2 - \mu)(\frac{1}{4}\mathrm{i}\kappa'' + \mathrm{i}\kappa^2) - 2\rho(1+\rho^2)(\mathrm{i}\rho^2\kappa'' + 4\mathrm{i}) - \frac{3}{4}\mathrm{i}(5\kappa'(1+\rho^2)^2 - \kappa'\mu)' \\
& + 6\mathrm{i}(\kappa'\rho^3 + \kappa'\rho^5)' + \frac{3}{2}\mathrm{i}\left(5\kappa(1+\rho^2)^2 - \kappa\mu\right)'' - 12\mathrm{i}\left(\rho^2(1+\rho^2)\right)'', \\
\varphi_3 = {} & \frac{5}{4}\kappa\kappa'(5(1+\rho^2)^2 - \mu) - 10\rho^2\kappa'(1+\rho^2) - \frac{3}{2}(5\kappa^2(1+\rho^2)^2 - \kappa^2\mu)' \\
& + 12\rho'(1+3\rho^2) + 4\left(\rho\rho'(1+2\rho^2)\right)'' - \frac{1}{4}\left(5\kappa(1+\rho^2)^2 - \kappa\mu\right)''',
\end{aligned}$$

$$\begin{aligned}\varphi' = &\left(\left(\frac{3}{2}\kappa(5(1+\rho^2)^2-\mu)-12\rho^2(1+\rho^2)+5(\rho\rho'(1+\rho^2))'-2(\rho^3(1+\rho^2))''\right)p_1\right)'\\
&+\left((-5\rho\rho'(1+\rho^2)+2(\rho^3(1+\rho^2))')p_1'\right)'+\left(\left(\frac{1}{4}(5(1+\rho^2)^2-\mu)-2\rho^3(1+\rho^2)\right)p_1''\right)'\\
&+\left(\left(\frac{3}{4}\mathrm{i}\kappa'(5(1+\rho^2)^2-\mu)-6\mathrm{i}\kappa'\rho^3(1+\rho^2)-\frac{3}{2}\mathrm{i}\left(5\kappa(1+\rho^2)^2-\kappa\mu\right)'+12\mathrm{i}\left(\rho^2(1+\rho^2)\right)'\right)p_2\right)'\\
&+\left(\left(\frac{3}{2}\mathrm{i}\kappa(5(1+\rho^2)^2-\mu)-12\mathrm{i}\rho^2(1+\rho^2)\right)p_2'\right)'\\
&+\left(\left(\frac{3}{2}\kappa^2(5(1+\rho^2)^2-\mu)-12\rho(1+\rho^2)+\frac{1}{4}\left(5\kappa(1+\rho^2)^2-\kappa\mu\right)''-4\left(\rho\rho'(1+2\rho^2)\right)'\right)p_3\right)'\\
&+\left(\left(-\frac{1}{4}\left(5\kappa(1+\rho^2)^2-\kappa\mu\right)'+4\rho\rho'(1+2\rho^2)\right)p_3'\right)'+\left(\left(\frac{1}{4}\kappa(5(1+\rho^2)^2-\mu)-2\rho^2(1+\rho^2)\right)p_3''\right)'.\end{aligned} \tag{3.8}$$

Proof If an isotropic curve γ defines an elastic strip, then we have

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left(\kappa_t^2(1+\rho_t^2)^2-\mu\right)\kappa_t^{\frac{1}{2}}|_{t=0}\\
&=\frac{1}{4}((1+\rho^2)^2-\mu)(\delta\kappa)+(1+\rho^2)^2(\delta\kappa)+2\rho(1+\rho^2)(\delta\rho).\end{aligned} \tag{3.9}$$

