TÜBİTAK

Turkish Journal of Mathematics

Volume 48 | Number 3 Article 8

5-1-2024

Topogenous orders and closure operators on posets

JOSEF SLAPAL
qmslapal@vutbr.cz

TOM RICHMOND
tom.richmond@wku.edu

MINANI IRAGI
84miragi2016@gmail.com

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2024) 48: 469 – 476
© TÜBİTAK
doi:10.55730/1300-0098.3519

Topogenous orders and closure operators on posets

Josef ŠLAPAL[1,*], **Tom RICHMOND**[2], **Minani IRAGI**[1]
[1]Institute of Mathematics, Brno University of Technology, Brno, Czech Republic
[2]Department of Mathematics, Western Kentucky University, Bowling Green, KY, USA

Received: 19.07.2023 • **Accepted/Published Online:** 28.02.2024 • **Final Version:** 10.05.2024

Abstract: We introduce the notion of topogenous orders on a poset X to be certain endomaps on X. We build on a Galois connection between endomaps and binary relations on X and study relationships between endomap properties and corresponding relational properties. In particular, we determine the topogenous orders that are in a one-to-one correspondence with (idempotent) closure operators. Extending our considerations to the categorical level, we find a cartesian closed category of topogenous systems.

Key words: Poset, meet-complete semilattice, topogenous order, closure operator, cartesian closed category

1. Introduction

Correspondences between topologies and binary relations were studied by many authors. Such a natural correspondence is obtained by assigning, to every topology τ on a set X, the binary relation ρ on the power set of X given by $A\rho B \Leftrightarrow B = uA$ where u is the Kuratowski closure operator associated with τ. However, such a correspondence is inefficient because it just provides relational equivalents to topological properties of Kuratowski closure operators (it is easy to formulate axioms on ρ equivalent to the Kuratowsky axioms so that we obtain an isomorphism between the lattice of the relations ρ on the power set of X satisfying these axioms and the lattice all topologies on X). To obtain a more efficient correspondence, Császár [3, 4] employed the one given by $A\rho B \Leftrightarrow A \subseteq iB$ where i denotes the interior operator associated with τ. This correspondence is, under some natural conditions on ρ, equivalent to the correspondence associating with every topology τ on a set X the relation σ on the power set of X given by $A\sigma B \Leftrightarrow uA \subseteq B$. Császár called his relation ρ, subject to certain axioms, a topogenous order and showed that it may be used as a common tool for the study of topological, uniform, and proximity spaces. In his paper [13], Šlapal studied the correspondence based on the relation σ given by $A\sigma B \Leftrightarrow B \subseteq uA$, hence a correspondence "dual" to the previous one. He investigated such a correspondence extended to closure operators u that are more general than the Kuratowski ones. And such closure operators are dealt with in the present note. However, while the usual closure operators on a set X are certain endomaps on the power set of X, we will define closure operators to be endomaps on posets (i.e., partially ordered sets). For such endomaps u, we study the correspondence $x\sigma y \Leftrightarrow u(x) \leq y$. We show that this correspondence gives rise to a Galois connection between naturally ordered sets of binary relations and endomaps on a meet-complete semilattice (i.e., a poset with meets of all nonempty subsets). We determine corresponding pairs of topogenous and closure axioms (after extending them to relations and endomaps, respectively, on meet-

*Correspondence: slapal@fme.vutbr.cz
2010 *AMS Mathematics Subject Classification:* 54A05, 54B30.

complete semilattices). It follows that (idempotent) closure operators correspond to (interpolative) topogenous orders. This fact is then used to introduce a cartesian closed category of topogenous systems.

Categorical closure operators (see [5] and the references there) and categorical topogenous orders (cf. [9]) are defined on certain complete lattices, namely the subobject lattices of the objects (subject to an axiom of functoriality). Our approach may be considered to be a generalization of the categorical one because we define closure operators and topogenous orders on posets.

2. Preliminaries

For the lattice-theoretic concepts used see [8] and for the the topological ones see [6] or [11]. If X, Y are posets and a map $f : X \to Y$ is a left adjoint, then the corresponding right adjoint is denoted by f^{-1}, hence $f^{-1} : Y \to X$. By a *meet-complete semilattice*, we understand a poset $X = (X, \leq)$ such that each of its nonempty subsets has a meet. If a meet-complete semilattice is a lattice, then we call it a *meet-complete lattice*. The smallest element of a poset X (provided it exists) is denoted by 0. A subset A of a poset $X = (X, \leq)$ is called a *stack* if, for all $x, y \in X$, $x \in A$ and $x \leq y$ imply $y \in A$. A *principal filter* in a poset X is any set $\{y \in X;\ x \leq y\}$, where $x \in X$.

