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Some new results on generalized Hyers-Ulam stability in modular function spaces

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Abstract: In this work, we present a new weighted method for proving the generalized Hyers-Ulam stability for nonlinear Volterra integral equations in modular spaces. Using the same technique, we also prove the generalized Hyers-Ulam stability for nonlinear functional equations under Δ_2 conditions. Fixed-point theorems in modular spaces form the foundation of our main conclusions.

Key words: Fixed point, modular function space, nonlinear functional equation, Volterra integral equation, weighted space method

1. Introduction

The theory of modular spaces has a wide range of applications, particularly in interpolation [30]. Nakano introduced modular spaces as a generalization of metric spaces in 1950 [35]. Later, Luxemburg extended the concept of modular spaces by equipping them with a norm [29]. The Musielak-Orlicz space was another generalization of the modular spaces introduced in 1959 [31, 33, 34]. For more detailed information, please refer to the book by Kannappan [37].

The concept of stability is a qualitative aspect of dynamic systems. In 1954, Ulam developed the idea of stability by posing queries regarding the degree to which an approximate solution of an equation approaches the exact solution [36]. A year later, Hyers investigated the stability of linear functional equations [19]. For this reason, the acquired characteristic is now referred to as the Hyers-Ulam stability [41]. Rassias, in 1978, investigated the stability of linear mapping in Banach space [38]. References [1, 3, 7, 13–16, 18, 23, 28, 32, 42] provide a wealth of information regarding the stability of Ulam-Hyers and Hyres-Ulam-Rassias. In 2017, Castro and Simões introduced a new type of stability called semi-Hyers-Ulam stability [10]. They investigated this type of stability for a class of integrodifferential equations in a generalized metric space endowed with Bielecki metric and fixed point theorems [11]. Moreover, stability of the Apollonius-type additive functional equation [27], refined stability of additive and quadratic functional equations [26], fixed point approach to functional equation stability [39], generalized Ulam-Hyers-Rassias stability for quartic functional equation [44], and generalized Hyers-Ulam stability of Cauchy mappings [16] are noteworthy collaborations in the field of stability in modular spaces.

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The Volterra integral equation is one of the significant equations whose stability has recently drawn attention. In 1896, Vito Volterra started studying equations [43]. Practical fields, including fluid flow, semiconductors, heat conduction, elasticity, chemical reactions, scattering theory, population dynamics, and seismology, all make use of this equation [9]. Furthermore, Burton released a book in 2005 that delves further into the application of this kind of integral equation in differential equations [6]. See [12, 17, 24, 40] for additional details on this kind of integral equation. The initial section of this paper explores the generalized Hyres-Ulam stability for a particular kind of this equation. If $\hbar$ is a continuous function and constant $a \in \mathbb{R}$, then the following integral equation:

$$\hbar(\kappa) = \int_a^\kappa \mathcal{T}(\tau, \hbar(\tau))\, \mathrm{d}\tau \tag{1.1}$$

is called a Volterra integral equation. In 2007, Jung conducted a study to analyze the stability of Hyers-Ulam and Hyres-Ulam-Rassias concerning this equation [24].Additionally, in 2009, Castro and his colleagues investigated the stability of the nonlinear Volterra integral equation in the Banach space of the form:

$$\hbar(\kappa) = \int_a^\kappa \mathcal{T}(\kappa, \tau, \hbar(\tau))\, \mathrm{d}\tau, \qquad -\infty < a \leq \kappa \leq b < +\infty, \tag{1.2}$$

such that $a, b \in \mathbb{R}$ and $\hbar$ is a continuous function [9]. It is worth noting that Eq. (1.2) is more global than Eq. (1.1). The stability of the nonlinear Volterra integral equation using the weighted space method has not yet been investigated. In this work, we will first explore the generalized Hyres-Ulam stability of the following Volterra integral equation:

$$\hbar(r) = \mathbf{f}(r) + \lambda \int_a^x \mathcal{U}(r, s, \hbar(r)),\ \forall r, s \in \mathcal{J} = [a, b]; \tag{1.3}$$

where $\mathcal{U} : \mathcal{J} \times \mathcal{J} \times \mathcal{L}^\varphi \to \mathcal{L}^\varphi$, $\hbar : \mathcal{J} \to \mathcal{L}^\varphi$ is a continuous, $\mathbf{f}$ is a real-valued function, and $\lambda \in \mathbb{R}$. Subsequently, we will investigate the generalized Hyers-Ulam stability for the following nonlinear equation:

$$\hbar(r) = \mathcal{T}\big(r, \hbar(r), \hbar(\eta(r))\big) \tag{1.4}$$

where $\mathcal{T} : \mathcal{S} \times \mathcal{G}_\mathbf{m} \times \mathcal{G}_\mathbf{m} \to \mathcal{G}_\mathbf{m}$ and $\eta : \mathcal{S} \to \mathcal{S}$ are given mappings and $\mathcal{S} \neq \emptyset$, $\mathcal{G}_\mathbf{m}$ is a complete modular space under Δ_2-conditions by a new weighted space method in modular function spaces. Liviu Cadariu first introduced this method [7]. For information regarding the latest contribution of this new method, please refer to references [2], [39], and [37]. In the following, we use this new approach to check the generalized Hyers-Ulam stability of the two mentioned integral and functional equations and present new results that generalize some previous results.

2. Preliminaries

Definition 2.1 *If $\forall \kappa \geq 0$ and for every continuous function $\hbar(\kappa)$ the following inequality holds:*

$$\left|\hbar(\kappa) - \int_a^\kappa \mathcal{T}(\tau, \hbar(\tau))\, \mathrm{d}\tau\right| \leq \rho(\kappa)$$

such that $\rho(\kappa) \geq 0$, there exists a solution $\hbar_*(\kappa)$ of Eq. (1.1), and $\exists \mathcal{A} > 0$ such that for all κ, we have:

$$|\hbar(\kappa) - \hbar_*(\kappa)| \leq \mathcal{A}\rho(\kappa)$$

where $\mathcal{A}$ is independent of $\hbar(\kappa)$ and $\hbar_*(\kappa)$, then we say that the Eq. (1.1) has the HUR stability. In addition, if $\rho(\kappa)$ is a constant function, then Eq. (1.1) has the HU stability.