Now we want to express (3.9) in terms of the functions $p_i (i=1,2,3)$ and its derivatives. By using (3.6) and long computing, the right hand side of equation (3.9) can be written as follows:

$$\begin{aligned}&\left(\frac{1}{4}\kappa'(5(1+\rho^2)^2-\mu)-2\kappa'\rho^3(1+\rho^2)\right)p_1+\left(\frac{3}{2}\kappa(5(1+\rho^2)^2-\mu)-12\rho^2(1+\rho^2)\right)p_1'\\
&+\left(\frac{1}{4}(5(1+\rho^2)^2-\mu)-2\rho^3(1+\rho^2)\right)p_1'''\\
&+\left((5(1+\rho^2)^2-\mu)(\frac{1}{4}\mathrm{i}\kappa''+\mathrm{i}\kappa^2)-2\rho(1+\rho^2)(\mathrm{i}\rho^2\kappa''+4\mathrm{i})\right)p_2\\
&+\left(\frac{3}{4}\mathrm{i}\kappa'(5(1+\rho^2)^2-\mu)-6\mathrm{i}\kappa'\rho^3(1+\rho^2)\right)p_2'+\left(\frac{3}{2}\mathrm{i}\kappa(5(1+\rho^2)^2-\mu)-12\mathrm{i}\rho^2(1+\rho^2)\right)p_2''\\
&+\left(\frac{5}{4}\kappa\kappa'(5(1+\rho^2)^2-\mu)-10\rho^2\kappa'(1+\rho^2)\right)p_3+\left(\frac{3}{2}\kappa^2(5(1+\rho^2)^2-\mu)-12\rho(1+\rho^2)\right)p_3'\\
&+\left(\frac{1}{4}\kappa(5(1+\rho^2)^2-\mu)-2\rho^2(1+\rho^2)\right)p_3'''.\end{aligned} \tag{3.10}$$

We put

$$\begin{aligned}\mathcal{A} :&= \varphi_1 p_1 + (f_1 p_1)' + (g_1 p_1')' + \left(\left(\frac{1}{4}(5(1+\rho^2)^2-\mu) - 2\rho^3(1+\rho^2)\right) p_1''\right)' \\ &= (f_1' + \varphi_1)p_1 + (f_1 + g_1')p_1' + \left(g_1 + \left(\frac{1}{4}(5(1+\rho^2)^2-\mu) - 2\rho^3(1+\rho^2)\right)'\right) p_1'' \\ &\quad + \left(\frac{1}{4}(5(1+\rho^2)^2-\mu) - 2\rho^3(1+\rho^2)\right) p_1''',\end{aligned} \tag{3.11}$$

where f_1, g_1 and φ_1 are smooth functions with the variable s.

By comparing the coefficients of p_1'' in (3.10) and (3.11), we have

$$g_1 = -5\rho\rho'(1+\rho^2) + 2(\rho^3(1+\rho^2))', \tag{3.12}$$

it follows that we can find f_1 and φ_1 with the help of (3.10), (3.11), and (3.12) as follows:

$$\begin{aligned} f_1 &= \frac{3}{2}\kappa(5(1+\rho^2)^2-\mu) - 12\rho^2(1+\rho^2) + 5(\rho\rho'(1+\rho^2))' - 2(\rho^3(1+\rho^2))'', \\ \varphi_1 &= \frac{1}{4}\kappa'(5(1+\rho^2)^2-\mu) - 2\kappa'\rho^3(1+\rho^2) - \frac{3}{2}(5\kappa(1+\rho^2)^2 - \kappa\mu)' + 12(\rho^2(1+\rho^2))' \\ &\quad - 5(\rho\rho'(1+\rho^2))'' + 2(\rho^3(1+\rho^2))'''. \end{aligned} \tag{3.13}$$