Recall that a *Galois connection* between partially ordered sets $(G, \leq)$ and $(H, \leq')$ is a pair (g, h) of order-reversing maps $g : G \to H$ and $h : H \to G$ such that $x \leq h(g(x))$ for every $x \in G$ and $y \leq' g(h(y))$ for every $y \in H$. Of the properties of a Galois connection (g, h) between $(G, \leq)$ and $(H, \leq')$, let us mention that the restrictions $\bar{g} : h(H) \to g(G)$ and $\bar{h} : g(G) \to h(H)$ of g and h, respectively, are dual order isomorphisms inverse to each other.

Definition 1 Let X be a poset and c be an endomap on X, i.e., a map $c : X \to X$. Then c is called:

(1) *extensive* if $m \leq c(m)$ for every $m \in X$,

(2) *monotonic* if $m \leq n \Rightarrow c(m) \leq c(n)$ for all $m, n \in X$,

(3) *idempotent* if $c(c(m)) = c(m)$ for all $m \in X$,

(4) *additive* if X is a join-semilattice and $c(m \vee n) = c(m) \vee c(n)$ for all $m, n \in X$,

(5) *grounded* if X has a smallest element 0 and $c(0) = 0$.

An endomap c on a poset X is called a *closure operator* on X if it is extensive and monotonic. A grounded, idempotent, and additive closure operator is called a *Kuratowski* closure operator. If c is a closure operator on a poset X, then the fixed points of c (i.e., elements of $x \in X$ with $c(x) = x$) are called the *closed elements*.

Definition 2 Let c and d be endomaps on posets X and Y, respectively. A left adjoint $f : X \to Y$ is called *continuous* if $f^{-1}(d(n)) \geq c(f^{-1}(n))$ for all $n \in Y$.

Note that, for monotonic endomaps c and d on X and Y, respectively, a left adjoint $f : X \to Y$ is continuous if and only if $f(c(m)) \leq d(f(m))$ for all $m \in X$.

3. Realization of endomaps on a poset by binary relations

Definition 3 Let X be a poset and ρ be a binary relation on X, i.e., $\rho \subseteq X \times X$. Then ρ is called

(1) *weakly reflexive* if $x\rho x$ whenever x is the smallest or greatest element of X (provided such an element exists),

(2) *minor* if $m\rho n \Rightarrow m \leq n$ for all $m, n \in X$,

(3) *extendable* if $m' \leq m$, $m\rho n$, and $n \leq n'$ imply $m'\rho n'$ for all $m, m', n, n' \in X$,

(4) $\bigwedge$*-stable* if X is a meet-complete semilattice (i.e., has meets of all nonempty subsets) and, whenever $m\rho n_i$ for every $i \in I \neq \emptyset$ ($m \in X$ and $n_i \in X$ for all $i \in I$), $m\rho \bigwedge_{i\in I} n_i$,

(5) *interpolative* if, for all $m, n \in X$, $m\rho n$ implies that there is $p \in X$ such that $m\rho p$ and $p\rho n$,

(6) *join-preserving* if X is a join-semilattice and, for all $m, m', n, n' \in X$, $m\rho m'$ and $n\rho n'$ imply $(m \vee n)\rho(m' \vee n')$.

If a binary relation ρ on a poset X is minor and extendable, then we call it a *topogenous order* in accordance with [9] and [10]. (This concept of a topogenous order differs from the one in [3], which is defined to be a binary relation on a power set that is not only minor and extendable but also weakly reflexive, union-preserving and intersection-preserving.) Note that a topogenous order is transitive but need not be reflexive or antisymmetric, hence need not be a (partial) order.

Definition 4 Let X, Y be posets and ρ, σ be binary relations on X and Y, respectively (hence, $\rho \subseteq X \times X$ and $\sigma \subseteq Y \times Y$). A left adjoint $f : X \to Y$ is called *compatible* if, for all $p, q \in Y$, $p\sigma q \Rightarrow f^{-1}(p)\rho f^{-1}(q)$.