Definition 2.2 *[35] Assume that $\mathbf{F}$ is a Field ($\mathbf{F} = \mathbb{R} \vee \mathbb{C}$), and $\mathcal{G}$ is a vector space over $\mathbf{F}$. Moreover, let $\mathbf{m} : \mathcal{G} \to [0, +\infty]$ be a function that is satisfied in the following conditions:*

- *$\mathbf{m}(g) = 0$, if and only if $g = 0$,*
- *For all ξ, s.t $|\xi| = 1$, we have $\mathbf{m}(\xi g) = \mathbf{m}(g)$,*
- *For all $g_1, g_2 \in \mathcal{G}$, where $\xi, \delta \geq 0$ and $\xi + \delta = 1$, we have $\mathbf{m}(\xi g_1 + \delta g_2) \leq \mathbf{m}(g_1) + \mathbf{m}(g_2)$,*

In this case, we called $\mathbf{m}$ a modular. Furthermore, if the third condition is as follows:

- *For all $g_1, g_2 \in \mathcal{G}$, where $\xi, \delta \geq 0$ and $\xi + \delta = 1$, we have $\mathbf{m}(\xi g_1 + \delta g_2) \leq \xi\mathbf{m}(g_1) + \delta\mathbf{m}(g_2)$,*

then we call $\mathbf{m}$ convex modular.

Definition 2.3 *The modular $\mathbf{m}$ itself defines a modular space, namely $\mathcal{G}_{\mathbf{m}}$, as follows:*

$$\mathcal{G}_{\mathbf{m}} = \{g \in \mathcal{G} : \ \mathbf{m}(\beta x) \to 0 \ \text{as} \ \beta \to 0\}.$$

For example, if $(\mathcal{G}, \|.\|)$ is a normed space, then $\|.\|$ is a convex modular on $\mathcal{G}$, but the converse is not necessarily true [34].

Definition 2.4 *[29] Suppose that $\mathbf{m}$ is a convex modular, then $\mathcal{G}$ equiped with the Luxemburg norm is defined as follows:*

$$\|g\|_{\mathbf{m}} = \inf\{\sigma > 0 : \mathbf{m}(\frac{g}{\sigma}) \leq 1\}.$$

Definition 2.5 *[37] For a sequence $\{g_n\}$ in the modular space $\mathcal{G}_{\mathbf{m}}$, the following are defined.*

- *$\{g_n\}$ in $\mathcal{G}_{\mathbf{m}}$ is called $\mathbf{m}$-convergent to g if $\mathbf{m}(g_n - g) \to 0$ as $n \to \infty$.*
- *$\{g_n\}$ in $\mathcal{G}_{\mathbf{m}}$ is called $\mathbf{m}$-Cauchy if $\mathbf{m}(g_n - g_m) \to 0$ as $n, m \to \infty$.*
- *The modular space $\mathcal{G}_{\mathbf{m}}$ is $\mathbf{m}$-complete if every $\mathbf{m}$-Cauchy sequence is $\mathbf{m}$-convergent.*

Definition 2.6 *(**Fatou property**) The modular $\mathbf{m}$ has the Fatou property if for sequences $\{g_n\}, \{h_n\} \in \mathcal{G}_{\mathbf{m}}$, the following inequality holds true*

$$\mathbf{m}(g - h) \leq \liminf \mathbf{m}(g_n - h_n),$$

such that $\mathbf{m}(g_n - g) \to 0$, $\mathbf{m}(h_n - h) \to 0$, as $n \to \infty$.

Definition 2.7 *(**Δ_2-condition**) The modular $\mathbf{m}$ satisfies the Δ_2-condition if $\exists k \in [0, \infty)$ such that*

$$\mathbf{m}(2g) \leq k\mathbf{m}(g) \quad \text{for all} \quad g \in \mathcal{G}_{\mathbf{m}}.$$

Definition 2.8 *Let $\mathcal{H} \subseteq \mathcal{G}_{\mathbf{m}}$, and $\{h_n\}$ be a sequence in $\mathcal{H}$, then closed and bounded sets in modular space are defined as follows:*

- *If for any* $\{h_n\}$, *where* $h_n \to h$, *we have* $h \in \mathcal{H}$, *then* $\mathcal{H}$ *is called* $\mathbf{m}$*-closed.*
- *If the* $\mathbf{m}$*-diameter of* $\mathcal{H}$ *is infinity, namely* $\mathfrak{D}_{\mathbf{m}} < \infty$, *then* $\mathcal{H}$ *is called* $\mathbf{m}$*-bounded. Also, the* $\mathbf{m}$*-diameter of* $\mathcal{H}$ *is expressed as follows:*

$$\mathfrak{D}_{\mathbf{m}}(\mathcal{H}) = \sup\{\mathbf{m}(g-h) : \quad g, h \in \mathcal{H}\} < \infty.$$

Definition 2.9 *Let* $\hbar$ *be a real-valued and measurable function, then the Orliez modular is formulated as follows:*

$$\mathbf{m}(\hbar) = \int_{\mathbb{R}} \varphi(|\hbar|)\,\mathrm{d}\mu,$$

such that $\varphi : \mathbb{R} \to [0,\infty)$ *is continuous, and* $\varphi(t) = 0$ *iff* $t = 0$, *also* μ *denotes the Lebesgue measure on* $\mathbb{R}$. *Moreover, the modular space induced by this modularity is called the Orlicz space.*

Remark 2.10 *[27] Assume that* $\mathbf{m}$ *is a convex modular, and* $\sum_{i=1}^{n} \kappa_i \leq 1$ *where* $\kappa_i \geq 0$. *Then, for all* $g_i \in \mathcal{G}_{\mathbf{m}}$, *the following inequality hold:*

$$\mathbf{m}(\sum_{i=1}^{n} \kappa_i g_i) \leq \sum_{i=1}^{n} \kappa_i \mathbf{m}(g_i).$$

Definition 2.11 *[40] Let* $\mathcal{J} = [a, b]$ *and* $g : \mathcal{J} \to \mathcal{L}^{\varphi}$. *The function* g *is said to be continuous at* $s_0 \in \mathcal{J}$ *if for* $s_n \in \mathcal{J}$ *and* $s_n \to s_0$, *we have*

$$\mathbf{m}(g(s_n) - g(s_0)) \to 0 \quad \textit{as} \quad n \to \infty.$$

Definition 2.12 *[25] Let* $\mathcal{C}$ *be a subset of a modular function space* $\mathcal{G}_{\mathbf{m}}$. *A mapping* $\mathbf{K} : \mathcal{C} \to \mathcal{C}$ *is called* $\mathbf{m}$*-strict contraction if there exists* $\lambda < 1$ *where:*

$$\mathbf{m}(\mathbf{K}g_1 - \mathbf{K}g_2) \leq \lambda \mathbf{m}(g_1 - g_2),$$

for all $g_1, g_2 \in \mathcal{C}$.