Applying the same algebraic method as above, we also obtain

$$\begin{aligned} \mathcal{B} &:= \varphi_2 p_2 + (f_2 p_2)' + \left(\left(\frac{3}{2}\mathrm{i}\kappa(5(1+\rho^2)^2-\mu) - 12\mathrm{i}\rho^2(1+\rho^2)\right) p_2'\right)', \\ \mathcal{C} &:= \varphi_3 p_3 + (f_3 p_3)' + (g_3 p_3')' + \left(\left(\frac{1}{4}\kappa(5(1+\rho^2)^2-\mu) - 2\rho^2(1+\rho^2)\right) p_3''\right)', \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \varphi_2 &= (5(1+\rho^2)^2-\mu)(\frac{1}{4}\mathrm{i}\kappa'' + \mathrm{i}\kappa^2) - 2\rho(1+\rho^2)(\mathrm{i}\rho^2\kappa'' + 4\mathrm{i}) - \frac{3}{4}\mathrm{i}(5\kappa'(1+\rho^2)^2 - \kappa'\mu)' \\ &\quad + 6\mathrm{i}(\kappa'\rho^3 + \kappa'\rho^5)' + \frac{3}{2}\mathrm{i}\left(5\kappa(1+\rho^2)^2 - \kappa\mu\right)'' - 12\mathrm{i}\left(\rho^2(1+\rho^2)\right)'', \\ \varphi_3 &= \frac{5}{4}\kappa\kappa'(5(1+\rho^2)^2-\mu) - 10\rho^2\kappa'(1+\rho^2) - \frac{3}{2}(5\kappa^2(1+\rho^2)^2 - \kappa^2\mu)' \\ &\quad + 12\rho'(1+3\rho^2) + 4\left(\rho\rho'(1+2\rho^2)\right)'' - \frac{1}{4}\left(5\kappa(1+\rho^2)^2 - \kappa\mu\right)''', \end{aligned} \tag{3.15}$$

$$\begin{aligned} f_2 &= \frac{3}{4}\mathrm{i}\kappa'(5(1+\rho^2)^2-\mu) - 6\mathrm{i}\kappa'\rho^3(1+\rho^2) - \frac{3}{2}\mathrm{i}\left(5\kappa(1+\rho^2)^2 - \kappa\mu\right)' + 12\mathrm{i}\left(\rho^2(1+\rho^2)\right)', \\ f_3 &= \frac{3}{2}\kappa^2(5(1+\rho^2)^2-\mu) - 12\rho(1+\rho^2) + \frac{1}{4}\left(5\kappa(1+\rho^2)^2 - \kappa\mu\right)'' - 4\left(\rho\rho'(1+2\rho^2)\right)', \\ g_3 &= -\frac{1}{4}\left(5\kappa(1+\rho^2)^2 - \kappa\mu\right)' + 4\rho\rho'(1+2\rho^2). \end{aligned} \tag{3.16}$$

Thus (3.12), (3.13), (3.15), and (3.16) imply (3.7). □

Theorem 3.4 *1. The critical points of the modified Sadowsky functional S_μ are characterized by the Euler-Lagrange equations $\varphi_1 = \varphi_2 = \varphi_3 = 0$.*

2. If the isotropic curve γ is a critical point of S_μ, then for each variation of γ with respect to the integrand $(\kappa_t^2(1+\rho_t^2)^2 - \mu)\kappa_t^{\frac{1}{2}}$ of the modified Sadowsky functional, we have $\varphi' = 0$.

Proof 1. If γ is a critical point of the modified Sadowsky functional S_μ and it is parametrized by pseudo arc length, then from (3.7) one finds

$$\begin{aligned} 0 = \frac{\partial}{\partial t}|_{t=0} S_\mu(\gamma_t) &= \int_0^L (p_1\varphi_1 + p_2\varphi_2 + p_3\varphi_3 + \varphi')ds \\ &= \int_0^L (p_1\varphi_1 + p_2\varphi_2 + p_3\varphi_3)ds + \varphi(L) - \varphi(0). \end{aligned}$$

Since $\varphi(L) = \varphi(0) = 0$, we can obtain the required Euler-Lagrange equations $p_1 = p_2 = p_3 = 0$.

2. Assume that the isotropic curve γ is a critical point of S_μ. Then the curve satisfies the Euler-Lagrange equations. In this case, the invariance of $(\kappa_t^2(1+\rho_t^2)^2 - \mu)\kappa_t^{\frac{1}{2}}$ with respect to t leads to

$$\begin{aligned} 0 &= \frac{1}{2}\frac{\partial}{\partial t}\left(\kappa_t^2(1+\rho_t^2)^2 - \mu\right)\kappa_t^{\frac{1}{2}}|_{t=0} \\ &= p_1\varphi_1 + p_2\varphi_2 + p_3\varphi_3 + \varphi'(s) \\ &= \varphi'(s). \end{aligned}$$

□

4. Conservation laws

In this section, we study conservation laws to characterize elastic strips of isotropic curves in complex 3-space $\mathbb{C}^3$.