Let X be a meet-complete lattice. We denote by C_X the set of all endomaps on X and by R_X the set of all binary relations on X.

Let $\preceq$ be the binary relation on C_X defined by $c \preceq d$ if and only if $c(m) \leq d(m)$ for all $m \in X$. Evidently, $\preceq$ is a partial order on C_X. Further, let $\trianglelefteq$ be the binary relation on R_X defined by $\rho \trianglelefteq \sigma$ if and only if $m\rho n \Rightarrow m\sigma n$ for all $m, n \in X$. Clearly, $\trianglelefteq$ is a partial order on R_X.

For every $c \in C_X$, let ρ^c be the binary relation on X given by $m\rho^c n \Leftrightarrow c(m) \leq n$ whenever $m, n \in X$. We denote by $H : C_X \to R_X$ the map defined by $H(c) = \rho^c$ for all $c \in C_X$. Any restriction of H will also be denoted by H.

For every $\rho \in R_X$, let c^ρ be the endomap on X given by $c^\rho(m) = \bigwedge\{n \in X;\ m\rho n\}$ for all $m \in X$. We denote by $G : R_X \to C_X$ the map defined by $G(\rho) = c^\rho$ for all $\rho \in R_X$. Any restriction of G will also be denoted by G.

Theorem 1 *Let X be a meet-complete semilattice. Then (G, H) is a Galois connection between $(R_X, \trianglelefteq)$ and $(C_X, \preceq)$ such that $G \circ H = id_{C_X}$.*

Proof Let $\rho, \sigma \in R_X$, $\rho \trianglelefteq \sigma$, and let $m \in X$. Since $m\rho n \Rightarrow m\sigma n$ for all $n \in X$, we have $\{n \in X;\ m\rho n\} \subseteq \{n \in X;\ m\sigma n\}$. Consequently, $c^\rho(m) = \bigwedge\{n \in X;\ m\rho n\} \geq \bigwedge\{n \in X;\ m\sigma n\} = c^\sigma(m)$. Hence, $G(\rho) = c^\rho \preceq c^\sigma = G(\sigma)$; therefore, G is order reversing.

Let $c, d \in C_X$, $c \preceq d$, and let $m, n \in X$. If $m\rho^d n$, then $c(m) \leq d(m) \leq n$, hence $m\rho^c n$. Thus, $H(d) = \rho^d \trianglelefteq \rho^c = H(c)$; therefore, H is order reversing.

Let $c \in C_X$ and $m \in X$. Then $c^{\rho^c} = \bigwedge\{n \in X;\ m\rho^c n\} = \bigwedge\{n \in X;\ c(m) \leq n\} = c(m)$. Thus, $G(H(c)) = c^{\rho^c} = c$ and, consequently, $G \circ H = \mathrm{id}_{C_X}$.

Let $\rho \in R_X$ and $m, n \in X$. Then $m\rho n \Rightarrow \bigwedge\{p \in X;\ m\rho p\} = c^\rho(m) \leq n \Leftrightarrow m\rho^{c^\rho} n$. Therefore, $\rho \trianglelefteq \rho^{c^\rho} = H(G(\rho))$. The proof is complete.

Corollary 1 *For every meet-complete semilattice X, C_X is dually order isomorphic to the subset of R_X whose elements are the binary relations ρ on X that satisfy the following condition:*

($\star$) For every $m \in X$, the set $\{n \in X;\ m\rho n\}$ is a principal filter of X.

Proof Denote by $R_X^\star$ the subset of R_X whose elements are the binary relations ρ on X that satisfy the condition ($\star$). Let $c \in C_X$ and $m \in X$. Since $\{n \in X;\ m\rho^c m\} = \{n \in X;\ c(m) \leq n\}$, $\{n \in X;\ m\rho^c n\}$ is a principal filter of X (with the smallest element $c(m)$). Hence, $H(c) = \rho^c \in R_X^\star$.

Let ρ be binary relations on X that satisfies the condition ($\star$) and let $m, n \in X$. Then, $m\rho^{c^\rho} n$ is equivalent to $c^\rho(m) = \bigwedge\{p \in X;\ m\rho p\} \leq n$, which is equivalent to $m\rho n$. Hence, $H(G(\rho)) = \rho^{c^\rho} = \rho$. Therefore, $H : C_X \to R_X^\star$ is surjective. By Theorem 1, $H : C_X \to R_X^\star$ is a dual order isomorphism (with the inverse orderisomorphism being G).