Theorem 2.13 *[25] Assume that* $\mathcal{C}$ *is an* $\mathbf{m}$*-bounded,* $\mathbf{m}$*-complete subset of* $\mathcal{L}_{\mathbf{m}}$ *and* $\mathbf{K} : \mathcal{C} \to \mathcal{C}$ *is an* $\mathbf{m}$*-strict contraction. Then* $\mathbf{K}$ *has a unique fixed point* $g \in \mathcal{C}$. *Furthermore,* g *is the* $\mathbf{m}$*-limit of the iterate of any point in* $\mathcal{C}$ *under the action of* $\mathbf{K}$.

3. Main results

3.1. Stability of nonlinear Volterra integral equations

In this section, we introduce some methods for proving the stability of integral equations in modular function spaces and examine the methods for the Volterra integral equation as a common appropriate example. Let $\mathcal{C}(\mathcal{J}, \mathcal{L}^{\varphi})$ be the set of all continuous mappings from $\mathcal{J} = [a, b]$ into $\mathcal{L}^{\varphi}$ where $\mathcal{L}^{\varphi}$ is the Musielak–Orlicz space. Moreover, the Musielak–Orlicz space is defined as follows: Let $(\Omega, \Sigma, \boldsymbol{\mu})$ be a measure space where $\boldsymbol{\mu}$ is σ-finite measure on Ω. Also, consider the modular $\mathbf{m}$ as follows:

$$\mathbf{m}(\hbar) = \int_{\Omega} \varphi(r, |\hbar(r)|)\,\mathrm{d}\boldsymbol{\mu}(r),$$

where $\hbar : \Omega \to \mathbb{R}$, $\varphi : \Omega \times \mathbb{R} \to [0, \infty)$ and the following statements hold:

- For all $a \in \mathbb{R}$, the function $\varphi(r,d)$ is measurable.
- For all $r \in \Omega$, the function $\varphi(r,d)$ is convex.
- The function $\varphi(r,d)$ is countinuous, nondecreasing for $d > 0$, $\varphi(r,0) = 0$, $\varphi(r,d) > 0$ for $d \neq 0$ and $\varphi(r,d) \to \infty$ as $d \to \infty$.

It is clear that this modular with the mentioned properties is a convex. Furthermore, the corresponding modular space is called the Musielak–Orlicz space. Also, if we consider $g :$, then the continuity of $g : \mathcal{J} \to \mathcal{L}^{\varphi}$ at s_0 is equivalent to:

$$(s_n \to s_0) \Rightarrow \|g(s_n) - g(s_0)\|_{\mathbf{m}} \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem 3.1 *Let α, λ be positive constants with $0 < \alpha\lambda < 1$ and*

$$\int_a^x P(r,s)\psi(s) \leq \alpha\psi(r) \quad \textit{for all} \quad r,s \in \mathcal{I}.$$

Also, suppose that $\mathcal{U} : \mathcal{J} \times \mathcal{J} \times \mathcal{L}^{\varphi} \to \mathcal{L}^{\varphi}$ is a continuous function such that

$$\mathbf{m}\Big(2\big(\mathcal{U}(r,s,g_1(s)) - \mathcal{G}(r,s,g_2(s))\big)\Big) \leq P(r,s)\mathbf{m}\big(2(g_1(s) - g_2(s))\big),$$

for all $r,s \in \mathcal{J}$ and $g_1, g_2 \in \mathcal{C}(\mathcal{J}, \mathcal{L}^{\varphi})$. If the continuous function $\hbar : \mathcal{J} \to \mathcal{L}^{\varphi}$ has the following property

$$\mathbf{m}\Big(2\big(\hbar(r) - \mathbf{f}(r) - \lambda\int_a^x \mathcal{U}(r,s,\hbar(s))\,\mathrm{d}s\big)\Big) \leq \psi(r) \quad \textit{for all} \quad r \in \mathcal{J}.$$

Then, there exists a unique solution $\hbar_0 \in \mathcal{C}(\mathcal{J}, \mathcal{L}^{\varphi})$, where

$$\hbar_0(r) = \mathbf{f}(r) + \lambda\int_a^x \mathcal{U}(r,s,\hbar_0(r)),\ \forall r \in \mathcal{J};$$

and the following inequality:

$$\mathbf{m}(\hbar - \hbar_0) \leq \frac{\psi(r)}{1-\alpha\lambda} \quad \textit{for all} \quad r \in \mathcal{J} \tag{3.1}$$

holds.

Proof Let $g_1 \in \mathcal{C}(\mathcal{J}, \mathcal{L}^{\varphi})$ and $\{s_0, s_1, \ldots, s_n\}$ be a partition of $[a,x]$. Now, suppose that

$$\sup\big\{|s_{i+1} - s_i|,\ i = 0,1,\cdots,n-1\big\} \to 0 \quad \text{as} \quad n \to \infty.$$

Then,

$$\Big\|\sum_{i=0}^{n-1}\lambda(s_{i+1} - s_i)\mathcal{U}(r,s_i,g_1(s_i)) - \int_a^x \lambda\mathcal{U}(r,s,g_1(s))ds\Big\|_{\mathbf{m}} \to 0.$$

By using the Δ_2-condition of $\mathbf{m}$

$$\mathbf{m}\Big(2\sum_{i=0}^{n-1}\lambda(s_{i+1} - s_i)\mathcal{U}(r,s_i,g_1(s_i)) - 2\int_a^x \lambda\mathcal{U}(r,s,g_1(s))ds\Big) \to 0.$$

According to the Fatou property, we have

$$\mathbf{m}\Big(2\int_a^x \lambda\mathcal{U}(r,s,g_1(s))ds\Big) \le \liminf \mathbf{m}\Big(2\sum_{i=0}^{n-1}\lambda(s_{i+1}-s_i)\mathcal{U}(r,s_i,g_1(s_i))\Big). \tag{3.2}$$

Let $q \in \mathbb{N}$ given. So,

$$\sum_{i=0}^{n-1}\lambda e^{\frac{s_i}{q}}(s_{i+1}-s_i) \le \lambda\int_a^x e^{\frac{s}{q}}ds = \lambda q(e^{\frac{x}{q}} - e^{\frac{a}{q}}) \le \lambda q(e^{\frac{b}{q}} - e^{\frac{a}{q}}).$$