Consider a variation

$$\gamma_t(s) = \gamma(s) + t\mathcal{B}$$

with the variation vector field

$$\begin{aligned} \dot{\gamma} = \mathcal{B} &= \langle \mathcal{B}, \mathbf{e}_3\rangle \mathbf{e}_1 + \langle \mathcal{B}, \mathbf{e}_2\rangle \mathbf{e}_2 + \langle \mathcal{B}, \mathbf{e}_1\rangle \mathbf{e}_3 \\ &= p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3, \end{aligned} \tag{4.1}$$

where $\mathcal{B}$ is an arbitrary point in $\mathbb{C}^3$.

Before dealing with the first conservation law of elastic strips for an isotropic curve, we work out the following:

$$\begin{aligned} p_1' &= -\mathrm{i}\kappa\langle \mathcal{B}, \mathbf{e}_2\rangle, \\ p_1'' &= \kappa^2\langle \mathcal{B}, \mathbf{e}_1\rangle - \mathrm{i}\kappa'\langle \mathcal{B}, \mathbf{e}_2\rangle + \kappa\langle \mathcal{B}, \mathbf{e}_3\rangle, \\ p_2' &= \mathrm{i}\kappa\langle \mathcal{B}, \mathbf{e}_1\rangle + \mathrm{i}\langle \mathcal{B}, \mathbf{e}_3\rangle, \\ p_3' &= -\mathrm{i}\langle \mathcal{B}, \mathbf{e}_2\rangle, \\ p_3'' &= \kappa\langle \mathcal{B}, \mathbf{e}_1\rangle + \langle \mathcal{B}, \mathbf{e}_3\rangle. \end{aligned} \tag{4.2}$$

Substituting (4.2) into (3.8), we can arrange as follows:

$$\varphi = \langle \mathcal{B}, Q_0 \rangle,$$

where

$$\begin{aligned} Q_0 = &\left(\frac{\kappa^2}{2}(5(1+\rho^2)^2 - \mu) - 2\rho(1+\rho^2)(\kappa\rho - 2)(\kappa\rho - 3) - 4(\rho\rho'(1+\rho^2))' + \frac{1}{4}(5\kappa(1+\rho^2)^2 - \kappa\mu)'' \right) \mathbf{e}_1, \\ &+ \left(\frac{\mathrm{i}}{2}\kappa'(5(1+\rho^2)^2 - \mu) + \mathrm{i}\rho(1+\rho^2)(5\kappa\rho' - 4\kappa'\rho^2 - 4\rho') - 4\mathrm{i}\rho^3\rho' + 12\mathrm{i}(\rho^2(1+\rho^2))' \right. \\ &\left. - 2\mathrm{i}\kappa(\rho^3(1+\rho^2))' - \frac{5}{4}\mathrm{i}(5\kappa(1+\rho^2)^2 - \kappa\mu)' \right) \mathbf{e}_2 \\ &+ \left(\frac{\kappa}{4}(5(1+\rho^2)^2 - \mu) - 12\kappa\rho^3(1+\rho^2) + 5(\rho\rho'(1+\rho^2))' - 2(\rho^3(1+\rho^2))'' \right) \mathbf{e}_3. \end{aligned} \tag{4.3}$$

We know that if the isotropic curve γ is a critical point of the modified Sadowsky functional S_μ, φ is a constant. Therefore, Q_0 is a constant for any $\mathcal{B} \in \mathbb{C}^3$. Thus, we characterize elastic strips with isotropic curves in terms of Q_0 and we have the following result.