Proposition 1 *Let X be a meet-complete semilattice and $\rho \in R_X$ be an extendable element. Then ρ satisfies the condition ($\star$) in Corollary 1 if and only if ρ is $\bigwedge$-stable.*

Proof Let ρ satisfy ($\star$) and let $m\rho n_i$ for all $i \in I (\neq \emptyset)$. Then $\bigwedge_{i\in I} n_i \geq \bigwedge\{n \in X;\ m\rho n\}$ and $m\rho \bigwedge\{n \in X;\ m\rho n\}$, which yields $m\rho \bigwedge_{i\in I} n_i$ by the extendability. Conversely, let ρ be $\bigwedge$-stable. Then, for every $m \in X$, $m\rho \bigwedge\{n \in X;\ m\rho n\}$, hence $\{n \in X;\ m\rho n\}$ is a principal filter in X.

Remark 1 (a) Clearly, if ρ is extendable, then the condition ($\star$) in Corollary 1 is equivalent also to the following condition: For every pair $m, n \in X$, $\bigwedge\{p \in X;\ m\rho p\} \leq n \Leftrightarrow m\rho n$.

(b) Let $m \in X$. Since $c(m)$ is the smallest element of the principal filter $\{n \in X;\ m\rho^c n\}$, for every element $\rho \in R_X$ satisfying the condition ($\star$) in Corollary 1, $c^\rho(m)$ is the smallest element of the principal filter $\{n \in X;\ m\rho n\}$.

Proposition 2 *Let X be a meet-complete semilattice. An element $c \in C_X$ is*

(1) extensive if and only if ρ^c is minor,

(2) monotonic if and only if ρ^c is extendable,

(3) grounded if and only if $0\rho^c 0$.

Proof (1) If c is extensive, then we have $m\rho^c n \Rightarrow c(m) \leq n \Rightarrow m \leq n$ for all $m, n \in X$. Conversely, if $m\rho^c n \Rightarrow m \leq n$ for all $m, n \in X$, then $c(m) = \bigwedge\{n;\ m\rho^c n\} \geq \bigwedge\{n;\ m \leq n\} \geq m$ whenever $m \in X$.

(2) If c is monotonic and $m, m', n, n' \in X$ are elements with $m' \leq m$, $m\rho^c n$, and $n \leq n'$, then $c(m') \leq c(m) \leq n \leq n'$. Therefore, $c(m') \leq n'$, which yields $m'\rho^c n'$. Conversely, let $m' \leq m$, $m\rho^c n$, and $n \leq n'$ imply $m'\rho^c n'$ for all $m, m', n, n' \in X$ and let $m \leq n$ $(m, n \in X)$. Then $c(n) = \bigwedge\{p;\ n\rho^c p\} \geq \bigwedge\{p;\ m \leq p\} = c(m)$.

(3) is clear.

Corollary 2 *Let X be a meet-complete semilattice. An element $c \in C_X$ is a closure operator on X if and only if ρ^c is a topogenous order on X.*

Proposition 3 *Let X be a meet-complete semilattice. A closure operator $c \in C_X$ is idempotent if and only if ρ^c is interpolative.*

Proof Suppose that c is idempotent and let $m, n \in X$, $m\rho^c n$. Putting $p = c(m)$, we get $c(m) \leq p$, so $m\rho^c p$. We have $c(p) = c(c(m)) = c(m) \leq n$, so $p\rho^c n$ and thus ρ^c is interpolative.

Conversely, let ρ^c be interpolative and let $m \in X$. Then, for every $n \in X$ with $m\rho^c n$, there exists $p \in X$ such that $m\rho^c p$ and $p\rho^c n$. Hence, $c(m) \leq p\rho^c n \leq n$, which yields $c(m)\rho^c n$ by Proposition 2(2). We have shown that $\{n;\ m\rho^c n\} \subseteq \{n;\ c(m)\rho^c n\}$; therefore, $c(m) = \bigwedge\{n;\ m\rho^c n\} \leq \bigwedge\{n;\ c(m)\rho^c n\} = c(c(m))$. Since c is extensive, we have $c(c(m)) = c(m)$.