Since $\lim_{q\to\infty}\big(\lambda(e^{\frac{b}{q}} - e^{\frac{a}{q}}) - \frac{1}{q}\big) = 0$; then, $\forall \epsilon > 0,\ \exists q_\epsilon \in \mathbb{N}$, where

$$\lambda(e^{\frac{b}{q}} - e^{\frac{a}{q}}) - \frac{1}{q} \le \epsilon$$

for all $q > q_\epsilon$. That is,

$$\lambda(e^{\frac{b}{q}} - e^{\frac{a}{q}}) \le \frac{1}{q} + \epsilon$$

for all $q > q_\epsilon$. So, we get

$$\sum_{i=0}^{n-1}\lambda e^{\frac{s_i}{q}}(s_{i+1}-s_i) \le q(\frac{1}{q}+\epsilon)$$

for all $q > q_\epsilon$. Since $\epsilon > 0$ is arbitrary, then

$$\sum_{i=0}^{n-1}|\lambda| e^{\frac{s_i}{q}}(s_{i+1}-s_i) \le 1.$$

Hence,

$$\sum_{i=0}^{n-1}\lambda e^{\frac{s_i}{q}}(s_{i+1}-s_i) \le 1.$$

It follows from the convexity of $\mathbf{m}$:

$$\begin{aligned}\mathbf{m}\big(\sum_{i=0}^{n-1}\lambda(s_{i+1}-s_i)2\mathcal{U}(r,s_i,\hbar(s_i))\big) &= \mathbf{m}\big(\sum_{i=0}^{n-1}\lambda e^{\frac{t_i}{q}}(s_{i+1}-s_i)e^{-\frac{s_i}{q}}2\sqcap(r,s_i,\hbar(s_i))\big)\\ &\le \sum_{i=0}^{n-1}(s_{i+1}-s_i)|\lambda|e^{\frac{s_i}{q}}\mathbf{m}\big(2e^{-\frac{s_i}{q}}\mathcal{U}(r,s_i,\hbar(s_i))\big)\\ &\le \int_a^x \lambda e^{\frac{s}{q}}\mathbf{m}\big(2e^{-\frac{s}{q}}\mathcal{U}(r,s,\hbar(r))\big)\,\mathrm{d}s.\end{aligned}$$

As $e^{-\frac{s}{q}} \le 1$, we have $\mathbf{m}\big(2e^{-\frac{s}{q}}\mathcal{U}(r,s,\hbar(s))\big) \le \mathbf{m}\big(2\mathcal{U}(r,s,\hbar(s))\big)$. Hence, the above inequality reduces to

$$\mathbf{m}\big(2\sum_{i=0}^{n-1}|\lambda|(s_{i+1}-s_i)\mathcal{U}(r,s_i,\hbar(s_i))\big) \le \int_a^x \lambda e^{\frac{s}{q}}\mathbf{m}\big(2\mathcal{U}(r,s,\hbar(s))\big)\,\mathrm{d}s. \tag{3.3}$$

Since $e^{\frac{s}{q}} \to 1$ as $q \to \infty$, putting limit on both sides of (3.3), we obtain

$$\mathbf{m}\Big(\sum_{i=0}^{n-1} \lambda(s_{i+1} - s_i)2\mathcal{U}(r, s_i, \hbar(s_i))\Big) \leq \int_a^x \lambda\mathbf{m}\big(2\mathcal{U}(r, s, \hbar(s))\big)\,\mathrm{d}s. \tag{3.4}$$

Taking $\lambda = 0$, one has

$$\begin{aligned}\mathbf{m}_Y(\mathbf{K}g_1 - \mathbf{K}g_2) &= \sup_{x\in\mathcal{J}} \frac{\mathbf{m}\big(\int_a^x 2\lambda(\mathcal{U}(r, s, g_1(s)) - \mathcal{U}(r, s, g_2(r)))\,\mathrm{d}s\big)}{\psi(r)} \\ &= 0 \leq \alpha|\lambda|\mathbf{m}_Y(g_1 - g_2),\end{aligned}$$

and, we have nothing to prove. So suppose that $\lambda \neq 0$. Now since $|\frac{\lambda}{|\lambda|}| = 1$; then,

$$\begin{aligned}\mathbf{m}\big(2\sum_{i=0}^{n-1} \lambda(s_{i+1} - s_i)\mathcal{U}(r, s_i, \hbar(s_i))\big) &= \mathbf{m}\Big(\frac{\lambda}{|\lambda|}2\sum_{i=0}^{n-1} |\lambda|(s_{i+1} - s_i)\mathcal{U}(r, s_i, \hbar(s_i))\Big) \\ &= \mathbf{m}\big(2\sum_{i=0}^{n-1} |\lambda|(s_{i+1} - s_i)\mathcal{U}(r, s_i, \hbar(s_i))\big).\end{aligned}$$

Therefore, (3.4) yields

$$\mathbf{m}\big(2\sum_{i=0}^{n-1} \lambda(s_{i+1} - s_i)\mathcal{U}(r, s_i, y(s_i))\big) \leq \int_a^x |\lambda|\mathbf{m}\big(2\mathcal{U}(r, s, \hbar(s))\big)\,\mathrm{d}s.$$

Utilizing (3.2), we have

$$\mathbf{m}\Big(\int_a^x 2\lambda\mathcal{U}(r, s, g_1(s))\,\mathrm{d}s\Big) \leq \int_a^x |\lambda|\mathbf{m}\big(2\mathcal{U}(r, st, g_1(s)\big)\,\mathrm{d}s.$$

Define $\mathbf{K} : C(\mathcal{I}, L^\varphi) \to C(\mathcal{I}, L^\varphi)$ by

$$(\mathbf{K}g_1)(r) := \mathbf{f}(r) + \lambda\int_a^x \mathcal{U}(r, s, g_1(r)) \quad \forall r \in \mathcal{J}.$$

Now, we have

$$\begin{aligned}
\mathbf{m}_Y(\mathbf{K}g_1 - \mathbf{K}g_2) &= \sup_{r\in\mathcal{J}} \frac{\mathbf{m}\big(\int_a^x 2\lambda(\mathcal{U}(r,s,g_1(s)) - \mathcal{U}(r,s,g_2(s)))\,\mathrm{d}s\big)}{\psi(r)} \\
&\leq \sup_{r\in\mathcal{J}} \frac{\int_a^x |\lambda| \mathbf{m}\big(2(\mathcal{U}(r,s,g_1(s)) - \mathcal{U}(r,s,g_2(s)))\big)\,\mathrm{d}s}{\psi(r)} \\
&\leq \sup_{r\in\mathcal{I}} \frac{\int_a^x |\lambda| P(r,s)\mathbf{m}\big(2(g_1(s) - g_2(s))\big)\,\mathrm{d}s}{\psi(r)} \\
&= \sup_{r\in\mathcal{J}} \frac{\int_a^x |\lambda| P(r,s)\psi(s)\frac{\mathbf{m}\left(2(g_1(s)-g_2(s))\right)}{\psi(s)}\,\mathrm{d}s}{\psi(r)} \\
&\leq \left(\sup_{s\in\mathcal{J}} \frac{\mathbf{m}\big(2(g_1(s) - g_2(s))\big)}{\psi(s)}\right) \sup_{r\in\mathcal{J}} \frac{\int_a^x |\lambda| P(r,s)\psi(s)\,\mathrm{d}s}{\psi(r)} \\
&\leq \alpha|\lambda|\mathbf{m}_Y(g_1 - g_2),
\end{aligned}$$

in which $\alpha|\lambda| < 1$. On the other hand (3.1), we have $\mathbf{m}_Y(\hbar - \mathbf{K}\hbar) < 1$. Now by Theorem 2.13, there exists a mapping $\hbar_0$, where