Theorem 4.1 (First conservation law of elastic strips with an isotropic curve) *A strip with an isotropic curve in complex 3-space $\mathbb{C}^3$ is elastic if and only if the force vector Q_0 given by (4.3) is a constant.*

Proof Let us set the force vector Q_0 given by (4.3) as follows:

$$Q_0 = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3. \tag{4.4}$$

We now use the Frenet formula (2.4) to obtain

$$Q_0' = (q_1' + \mathrm{i}\kappa q_2)\mathbf{e}_1 + (q_2' - \mathrm{i}q_1 - \mathrm{i}\kappa q_3)\mathbf{e}_2 + (q_3' + \mathrm{i}q_2)\mathbf{e}_3.$$

By a long computation, we can show that the coefficients of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ can be expressed as

$$\begin{aligned} q_1' + \mathrm{i}\kappa q_2 &= \varphi_3, \\ q_2' - \mathrm{i}q_1 - \mathrm{i}\kappa q_3 &= \varphi_2, \\ q_3' + \mathrm{i}q_2 &= \varphi_1. \end{aligned}$$

We know that if the strip Γ_γ with the isotropic curve γ is elastic, then $\varphi_1 = \varphi_2 = \varphi_3 = 0$, which implies that the force vector Q_0 is a constant. If Q_0 is a constant, the last equations lead to $\varphi_1 = \varphi_2 = \varphi_3 = 0$, it follows that the strip Γ_γ is elastic, thus the proof is completed. □

Proposition 4.2 *If an isotropic curve γ defines an elastic strip such that q_3 is a constant, then γ is an isotropic helix.*

Proof For an elastic strip, Q_0 is a constant. Since $\langle Q_0, \mathbf{e}_1 \rangle = q_3$ is a constant, γ is an isotropic helix. □

By taking into account that

$$\frac{\partial}{\partial t}A_t(\gamma(s))|_{t=0} = \mathcal{C}\times\gamma(s) = \langle \mathcal{C}\times\gamma(s), \mathbf{e}_3\rangle \mathbf{e}_1 + \langle \mathcal{C}\times\gamma(s), \mathbf{e}_2\rangle \mathbf{e}_2 + \langle \mathcal{C}\times\gamma(s), \mathbf{e}_1\rangle \mathbf{e}_3,$$
$$= p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3.$$

for $\mathcal{C}\in\mathbb{C}^3$ and $A_t \in SU(3)$. Using a similar method as in the first conservation law, we have

$$\varphi = \langle \mathcal{C}, Q_1\rangle,$$

where

$$\begin{aligned} Q_1 = &\left(-\frac{3\kappa}{4}(5(1+\rho^2)^2-\mu) - 2\rho^2(1+\rho^2)(2\kappa\rho-5)\right)\mathbf{e}_1 \\ &+ (5\mathrm{i}\rho\rho'(1+\rho^2) - 2\mathrm{i}(\rho^3(1+\rho^2))')\mathbf{e}_2 \\ &+ \left(\frac{1}{4}(5(1+\rho^2)^2-\mu) - 2\rho^3(1+\rho^2)\right)\mathbf{e}_3 - \gamma\times Q_0. \end{aligned} \tag{4.5}$$

Now, we characterize elastic strips with isotropic curves in terms of Q_1 and we have the following result.

Theorem 4.3 (Second conservation law of elastic strips with an isotropic curve) *A strip with an isotropic curve in complex 3-space $\mathbb{C}^3$ is elastic if and only if the torque vector Q_1 given by (4.5) is a constant.*

Proof Let us put the torque vector Q_1 given by (4.5) as follows:

$$Q_1 = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 - \gamma\times Q_0,$$

it follows that from (4.4) we have

$$\begin{aligned} Q_1' &= (u_1' + \mathrm{i}\kappa u_2)\mathbf{e}_1 + (u_2' - \mathrm{i}u_1 - \mathrm{i}\kappa u_3)\mathbf{e}_2 + (u_3' + \mathrm{i}u_2)\mathbf{e}_3 - \mathbf{e}_1\times Q_0 - \gamma\times Q_0' \\ &= (u_1' + \mathrm{i}\kappa u_2 - \mathrm{i}q_2)\mathbf{e}_1 + (u_2' - \mathrm{i}u_1 - \mathrm{i}\kappa u_3 - \mathrm{i}q_3)\mathbf{e}_2 + (u_3' + \mathrm{i}u_2)\mathbf{e}_3 - \gamma\times Q_0'. \end{aligned}$$

By using (4.3) and (4.5), we show that the coefficients of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are identically zero, that is, the last equation is reduced to

$$Q_1' = -\gamma\times Q_0'.$$

Thus, Q_0 is a constant if and only if the isotropic curve γ defines the elastic strip. □

5. Examples of elastic strips in $\mathbb{C}^3$

In this section, we give examples to indicate elastic strips along isotropic helices as isotropic curves in a complex 3-space.