Proposition 4 *Let X be a meet-complete lattice. A monotonic element $c \in C_X$ is additive if and only if ρ^c is join-preserving.*

Proof Let $c \in C_X$ be a monotonic element and suppose that it is additive. Let $m, n, p, q \in X$ be elements with $m\rho^c n, p\rho^c q$. Then $c(m) \leq n, c(p) \leq q$, so $c(m \vee p) = c(m) \vee c(p) \leq n \vee q$, hence $(m \vee p)\rho^c(n \vee q)$. Therefore, ρ^c is join-preserving.

Conversely, let ρ^c be join-preserving and suppose that c is not additive. Then there exist $m, p \in X$ such that $c(m \vee p) \neq s = c(m) \vee c(p)$. Monotonicity implies $c(m) \vee c(p) = s \leq c(m \vee p)$, so $c(m \vee p) > s$. Now $c(m) \leq s$ and $c(p) \leq s$ imply $m\rho^c s$ and $p\rho^c s$, hence $(m \vee p)\rho^c s$. Therefore, $c(m \vee p) \leq s$, which is a contradiction with $s < c(m \vee p)$. Therefore, c is additive.

Proposition 5 *Let X be a coatomic complete lattice and $c \in C_X$. Then c is additive if and only if, for all $m, n \in X$ and every coatom $a \in X$, $(m \vee n)\rho^c a \Leftrightarrow (m\rho^c a$ and $n\rho^c a)$.*

Proof Let $m, n \in X$. Then, for every co-atom $a \in X$, $c(m \vee n) = c(m) \vee c(n)$ is equivalent to $c(m \vee y) \leq a \Leftrightarrow (c(m) \vee c(n)) \leq a$, which is equivalent to $(m \vee n)\rho^c a \Leftrightarrow (c(m) \leq a$ and $c(n) \leq a)$. Since the right side of the last equivalence is equivalent to the conjunction of $m\rho^c a$ and $n\rho^c a$, the proof is complete.

Proposition 6 *Let X, Y be meet-complete semilattices. If $c \in C_X$, $d \in C_Y$, and $f : (X, c) \to (Y, d)$ is a continuous map, then $f : (X, \rho^c) \to (Y, \rho^d)$ is compatible. Conversely, if $\rho \in R_X$, $\sigma \in R_Y$, and $f : (X, \rho) \to (Y, \sigma)$ is a compatible map, then $f : (X, c^\rho) \to (Y, c^\sigma)$ is continuous.*

Proof Let $f : (X, c) \to (Y, d)$ be continuous and let $m, n \in Y$, $m\rho^d n$. Then $d(m) \leq n$, hence $c(f^{-1}(m)) \leq f^{-1}(d(m)) \leq f^{-1}(n)$. This yields $f^{-1}(m)\rho^c f^{-1}(n)$, hence $f : (X, \rho^c) \to (Y, \rho^d)$ is compatible.

Conversely, let $f:(X,\rho)\to(Y,\sigma)$ be a compatible map and let $n\in Y$. Then $f^{-1}(c^{\sigma}(n))=f^{-1}(\bigwedge\{p\in Y;\ n\sigma p\})=\bigwedge\{f^{-1}(p)\in X;\ n\sigma p\}\geq\bigwedge\{f^{-1}(p)\in X;$
$f^{-1}(n)\rho f^{-1}(p)\}\geq\bigwedge\{m\in X;\ f^{-1}(n)\rho m\}=c^{\rho}(f^{-1}(n))$. Hence, f is continuous.

By Theorem 1, we have $c=c^{\rho^{c}}$ for every $c\in C_X$ where X is a meet-complete semilattice. Therefore, Proposition 6 results in

Corollary 3 *If X is a meet-complete semilattice, $c\in C_X$, and $d\in C_Y$, then a map $f:(X,c)\to(Y,d)$ is continuous if and only if $f:(X,\rho^{c})\to(Y,\rho^{d})$ is compatible.*

Proposition 7 *Let X be a meet-complete semilattice, $\rho\in R_X$ be a binary relation on X satisfying condition $(\star)$ in Corollary 1, and let $m\in X$. If m is a fixed point of c^{ρ}, then $m\rho m$. Conversely, if ρ is minor and $m\rho m$, then m is a fixed point of c^{ρ}.*

Proof Let m be a fixed point of c^{ρ}. Then $c^{\rho}(m)=\bigwedge\{n\in X;\ m\rho n\}=m$, thus $m\rho m$ because, by ρ satisfies the condition $(\star)$.