- $\hbar_0$ is the fixed point of $\mathbf{K}$, i.e.,

$$\hbar_0(r) = \mathbf{f}(r) + \lambda \int_a^x \mathcal{U}(r,s,\hbar_0(r)) \quad \forall r \in \mathcal{J}.$$

- $\mathbf{m}_Y(\mathbf{K}^n\hbar - \mathbf{K}\hbar_0) \to 0$ as $n \to \infty$, which implies that $\hbar_0(r) = \lim_{n\to\infty}(\mathbf{K}^n\hbar)(r)$ for all $\mathcal{J}$.

- $\mathbf{m}_Y(\hbar - \hbar_0) \leq \frac{1}{1-\ell}\mathbf{m}_Y(\hbar - \mathbf{K}\hbar)$, which implies the inequality

$$\mathbf{m}_Y(\hbar - \hbar_0) \leq \frac{1}{1-\ell}.$$

So, the estimate relation (3.1) is true. □

3.2. The generalized Hyers-Ulam stability of a nonlinear equation

In this section, we introduce a generalization of the Hyers-Ulam stability result for the following nonlinear equation

$$\hbar(r) = \mathcal{T}\big(r, \hbar(r), \hbar(\eta(r))\big),$$

in a modular function space via a new weighted space method. In the rest of this paper, the function $\hbar : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$ is a mapping by a specific property, $\mathcal{S} \neq \emptyset$, $\mathcal{G}_{\mathbf{m}}$ is a complete modular space, $\mathcal{T} : \mathcal{S} \times \mathcal{G}_{\mathbf{m}} \times \mathcal{G}_{\mathbf{m}} \to \mathcal{G}_{\mathbf{m}}$ and $\eta : \mathcal{S} \to \mathcal{S}$ are given mappings.

Theorem 3.2 *Let the modular* $\mathbf{m}$ *satisfies* Δ_2*-condition, there exist* $\ell \in [0,1)$ *and mappings* $\theta, \vartheta : \mathcal{S} \to [0,\infty)$ *such that*

$$\theta(r)\psi(r) + \vartheta(r)\psi(\eta(r)) \leq \ell\psi(r), \qquad \forall r \in \mathcal{S}, \tag{3.5}$$

for a given function $\psi : \mathcal{S} \to (0,\infty)$*. Also, there is a mapping* $\mathcal{T} : \mathcal{S} \times \mathcal{G}_{\mathbf{m}} \times \mathcal{G}_{\mathbf{m}} \to \mathcal{G}_{\mathbf{m}}$ *for all* $r \in \mathcal{S}$ *and for all* $g_1, g_2 \in \mathcal{G}_{\mathbf{m}}$ *that satisfies*

$$\mathbf{m}\Big(2\big(\mathcal{T}(r, g_1(r), g_1(\eta(r))) - \mathcal{T}(r, g_2(r), g_2(\eta(r)))\big)\Big) \leq \theta(r)\mathbf{m}(2(g_1(r) - g_2(r)))$$
$$+ \vartheta(r)\mathbf{m}\big(2(g_1(\eta(r)) - g_2(\eta(r)))\big). \tag{3.6}$$

If $\hbar : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$ *is a fixed mapping with the property*

$$\mathbf{m}(2(\hbar(r) - \mathcal{T}(r, \hbar(r), \hbar(\eta(r))))) \leq \psi(r), \qquad \forall r \in \mathcal{S}. \tag{3.7}$$

Then there exists a unique $\hbar_0 : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$ *such that*

$$\hbar_0(r) = \mathcal{T}\big(r, \hbar_0(r), \hbar_0(\eta(r))\big), \qquad \forall r \in \mathcal{S},$$

and the inequality

$$\mathbf{m}(\hbar(r) - \hbar_0(r)) \leq \frac{\psi(r)}{1-\ell} \tag{3.8}$$

holds for all $r \in \mathcal{S}$*.*

Proof Let

$$Y := \Big\{ g_1 : \mathcal{S} \to \mathcal{G}_{\mathbf{m}} \;\mid\; \sup_{p \in \mathcal{S}} \frac{\mathbf{m}(g_1(r) - \hbar(r))}{\psi(r)} < \infty \Big\},$$

which implies that Y is a complete modular space with the modular

$$\mathbf{m}_{Y(g_1-g_2)} = \sup_{r \in \mathcal{S}} \frac{\mathbf{m}(2(g_{1(r)} - g_{2(r)}))}{\psi(r)}.$$

Since $(\mathcal{G}_{\mathbf{m}}, \mathbf{m})$ is modular space and $\psi > 0$, then $\mathbf{m}_{Y(g_1)} \geq 0$ for every $g_1 \in Y$. Let $\mathbf{m}_{Y(g_1-g_2)} = 0$. Then,

$$0 \leq \frac{\mathbf{m}(2(g_1(r) - g_2(r)))}{\psi(r)} \leq \sup_{r \in \mathcal{S}} \frac{\mathbf{m}(2(g_1(r) - g_2(r)))}{\psi(r)} = \mathbf{m}_{Y(g_1-g_2)} = 0.$$

Therefore,

$$\frac{\mathbf{m}\big(2(g_1(r) - g_2(r))\big)}{\psi(r)} = 0 \qquad \text{for all} \quad r \in \mathcal{S}.$$

Since $\psi > 0$, then $\mathbf{m}(2(g_1(r) - g_2(r))) = 0$ for all $r \in \mathcal{S}$. Hence, $2(g_1(r) - g_2(r)) = 0$ for all $r \in \mathcal{S}$. Thus, $r_1 = r_2$. Now, let $g_1, g_2 \in Y$ and $\xi, \delta \geq 0$ such that $\xi + \delta = 1$. Then,

$$\mathbf{m}_{Y(\xi g_1 + \delta g_2)} = \sup_{r \in \mathcal{S}} \frac{\mathbf{m}\big(2(\xi g_1 + \delta g_2)(r)\big)}{\psi(r)} = \sup_{r \in \mathcal{S}} \frac{\mathbf{m}\big(\xi(2g_1(r)) + \delta(2g_2(r))\big)}{\psi(r)}.$$