Example 5.1 *Let us consider an isotropic curve*

$$\gamma(s) = \left(\frac{1}{3}(2\cos s + \sqrt{13}s), \mathrm{i}s, \frac{\mathrm{i}}{3}(\sqrt{13}\cos s + 2s)\right)$$

in complex 3-space. Then we have the Frenet frame of the curve γ *as follows:*

$$\mathbf{e}_1 = \left(\frac{1}{3}(-2\sin s + \sqrt{13}), \mathrm{i}\cos s, \frac{\mathrm{i}}{3}(-\sqrt{13}\sin s + 2)\right),$$

$$\mathbf{e}_2 = \left(-\frac{2\mathrm{i}}{3}\cos s, \sin s, \frac{\sqrt{13}}{3}\cos s\right),$$

$$\mathbf{e}_3 = \left(\frac{1}{3}(\sin s + \frac{\sqrt{13}}{2}), -\frac{\mathrm{i}}{2}\cos s, \frac{\mathrm{i}}{3}(\frac{\sqrt{13}}{2}\sin s + 1)\right).$$

In particular, the curve is an isotropic helix with the pseudo curvature $\kappa = -\frac{1}{2}$ *and the axis* $\mathbf{u} = (\frac{\sqrt{13}}{3}, 0, \frac{2\mathrm{i}}{3})$. *Thus, the strip with the isotropic helix is given by*

$$\Gamma_\gamma(s,t) = \left(\frac{1}{3}(2\cos s + \sqrt{13}s - \sqrt{13}t), \mathrm{i}\sin s, \frac{\mathrm{i}}{3}(\sqrt{13}\cos s + 2s - 2t)\right).$$

Since γ *satisfies the Euler-Lagrange equation, the strip* $\Gamma_\gamma(s,t)$ *is an elastic strip with the isotropic helix* γ. *Also, the critical point of the modified Sadowsky functional corresponds to the isotropic elastic curve.*

Example 5.2 *Consider an isotropic curve in complex 3-space as*

$$\gamma(s) = (\sinh s, \mathrm{i}\cosh s, \mathrm{i}s).$$

In this case, Frenet frames of the isotropic curve γ *are given by*

$$\mathbf{e}_1 = (\cosh s, \mathrm{i}\sinh s, \mathrm{i}),$$

$$\mathbf{e}_2 = (\mathrm{i}\sinh s, -\cosh s, 0),$$

$$\mathbf{e}_3 = (\frac{1}{2}\cosh s, \frac{\mathrm{i}}{2}\sinh s, -\frac{\mathrm{i}}{2}),$$

it follows that the curve is an isotropic helix with the pseudo curvature $\kappa = \frac{1}{2}$ *and the axis* $\mathbf{u} = (0, 0, -\mathrm{i})$. *Thus, the elastic strip is parametrized by*

$$\Gamma_\gamma(s,t) = (\sinh s, \mathrm{i}\cosh s, \mathrm{i}(s+t)).$$

6. Conclusion

First of all, we introduce the modified Sadowsky functional along an isotropic curve in complex 3-space corresponding to Sadowsky functional in Euclidean and Minkowski 3-spaces. From this, we study an elastic strip in terms of the modified Sadowsky functional and the associated variational vector field of an isotropic curve in complex 3-space. Furthermore, we show that the critical points of the modified Sadowsky functional are characterized by the Euler-Lagrange equations. Also, we used the force vector and the torque vector to give the physical properties of an elastic strip and we give some applications of soliton solutions of the elastic strips for the isotropic curves in complex 3-space.

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