Conversely, let ρ be minor and let $m\rho m$. Then $c^{\rho}(m)=\bigwedge\{n\in X;\ m\rho n\}\leq m$; hence, m is a fixed point of c^{ρ} because c^{ρ} is extensive by Proposition 2(1) and Corollary 1 (which yields $\rho^{c^{\rho}}=\rho$).

Thus, by Proposition 1 and Corollary 2, if ρ is a $\bigwedge$-stable topogenous order on a meet-complete semilattice X, then an element $m\in X$ is c^{ρ}-closed if and only if $m\rho m$. Moreover, Proposition 7 results in

Corollary 4 *If ρ is a $\bigwedge$-stable topogenous order on a meet-complete semilattice X and $m\in X$, then $c^{\rho}(m)=\bigwedge\{n\in X;\ m\leq n$ and $n\rho n\}$.*

Example 1 In [9], categorical neighborhood operators are studied in relationship to categorical topogenous orders. An analogous definition of a neighborhood operator in our poset-theoretic setting is as follows:

A *neighborhood operator* on a poset X is a map $\nu:X\to\exp X$ such that

(i) $\nu(m)$ is a stack for every $m\in X$,

(ii) $n\in\nu(m)\Rightarrow m\leq n$ for all $m,n\in X$,

(iii) $m\leq n\Rightarrow\nu(n)\subseteq\nu(m)$ for all $m,n\in X$.

The elements of $\nu(m)$ are called *neighborhoods* of m. Analogously to [9], it may easily be shown that, on an arbitrary poset X, there is a one-to-one correspondence between the set of all topogenous orders on X and that of all neighborhood operators on X (and the correspondence is even an order isomorphism between the two sets provided with naturally defined partial orders). Such a correspondence is obtained by assigning to a topogenous order ρ on X the neighborhood structure ν^{ρ} on X in the following way: for every $m\in X$, $\nu^{\rho}(m)=\{n\in X;\ m\rho n\}$. The inverse correspondence is obtained by assigning to a neighborhood structure ν on X the topogenous order ρ^{ν} on X in the following way: for every $m,n\in X$, $m\rho^{\nu}n\Leftrightarrow n\in\nu(m)$.

4. A cartesian closed category of topogenous systems

Recall [1] that a category $\mathcal{C}$ with finite products is *cartesian closed* if it possesses a well-behaved binary operation of exponentiation on the class of objects. This means that, for every pair of objects $A, B \in \mathcal{C}$, there is an object $B^A \in \mathcal{C}$ and a morphism $ev : A \times B^A \to B$ (the so-called *evaluation map*) having the property that, for every morphism $f : A \times C \to B$ in $\mathcal{C}$, there exists a unique morphism $f^* : C \to B^A$ such that $ev \circ (id_A \times f^*) = f$. The well-behaved operation of exponentiation of objects makes cartesian closed categories useful for numerous applications. They play a particularly important role in mathematical logic (cf. [12]) and the theory of programming where they serve as models of typed lambda-calculi, which are important foundational programming languages (cf. [2]). Since the category of topological spaces and continuous maps is not cartesian closed, it has to be often replaced by a category of topological structures more general than topological spaces, e.g., certain closure spaces.

If c is a closure operator on a meet-complete semilattice X, then the pair (X, c) is called a *closure system* (to distinguish it from a closure space (X, c), which usually means that c is an endomap on the power set of X). And (X, c) is said to be *idempotent* if c is idempotent. Similarly, if ρ is a topogenous order on a meet-complete semilattice X, then the pair (X, ρ) is called a *topogenous system* (to distinguish it from a topogenous space (X, ρ), which means [3] that ρ is a binary relation on the power set of X). And (X, ρ) is said to be $\bigwedge$*-stable* or *interpolative* if ρ is $\bigwedge$-stable or interpolative, respectively.

Given two closure systems (X, c) and (Y, d), a map $f : (X, c) \to (Y, d)$ is called *closed* if, for every closed element $m \in X$, the element $f(m)$ is closed. And, given two topogenous systems (X, ρ) and (Y, σ), a map $f : (X, \rho) \to (Y, \sigma)$ is called *loop-preserving* if, for every element $m \in X$, $m \rho m \Rightarrow f(m) \sigma f(m)$.