We have

$$\frac{\mathbf{m}\big(\xi(2g_1(r))+\delta(2g_2(r))\big)}{\psi(r)}\leq\frac{\mathbf{m}(2g_1(r))}{\psi(r)}+\frac{\mathbf{m}(2g_2(r))}{\psi(r)}\leq\mathbf{m}_Y(g_1)+\mathbf{m}_Y(g_2).$$

Therefore,

$$\mathbf{m}_Y(\xi g_1+\delta g_2)\leq\mathbf{m}_Y(g_1)+\mathbf{m}_Y(g_2).$$

Now, we claim that $\mathbf{m}_Y(\xi g_1)=\mathbf{m}_Y(g_1)$ for $g_1\in Y$, and $|\xi|=1$. We can write

$$\mathbf{m}_Y(\xi g_1)=\sup_{r\in\mathcal{S}}\frac{\mathbf{m}(2\xi g_1(r))}{\psi(r)}=\sup_{r\in\mathcal{S}}\frac{\mathbf{m}(\xi(2g_1(r)))}{\psi(r)}=\sup_{r\in\mathcal{S}}\frac{\mathbf{m}(2g_1(r))}{\psi(r)}=\mathbf{m}_Y(g_1).$$

Now, we prove that Y is $\mathbf{m}_Y$-complete modular space. Suppose that (g_{1_n}) be a $\mathbf{m}_Y$-Cauchy sequence in Y, then

$$\mathbf{m}_Y(g_{1_n}-g_{1_m})=\sup_{r\in\mathcal{S}}\frac{\mathbf{m}\big(2(g_{1_n}-g_{1_m})(r)\big)}{\psi(r)}\to 0,$$

as $m,n\to\infty$. Thus,

$$\mathbf{m}\big(2g_{1_n}(r)-2g_{1_m}(r)\big)\xrightarrow[m,n\to\infty]{}0\ \text{ for every }\ r\in\mathcal{S}.$$

Therefore, $(2g_{1_n}(r))$ is $\mathbf{m}$-Cauchy sequence for every $p\in\mathcal{S}$. Since $\mathcal{G}_{\mathbf{m}}$ is $\mathbf{m}$-complete modular space, then there exist $g_{1_0}:\mathcal{S}\to X$ such that $2g_{1_n}(r)\to g_{1_0}(r)$ as $n\to\infty$. Hence,

$$\mathbf{m}_Y(g_{1_n}-\frac{1}{2}g_{1_0})=\sup_{r\in\mathcal{S}}\frac{\mathbf{m}\big(2(g_{1_n}-\frac{1}{2}g_{1_0})(r)\big)}{\psi(r)}=\sup_{r\in\mathcal{S}}\frac{\mathbf{m}\big(2(g_{1_n}(r)-g_{1_0}(r)\big)}{\psi(r)}\xrightarrow[m,n\to\infty]{}0.$$

Then, $g_{1_n}\overset{\mathbf{m}_Y}{\to}\frac{1}{2}g_{1_0}$ as $n\to\infty$. Now, we define the mapping

$$(\mathbf{K}g_1)(r):=\mathcal{T}(r,g_1(r),g_1(\eta(r))).$$

Applying (3.6) and (3.5) for all $g_1,g_2\in Y$, we have

$$\begin{aligned}\frac{\mathbf{m}\big(2Tg_1(r)-2Tg_2(r)\big)}{\psi(r)}&=\frac{\mathbf{m}\Big(2\mathcal{T}\big(r,g_1(r),g_1(\eta(r))\big)-2\mathcal{T}\big(r,g_2(r),g_2(\eta(r))\big)\Big)}{\psi(r)}\\&\leq\frac{\theta(r)\mathbf{m}\big(2(g_1(r)-g_2(r))\big)+\vartheta(r)\mathbf{m}\big(2(g_1(\eta(r))-g_2(\eta(r))\big)}{\psi(r)}\\&=\theta(r)\frac{\mathbf{m}\big(2(g_1(r)-g_2(r))\big)}{\psi(r)}+\vartheta(r)\frac{\psi(\eta(r))}{\psi(r)}\\&\times\frac{\mathbf{m}\big(2(g_1(\eta(r))-g_2(\eta(r))\big)}{\psi(\eta(r))}\\&\leq\Big(\theta(r)+\vartheta(r)\times\frac{\psi(\eta(r))}{\psi(r)}\Big)\mathbf{m}_Y(g_1-g_2)\\&\leq\ell\mathbf{m}_Y(g_1-g_2).\end{aligned}$$

On the other hand, considering (3.7), we obtain

$$\begin{aligned}\frac{\mathbf{m}(\mathbf{K}g_1(r)-\hbar(r))}{\psi(r)} &= \frac{\mathbf{m}\frac{1}{2}(2(\mathbf{K}g_1(r)-\mathbf{K}Y(r)))+\frac{1}{2}(2(\mathbf{K}\hbar(r)-\hbar(r))))}{\psi(r)}\\ &\leq \frac{\mathbf{m}(2(\mathbf{K}g_1(r)-\mathbf{K}\hbar(r)))}{\psi(r)}+\frac{\mathbf{m}(2(\mathbf{K}\hbar(r)-\hbar(r)))}{\psi(r)}\\ &\leq \ell\mathbf{m}_Y(g_1-\hbar)+\frac{\psi(r)}{\psi(r)}=\ell\mathbf{m}_Y(g_1-\hbar)+1<\infty.\end{aligned}$$

If $g_1 \in Y$, then $\mathbf{K}g_1 \in Y$. Hence, the map $\mathbf{K}: Y \to Y$ is well-defined. Also, we obtain that

$$\mathbf{m}_Y(\mathbf{K}g_1-\mathbf{K}g_2)=\sup_{p\in\mathcal{S}}\frac{\mathbf{m}\big(2(\mathbf{K}g_1(r)-\mathbf{K}g_2(r))\big)}{\psi(p)}\leq \ell\mathbf{m}_Y(g_1-g_2).$$

Thus, $\mathbf{K}$ is strictly contractive self-mapping on Y, with the constant $\ell<1$.