As a consequence of the results of the previous section, particularly Proposition 7, we get:

Proposition 8 *Let (X, c) and (Y, d) be closure systems. Then a map $f : (X, c) \to (Y, d)$ is closed if and only if $f : (X, \rho^c) \to (Y, \rho^d)$ is loop-preserving.*

Theorem 2 *The category of $\bigwedge$-stable interpolative topogenous systems and loop-preserving maps is cartesian closed.*

Proof In [14], Theorem 3.2, it is proved that the category $\mathcal{C}$ of idempotent closure systems with closed maps as morphisms is cartesian closed, but the closure systems dealt with in [14] differ from those introduced in this note. Namely, in the definition of a closure systems (X, c) in [14], X is a poset with every principal filter being a meet-complete semilattice. Thus, every closure system in our sense is a closure system in the sense of [14]. Therefore, the category $\mathcal{D}$ of closure systems in the sense of this note with closed maps as morphisms is a full subcategory of $\mathcal{C}$. It may easily be seen in the proof of Theorem 3.2 in [14] that $\mathcal{D}$ inherits cartesian closedness from $\mathcal{C}$, i.e., that $\mathcal{D}$ is closed under the formation of cartesian products and power objects in $\mathcal{C}$. By Proposition 8 and the results of the previous section, the category of $\bigwedge$-stable interpolative topogenous systems and loop-preserving maps is isomorphic to $\mathcal{D}$, hence cartesian closed, too.

5. Conclusion

In [9], correspondences between topogenous structures and closure operators on categories are investigated, but these categorical topogenous structures and categorical closure opertors are nothing but certain binary relations and closure operators, respectively, on the (complete) subobject lattices of the given category. In this note, we have defined and discussed closure operators in a more general setting – not only on complete lattices but on arbitrary posets. Thus, the results obtained may be used, among others, when studying topogenous structures and closure operators on categories. Based on a Galois connection between binary relations and endomaps on a poset, we have specified the relational axioms that correspond to certain closure axioms in the connection. In particular, a condition is found under which topogenous orders correspond to closure operators. This result is then used to find a cartesian closed subcategory of the category of topogenous orders and compatible maps.

Acknowledgments

The first author was supported by Brno University of Technology from the Specific Research project no. FSI-S-23-8161.The second and third authors were supported by Brno University of Technology from the project MeMoV II no. CZ.02.2.69/0.0/0.0/18-053/0016962.

References

[1] Adámek J, Herrlich H, Strecker GE. Abstract and Concrete Categories. New York, USA: Wiley & Sons, 1990.

[2] Barendregt HP. The Lambda Calculus. Amsterdam, New York, and Oxford: North-Holland, 1984.

[3] Császár Á. Foundations of General Topology. Oxford, New York: Pergamon Press, 1963.

[4] Császár Á. Finite extensions of topogenities. Acta Mathematica Hungarica 2000; 89 (1-2): 55-69. https://doi.org/10.1023/A:1026725424906

[5] Dikranjan D, Tholen W. Categorical Structure of Closure Operators. Dordrecht, Netherlands: Kluwer Acadeic Publishers, 1995.

[6] Engelking R. General Topology. Berlin, Germany: Heldermann Verlag, 1989.

[7] Fletcher P, Lindgren WF. Quasi-Uniform Spaces. Lecture Notes in Pure and Applied Mathematics 77, New York, USA: Marcel Dekker, 1982.

[8] Grätzer G. General Lattice Theory. Basel, Switzerland: Birkhäuser Verlag, 1978.

[9] Holgate D, Iragi M, Razafindrakoto A. Topogenous and nearness structures on categories. Applied Categorical Structures 2016; 24 (5): 447-455. https://doi.org/10.1007/s10485-016-9455-x

[10] Holgate D, Iragi M. Quasi-uniform and syntopogenous structures on categories. Topology and its Applications 2019; 263: 16-25. https://doi.org/10.1016/j.topol.2019.05.024

[11] Richmond T. Genereal Topology: An Introduction. Berlin, Germany: De Gruyter, 2020.

[12] Lambek J, Scott PJ. Introduction to Higher Order Categorical Logic. Cambridge, UK: Cambridge University Press, 1988.

[13] Šlapal J. A note on F-topologies. Mathematische Nachrichten 1989; 141 (1): 283-287. https://doi.org/10.1002/mana.19891410126

[14] Šlapal J. On categories of ordered sets with a closure operator. Publicationes Mathematicae Debrecen 2011; 78 (1): 61-69. https://doi.org/10.5486/PMD.2011.4442