Now, we have to prove that Y is $\mathbf{m}_Y$-closed and $\mathbf{m}_Y$-bounded. Let $\{\hbar_n\}_{n\in\mathbb{N}}$ be a sequence in Y such that $\hbar_n \overset{\mathbf{m}}{\to} \hbar_0$ as $n\to\infty$. Since $\mathbf{m}$ satisfies the Δ_2-condition, then there exists $k>0$ such that $\mathbf{m}(2g)\leq k\mathbf{m}(g)$, for any $g\in\mathcal{G}_{\mathbf{m}}$. Therefore,

$$\begin{aligned}\frac{\mathbf{m}(\hbar_0(r)-\hbar(r))}{\psi(r)} &= \frac{\mathbf{m}\big(\hbar_0(r)-\hbar_n(r)+\hbar_n(r)-\hbar(r)\big)}{\psi(r)}\\ &= \frac{\mathbf{m}\Big(\frac{1}{2}\big(2(\hbar_0(r)-\hbar_n(r))\big)+\frac{1}{2}\big(2(\hbar_n(r)-\hbar(r))\big)\Big)}{\psi(r)}\\ &\leq \frac{\mathbf{m}\big(2(\hbar_0(r)-\hbar_n(r))\big)}{\psi(r)}+\frac{\mathbf{m}\big(2(\hbar_n(r)-\hbar(r))\big)}{\psi(r)}\\ &\leq k\frac{\mathbf{m}(\hbar_0(r)-\hbar_n(r))}{\psi(r)}+k\frac{\mathbf{m}(\hbar_n(r)-\hbar(r))}{\psi(r)}\\ &\leq k\sup\frac{\mathbf{m}(\hbar_n(r)-\hbar(r))}{\psi(r)}<\infty.\end{aligned}$$

Thus, $\sup_{p\in\mathcal{S}} \frac{\mathbf{m}(\hbar_0(p)-\hbar(p))}{\psi(p)} < \infty$. Hence, $\hbar_0 \in Y$. Let $g_1, g_2 \in Y$. Then,

$$\begin{aligned}
\frac{\mathbf{m}(2(g_1(r)-g_2(r))}{\psi(r)} &= \frac{\mathbf{m}(2(g_1(r)-\hbar(r))) + 2(\hbar(r)-g_2(r)))}{\psi(r)} \\
&= \frac{\mathbf{m}(\frac{1}{2}(4(g_1(r)-\hbar(r))) + \frac{1}{2}(4(\hbar(r)-g_2(r))))}{\psi(r)} \\
&\le \frac{\mathbf{m}(4(g_1(r)-\hbar(r))) + \mathbf{m}(4(\hbar(r)-g_2(r)))}{\psi(r)} \\
&\le \frac{k\mathbf{m}(2(g_1(r)-\hbar(r))}{\psi(r)} + \frac{k\mathbf{m}(2(\hbar(r)-g_2(r))}{\psi(r)} \\
&\le \frac{k^2\mathbf{m}(g_1(r)-\hbar(r))}{\psi(r)} + \frac{k^2\mathbf{m}(\hbar(r)-g_2(r))}{\psi(r)} \\
&\le k^2(\sup_{r\in\mathcal{S}} \frac{\mathbf{m}(g_1(r)-\hbar(r))}{\psi(r)} + \sup_{r\in\mathcal{S}} \frac{\mathbf{m}(\hbar(r)-g_2(r))}{\psi(r)}) = M < \infty
\end{aligned}$$

for every $r \in \mathcal{S}$. Therefore,

$$\mathbf{m}_Y(g_1 - g_2) = \sup_{r\in\mathcal{S}} \frac{\mathbf{m}(2(g_1(r)-g_2(r)))}{\psi(r)}) \le M,$$

for every $g_1, g_2 \in Y$. Then

$$\operatorname{diam}_{\mathbf{m}_Y} Y = \sup\{\mathbf{m}_Y(g_1-g_2); g_1, g_2 \in Y\} < \infty.$$

Thus, Y is $\mathbf{m}_Y$-bounded. In view of Theorem 2.13, there exists a mapping $\hbar_0 : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$, where

- $\hbar_0$ is the unique fixed point of K, i.e.,

$$\hbar_0(r) = (K\hbar_0)(r) = \mathcal{T}(r, \hbar_0(r), \hbar_0(\eta(r))), \qquad r \in \mathcal{S};$$

- $\mathbf{m}_Y(K^n_\hbar - \hbar_0) \xrightarrow[m,n\to\infty]{} 0$, which implies that

$$\hbar_0(r) = \lim_{n\to\infty} (k^n\hbar)(r), \qquad \forall r \in \mathcal{S};$$

- Since

$$\frac{\mathbf{m}(Kg_1(r)-\hbar(r))}{\psi(r)} \le \ell\mathbf{m}_Y(g_1-\hbar)+1,$$

  for every $g_1 \in Y$, then

$$\frac{\mathbf{m}(K\hbar_0(r)-\hbar(r))}{\psi(r)} = \frac{\mathbf{m}(\hbar_0(r)-\hbar(r))}{\psi(r)} \le \ell\mathbf{m}_Y(\hbar_0-\hbar)+1,$$

for every $r \in \mathcal{S}$. Therefore, $\mathbf{m}_Y(\hbar_0 - \hbar) \leq \ell \mathbf{m}_Y(\hbar_0 - \hbar) + 1$. So, we have

$$(1-\ell)\mathbf{m}_Y(\hbar_0 - \hbar) \leq 1.$$

Thus, $\mathbf{m}_Y(y_0 - y) \leq \dfrac{1}{1-\ell}$. It means that the estimation relation (3.8) holds.

□

Remark 3.3 *As an explanation, it is notable that Theorem 3.2 is the modular version of the Theorem 2.2 of [8].*

Corollary 3.4 *Let $\mathcal{S} \neq \emptyset$ and $(\mathcal{G}_{\mathbf{m}}, \mathbf{m})$ be a complete modular space where $\mathbf{m}$ satisfies the Δ_2-condition. Moreover, let $\eta : \mathcal{S} \to \mathcal{S}$ and $\mathcal{U} : \mathcal{G}_{\mathbf{m}} \times \mathcal{G}_{\mathbf{m}} \to \mathcal{G}_{\mathbf{m}}$ be given mappings. Additionally, $\lambda, \mu \in \mathbb{R}^+$ and the following condition holds:*

$$\mathbf{m}\big(2(\mathcal{U}(s,g_1) - \mathcal{U}(t,g_2))\big) \leq \lambda \mathbf{m}(2(s-t)) + \mu\big(\mathbf{m}(2(g_1 - g_2))\big), \qquad \forall s,t,g_1,g_2 \in \mathcal{G}_{\mathbf{m}}.$$

Let $\hbar : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$ be ψ-solution, for some given function $\psi : \mathcal{S} \to (0,\infty)$, in which

$$\mathbf{m}\Big(2\big(\hbar(r) - \mathcal{G}(\hbar(r), \hbar(\eta(r)))\big)\Big) \leq \psi(r), \qquad \forall r \in \mathcal{S}.$$

Also, for some $\ell \in [0,1)$,

$$\mu \times \psi(\eta(r)) + \lambda \times \psi(r) \leq \ell \times \psi(r), \qquad \forall r \in \mathcal{S}.$$

Then, there exists a unique function $\hbar_0 : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$ such that

$$\hbar_0(r) = \mathcal{U}(\hbar_0(r), \hbar_0(\eta(r))), \qquad \forall r \in \mathcal{S}$$

and

$$\mathbf{m}(\hbar(r) - \hbar_0(r)) \leq \frac{\psi(r)}{1-\ell}, \qquad \forall r \in \mathcal{S}.$$

It is not hard to check that the generalized Hyers-Ulam stability discussed in [14] can be derived from Theorem 3.2, as a direct consequence, for the following nonlinear equation:

$$\hbar(r) = \mathcal{T}\big(r, \hbar(\eta(r))\big), \tag{3.9}$$

which is expressed in the following corollary.

Corollary 3.5 *Let $\exists \ell \in [0,1)$ such that*

$$\psi(\eta(r)) \times \mathbf{m}\Big(2\big(\mathcal{T}(r, g_1(\eta(r))) - \mathcal{T}(r, g_2(\eta(r)))\big)\Big) \leq \ell \times \psi(r) \times \mathbf{m}\big(2(g_1(\eta(r)) - g_2(\eta(r)))\big),$$

holds for all $r \in \mathcal{S}$ and for all $g_1, g_2 \in \mathcal{G}_{\mathbf{m}}$. In addition, assume that for the function $\hbar : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$, the following inequality holds

$$\mathbf{m}\big(2(\hbar(r) - \mathcal{T}(r, \hbar(\eta(r))))\big) \leq \psi(r), \qquad \forall r \in \mathcal{S}.$$

Then, there exists a unique solution $\hbar_0 : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$ *such that*

$$\hbar_0(r) = \mathcal{T}(r, \hbar_0(\eta(r))), \qquad \forall r \in \mathcal{S},$$

and

$$\mathbf{m}(\hbar(r) - \hbar_0(r)) \leq \frac{\psi(r)}{1-\ell},$$

holds for all $r \in \mathcal{S}$*.*

Proof Put $\lambda(r) = 0$ and $\mu(r) = \dfrac{\ell.\psi(r)}{\psi(\eta(r))}$. Then,

$$\lambda(r)\psi(r) + \mu(r)\psi(\eta(r)) = 0.\psi(r) + \frac{\ell.\psi(r)}{\psi(\eta(r))} \times \psi(\eta(r)) = \ell \times \psi(r) \leq \ell \times \psi(0).$$

Also, we have

$$\begin{aligned}
\mathbf{m}(2(\mathcal{T}(r, g_1(\eta(r)) - \mathcal{T}(r, g_2(\eta(r)))))) &\leq \lambda(r)\mathbf{m}(2(g_1(r) - g_2(r))) \\
&+ \mu(r)\mathbf{m}(2(g_1(\eta(r)) - g_2(\eta(r)))) \\
&= 0 \times \mathbf{m}(2(g_1(r) - g_2(r))) \\
&+ \frac{\ell.\psi(r)}{\psi(\eta(r))}\mathbf{m}(2(g_1(\eta(r)) - g_2(\eta(r)))) \\
&= \frac{\ell.\psi(r)}{\psi(\eta(r))}\mathbf{m}(2(g_1(\eta(r)) - g_2(\eta(r))))
\end{aligned}$$

for all $r \in \mathcal{S}$ and for all $g_1, g_2 \in \mathcal{G}_{\mathbf{m}}$. Therefore,

$$\psi(\eta(r))\mathbf{m}(2(\mathcal{T}(r, g_1(\eta(r)))) - \mathcal{T}(r, g_2(\eta(r)))))) \leq \ell.\psi(r)\mathbf{m}(2(g_1(\eta(r)) - g_2(\eta(r)))),$$

for all $r \in \mathcal{S}$ and for all $g_1, g_2 \in \mathcal{G}_{\mathbf{m}}$. □

Remark 3.6 *Taking* $\psi(r) = \gamma > 0$ *in Corollary 3.5, we get the following result which is the modular version of ([5, Theorem 2]) and ([2, Theorem 13]) for the nonlinear equation (3.9).*

Corollary 3.7 *Assume that* $\mathcal{S} \neq \emptyset$ *and* $(\mathcal{G}_{\mathbf{m}}, \mathbf{m})$ *is a complete modular space where* $\mathbf{m}$ *satisfies the* Δ_2*-condition. Moreover,* $\eta : \mathcal{S} \to \mathcal{S}$ *and* $\mathcal{T} : \mathcal{S} \times \mathcal{G}_{\mathbf{m}} \to \mathcal{G}_{\mathbf{m}}$ *are some given mappings and* $\ell \in [0, 1)$*. Let*

$$\mathbf{m}\big(2(\mathcal{T}(r, g_1) - \mathcal{T}(r, g_2))\big) \leq \ell\mathbf{m}\big(2(g_1 - g_2)\big), \qquad \forall r \in \mathcal{S}, \forall g_1, g_2 \in \mathcal{G}_{\mathbf{m}}.$$

If the function $\hbar : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$ *satisfies the following relation*

$$\mathbf{m}\big(2(\hbar(r) - \mathcal{T}(r, g_1(\eta(r)))))\big) \leq \gamma, \qquad \forall r \in \mathcal{S},$$

with a positive constant γ*, then there exists a unique mapping* $\hbar_0 : \mathcal{S} \to \mathcal{G}_{\mathbf{m}}$ *that satisfies both the equation*

$$\hbar(r) = \mathcal{T}(r, g_1(\eta(r))), \qquad \forall r \in \mathcal{S}$$

and the inequality

$$\mathbf{m}(\hbar(r) - \hbar_0(r)) \leq \frac{\gamma}{1-\ell}, \qquad \forall r \in \mathcal{S}.$$

Proof Put $\psi(r) = \gamma$ in Corollary 3.5. Then, we have $\lambda(r) = 0$ and

$$\mu(r) = \frac{\ell.\psi(r)}{\psi(\eta(r))} = \frac{\ell \cdot \gamma}{\gamma} = \ell$$

□

4. Conclusion

Providing accurate models to interpret natural and physical phenomena is a current hot research topic. Interpreting models using differential and functional equations does not always lead to a unique solution. Therefore, checking the stability of equations is necessary and unavoidable. Since proving the stability of most functional equations in modular function spaces is hard, in the current research, we presented some new results about the generalized Hyers-Ulam stability of a nonlinear functional equation and the Volterra integral equation using weighted space methods. We have guaranteed the stability of the mentioned equations employing fixed point techniques in theorems 3.2 and 3.1, and we have also generalized the previous works in our corollaries. We hope that other researchers will examine the equations presented in this work about semi-Ulam-Hyers stability in the appropriate metric space.

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